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Solving the Perspective-Three-Point Problem Using Comprehensive Gröbner Systems*

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Abstract A complete solution classification of the perspective-three-point (P3P) problem is given by using the Gröbner basis method. The structure of the solution space of the polynomial system deduced by the P3P problem can be obtained by computing a comprehensive Gröbner system. Combining with properties of the generalized discriminant sequences, the authors give the explicit conditions to determine the number of distinct real positive solutions of the P3P problem. Several examples are provided to illustrate the effectiveness of the proposed conditions.

Keywords Comprehensive Gröbner system, parametric polynomials, perspective-three-point problem, real solutions.

1 Introduction

The Perspective-n-Point (PnP) problem comes from camera calibration, which is to determine the position of the camera with respect to a scene object from n corresponding points. It is a classical problem in many research fields such as image analysis, automated cartography, robotics, etc.

In 1981, Fischler and Bolles[1] gave the general mathematical definition of the problem as follows:
"Given the relative spatial locations of n control points, and given the angle to every pair of control points from an additional point called the Center of Perspective C_P, find the lengths of the line segments joining C_P to each of the control points".

In 1984, Ganapathy[2] proved that the position of the center of perspective was uniquely determined when the number of control points were equal or greater than six. Hu, et al.[3] showed that the P5P problem could have 2 solutions and P4P problem had at most 5 solutions.

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In [4], Wu, et al. revised the PnP problem, and gave a systematic investigation on it from both geometric and algebraic standpoints. Moreover, Fischler and Bolles[1] also noticed that the P3P problem had at most four possible solutions. Recently, there were some other researches about the PnP problem, including [5–10].

There were many algorithms designed for solving the P3P problem. In 1991, Haralick, et al.[11] reviewed the major direct solutions up to that time. Yang[12] gave a solution classification of the P3P problem under some non-degenerate conditions. Gao, et al.[13] used the Wu-Ritt's zero decomposition method, Descartes' rule of sign and the Sylvester-Habicht sequences to obtain a complete solution classification of the P3P problem for the first time. In [14], Reid, et al. introduced a symbolic-numeric method, which was based on the geometric theory of partial differential equations, for solving the problem of camera pose estimation. In 2008, Faugère, et al.[15] used the discriminant variety to give a full classification of the P3P problem in the case where the three control points formed an isosceles triangle. In [16], Yang gave an explicit criterion on determining the number of real roots of parametric polynomial systems.

In this article, we use the comprehensive Gröbner system (CGS) and the discriminant sequence to solve the P3P problem. The definition of CGS was first introduced by Weispfenning in 1992[17]. It can give the structure of solution space (finitely many, infinitely many, in which their dimension) of the parametric polynomial systems in the whole parametric space. It is similar to the properties of the Gröbner basis. The discriminant sequence[18] of a polynomial $f(x)$ is similar to the sturm sequence[19], which can give the number of distinct real roots of $f(x)$. By computing a minimal CGS of the parametric polynomial system deduced by the P3P problem, we can obtain the structure of the solution space of the P3P problem. The minimal CGS gives a disjoint partition of the parametric space. In every partition, the number of complex solutions counted with multiplicities of the polynomial system is decided. Then by using the discriminant sequence, the number of distinct real positive solutions of the P3P problem can be determined. As a result, we give a complete solution classification of the P3P problem by using the Gröbner basis method, and the explicit conditions under which the P3P problem has one, two, three, or four real positive solutions. Some values of parameters are given such that the P3P problem has different number of real positive solutions.

The rest of the paper is organized as follows. In Section 2, we describe the P3P problem and give the polynomial systems of this problem. Some notations and preliminaries about CGS and discriminant sequence are presented in Section 3. In Section 4, a minimal CGS of the parametric polynomial system deduced by P3P problem is computed. In Section 5, we give the specific analysis about the solution classification of the P3P problem. Conclusions are presented in Section 6.

2 Description of the Perspective-Three-Point Problem

Let P be the perspective point, A, B, C be the control points, l_{AB}, l_{BC}, l_{AC} be the length of the three sides AB, BC, AC respectively, and $p = 2\cos\angle BPC, q = 2\cos\angle APC, r = 2\cos\angle APB$. Let X, Y, Z be the distances between the point P and the points A, B, C, respectively.

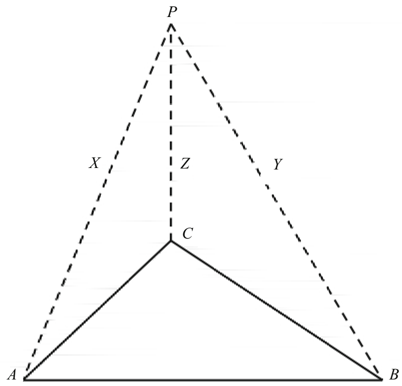

Figure 1 The P3P problem

From the triangles PBC, PAC, PAB and the Cosine Law, we obtain the following parametric polynomial system as in [13]:

$$\begin{cases} Y^2 + Z^2 - pYZ - l_{BC}^2 = 0, \\ X^2 + Z^2 - qXZ - l_{AC}^2 = 0, \\ X^2 + Y^2 - rXY - l_{AB}^2 = 0. \end{cases} \tag{1}$$

In (1), $l_{BC}, l_{AC}, l_{AB}, p, q, r$ are parameters, and X, Y, Z are variables. The real positive values of X, Y, Z are the solutions of the P3P problem. In the general case, the points P, A, B, C are not co-planar, that is, $p^2 + q^2 + r^2 - pqr - 4 \neq 0$, and $l_{BC} > 0, l_{AC} > 0, l_{AB} > 0$. As in [13], to simplify the equation system, we let $a = (\frac{l_{BC}}{l_{AB}})^2, b = (\frac{l_{AC}}{l_{AB}})^2, v = (\frac{l_{AB}}{Z})^2, x = \frac{X}{Z}, y = \frac{Y}{Z}$. Then, we have

$$\begin{cases} y^2 + 1 - py - av = 0, \\ x^2 + 1 - qx - bv = 0, \\ x^2 + y^2 - rxy - v = 0. \end{cases} \tag{2}$$

The last equation of (2) is equivalent to $v = x^2 + y^2 - rxy$. Eliminating v in the parametric polynomial systems (2), we obtain

$$\begin{cases} p_1 = (1-a)y^2 + arxy - py - ax^2 + 1 = 0, \\ p_2 = -by^2 + brxy + (1-b)x^2 - qx + 1 = 0, \end{cases} \tag{3}$$

where $a > 0, b > 0$ and $p^2 + q^2 + r^2 - pqr - 4 \neq 0$. The P3P problem is transformed to solve the real positive solutions of the quadratic parametric polynomial system (3).

3 Preliminaries About CGS and Discriminant Sequence

In this section, we review the definitions of the CGS and the discriminant sequence. The part of CGS is similar to that given by Kapur, et al.[20], and the notation of discriminant sequence follows that given in [18] and [21].

3.1 Comprehensive Gröbner System

Let k be a field, R be a polynomial ring $k[U]$ in parameters $U = \{u_1, u_2, \cdots, u_m\}$, and $R[X]$ be a polynomial ring over R in variables $X = \{x_1, x_2, \cdots, x_n\}$ where X and U are disjoint. For a polynomial $f \in R[X] = k[U][X]$, the leading power product and leading coefficient of f w.r.t. the ordering $\prec_X$ are denoted by $\text{lpp}_X(f)$ and $\text{lc}_X(f)$, respectively. Note that $\text{lc}_X(f) \in k[U]$.

Given a field L, a specialization of R is a homomorphism $\sigma : R \mapsto L$. We always assume L is an algebraically closed field containing k and we only consider the specializations induced by the elements in L^m. That is, for $\overline{a} \in L^m$, the induced specialization $\sigma_{\overline{a}}$ is defined by

$$\sigma_{\overline{a}} : f \mapsto f(\overline{a}),$$

for any $f \in R$. Every specialization $\sigma : R \mapsto L$ extends canonically to a specialization $\sigma : R[X] \mapsto L[X]$ by applying σ coefficient-wise.

For a parametric polynomial system, a CGS is defined below.

Definition 3.1 Let F be a subset of $R[X]$, A_1, A_2, $\cdots$, A_l be algebraically constructible subsets of L^m, S be a subset of L^m such that $S = A_1 \cup A_2 \cup \cdots \cup A_l$, and $G_1, G_2, \cdots, G_l$ be subsets of $R[X]$. A finite set $\mathcal{G} = \{(A_1, G_1), (A_2, G_2), \cdots, (A_l, G_l)\}$ is called a *comprehensive Gröbner system* on S for F, if $\sigma_{\overline{a}}(G_i)$ is a Gröbner basis for the ideal $\langle \sigma_{\overline{a}}(F) \rangle$ in $L[X]$ for any $\overline{a} \in A_i$, $i = 1, 2, \cdots, l$. Each (A_i, G_i) is called a branch of $\mathcal{G}$. If $S = L^m$, $\mathcal{G}$ is called a comprehensive Gröbner system for F.

For a set $A \subset R = k[U]$, the affine variety defined by A in L^m is denoted by $\mathbb{V}(A)$. Following [20], a constructible set A_i is defined to be of the form: $A_i = \mathbb{V}(E_i) \setminus \mathbb{V}(N_i)$, where E_i, N_i are subsets of $k[U]$. We call A_i the parametric constraint of the branch (A_i, G_i).

Definition 3.2 A comprehensive Gröbner system $\mathcal{G} = \{(A_1, G_1), (A_1, G_1), \cdots, (A_l, G_l)\}$ for F is said to be minimal, if for each $i = 1, 2, \cdots, l$,

1) $A_i \neq \emptyset$, and furthermore, for each $i, j = 1, 2, \cdots, l$, $A_i \cap A_j = \emptyset$ whenever $i \neq j$;

2) for each $g \in G_i$, $\sigma_{\overline{a}}(\text{lc}_X(g)) \neq 0$ for any $\overline{a} \in A_i$;

3) for all $g \in G_i$, $\text{lpp}_X(g)$ is not divisible by any leading power products of $G_i \setminus \{g\}$.

For any branch $(A_i, G_i) \in \mathcal{G}$, under the parametric constraint A_i, the total number of solutions of the system $\{g = 0 \mid g \in G_i\}$ can be decided by the following lemma. See Chapter 4 of [22] for details.

Lemma 3.3 *Let k be a field, L be an algebraically closed field containing k, and $I = \langle f_1, f_2, \cdots, f_s \rangle$ be a zero dimensional ideal in $k[X]$. Then the quotient ring $k[X]/I$ is a vector space over k. Furthermore, the total number of solutions of the system $\{f_1 = 0, f_2 = 0, \cdots, f_s = 0\}$ in L^n, counted with multiplicities, is the dimension of $k[X]/I$.*

Let I be a zero dimensional ideal, and G be a Gröbner basis of I w.r.t. a monomial order $<$. Since $\dim\ k[X]/I = \dim\ k[X]/\langle \text{lpp}(I) \rangle$, the dimension of $k[X]/I$ is equal to the number of the monomials which are not divided by any leading power products of polynomials in G, where $\langle \text{lpp}(I) \rangle$ is the ideal generated by the leading power product of polynomials in I.

We give an example to illustrate the minimal CGS and the above Lemma 3.3.

Example 3.4 Let $F_1 = \{x^2 - y^2, ab^2x^2 - b^2xy^2, y^2 - ax + 1\} = \{f_1, f_2, f_3\}$ be a subset of $k[a,b][x,y]$, where a, b are parameters and x, y are variables. We compute a minimal CGS for F_1, and determine the number of solutions (counted with multiplicities) of the system $F_2 = \{f_1 = 0, f_2 = 0, f_3 = 0\}$.

A minimal CGS for F_1 w.r.t. the lexicographic term ordering $x \succ y$ is

$$\begin{aligned}\mathcal{G} = \{&(\mathbb{V}(\emptyset) \setminus \mathbb{V}(b^2), \{1\}), (\mathbb{V}(b) \setminus \mathbb{V}(a), \{y^4 - a^2y^2 + 2y^2 + 1, ax - y^2 - 1\}),\\ &(\mathbb{V}(a,b) \setminus \mathbb{V}(\emptyset), \{y^2 + 1, x^2 + 1\})\}.\end{aligned}$$

The branch $(\mathbb{V}(\emptyset) \setminus \mathbb{V}(b^2), \{1\})$ indicates that a Gröbner basis of $\langle F_1 \rangle$ is $G_1 = \{1\}$ when $b \neq 0$. Thus, F_2 has no solution when $b \neq 0$.

The branch $(\mathbb{V}(b) \setminus \mathbb{V}(a), \{y^4 - a^2y^2 + 2y^2 + 1, ax - y^2 - 1\})$ indicates that a Gröbner basis of $\langle F_1 \rangle$ is $G_2 = \{y^4 - a^2y^2 + 2y^2 + 1, ax - y^2 - 1\}$ when $a \neq 0$ and $b = 0$. Since the monomials $1, y, y^2$ and y^3 are not divided by the leading power products of polynomials in G_2, F_2 has four solutions when $a \neq 0$ and $b = 0$.

The branch $(\mathbb{V}(a,b) \setminus \mathbb{V}(\emptyset), \{y^2 + 1, x^2 + 1\})$ indicates that a Gröbner basis of $\langle F_1 \rangle$ is $G_3 = \{y^2 + 1, x^2 + 1\}$ when $a = 0$ and $b = 0$. Since the monomials $1, x, y$, and xy are not divided by the leading power products of polynomials in G_3, F_2 also has four solutions when $a = 0$ and $b = 0$.

3.2 Discriminant Sequence

Given two parametric polynomials $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ and $g(x) = b_0x^m + b_1x^{m-1} + \cdots + a_m$ in $k[x]$, where $a_0b_0 \neq 0$, we define the resultant of f and g w.r.t. x to be the determinant

$$\mathrm{Res}(f,g,x) = \det \begin{pmatrix} a_0 & & & b_0 & & \\ a_1 & a_0 & & b_1 & b_0 & \\ & a_1 & \ddots & & b_1 & \ddots \\ \vdots & & \ddots\ a_0 & \vdots & & \ddots\ b_0 \\ & \vdots & a_1 & & \vdots & b_1 \\ a_n & & & b_m & & \\ & a_n & \vdots & & b_m & \vdots \\ & & \ddots & & & \ddots \\ & & a_n & & & b_m \end{pmatrix}_{(m+n)\times(m+n)}.$$

From the property of resultant, f and g have common zeros if and only if $\mathrm{Res}(f,g,x) = 0$.

Let f' be the derivative of f w.r.t. x, $r(x)$ be the remainder of $f'g$ divided by f. Assume $r(x) = c_1x^{n-1} + c_2x^{n-2} + \cdots + c_n$. The following $2n \times 2n$ matrix is called the generalized discrimination matrix of $f(x)$ with respect to $g(x)$:

$$\mathrm{Discr}(f,g)=\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & & \\ 0 & c_1 & c_2 & \cdots & c_n & & & & \\ & a_0 & a_1 & \cdots & a_{n-1} & a_n & & & \\ & 0 & c_1 & \cdots & c_{n-1} & c_n & & & \\ & & & & \vdots & \vdots & & & \\ & & & a_0 & a_1 & a_2 & \cdots & a_n & \\ & & & 0 & c_1 & c_2 & \cdots & c_n & \end{pmatrix}.$$

Let $D_0 = 1$ and denote the even principal minors of $\mathrm{Discr}(f, g)$ by $D_1, D_2, \cdots, D_n$. We call the $n + 1$ tuple

$$[D_0, D_1, \cdots, D_n]$$

the generalized discriminant sequence of $f(x)$ w.r.t. $g(x)$ and denote it by $\mathrm{GDL}(f, g)$. Particularly, we call $\mathrm{GDL}(f, 1)$ the discriminant sequence of $f(x)$.

Given a sequence $D = [D_0, D_1, \cdots, D_n]$, we call $\mathrm{Sign}(D) = [\mathrm{sign}(D_0), \mathrm{sign}(D_1), \cdots, \mathrm{sign}(D_n)]$ the sign list of the sequence D, where

$$\mathrm{sign}(D_i)=\begin{cases} 1, & \text{if } D_i > 0; \\ 0, & \text{if } D_i = 0; \\ -1, & \text{if } D_i < 0. \end{cases}$$

We construct the revised sign list $S = [s_0, s_1, \cdots, s_n]$ of D according to the following rule:

(i) If $[\mathrm{sign}(D_i), \mathrm{sign}(D_{i+1}), \cdots, \mathrm{sign}(D_{i+j})]$ is a section of $\mathrm{Sign}(D)$, where $\mathrm{sign}(D_i) \neq 0$, $\mathrm{sign}(D_{i+1}) = \cdots = \mathrm{sign}(D_{i+j-1}) = 0$, $\mathrm{sign}(D_{i+j}) \neq 0$, then we replace the subsection $[\mathrm{sign}(D_{i+1}), \cdots, \mathrm{sign}(D_{i+j-1})]$ with

$$[-\mathrm{sign}(D_i), -\mathrm{sign}(D_i), \mathrm{sign}(D_i), \mathrm{sign}(D_i), -\mathrm{sign}(D_i), -\mathrm{sign}(D_i), \mathrm{sign}(D_i), \mathrm{sign}(D_i), \cdots]$$

keeping the number of terms;

(ii) Otherwise, let $s_k = \mathrm{sign}(D_k)$.

For simplicity, we use the notation $\mathrm{RSGDL}(f, g)$ to denote the revised sign list of $\mathrm{GDL}(f, g)$.

Yang, et al.[18] gave a method to determine the number of distinct real roots of a polynomial.

Lemma 3.5 *Given two polynomials $f(x)$ and $g(x)$ with real coefficients, if the number of the sign changes of the revised sign list of* $\mathrm{GDL}(f, g)$ *is v, and the number of the non-vanishing members of the revised sign list is l, then*

$$l - 2v - 1 = \sharp\{x \in \mathbb{R} | f(x) = 0, g(x) > 0\} - \sharp\{x \in \mathbb{R} | f(x) = 0, g(x) < 0\},$$

where $\sharp\{\cdot\}$ represents the card of the set $\{\cdot\}$. In particular, if $g(x) = 1$, then the number of distinct real roots of $f(x)$ equals $l - 2v - 1$.

Since $\{x \in \mathbb{R} | f(x) = 0\} = \{x \in \mathbb{R} | f(x) = 0, g(x) > 0\} \cup \{x \in \mathbb{R} | f(x) = 0, g(x) = 0\} \cup \{x \in \mathbb{R} | f(x) = 0, g(x) < 0\}$, the following consequence[18] is obvious by Lemma 3.5.

Lemma 3.6 *For two polynomials $f(x)$ and $g(x)$ with real coefficients, let l_1, l_2, v_1, v_2 be the number of the non-vanishing members and the sign changes of* RSGDL$(f, 1)$, RSGDL(f, g) *respectively. If f and g have no common zero, then $\sharp\{x \in \mathbb{R} | f(x) = 0, g(x) > 0\} = \frac{1}{2}(l_1 + l_2 - 2v_1 - 2v_2 - 2)$.*

Let $f(x), g(x), h(x)$ be polynomials in $\mathbb{R}[x]$. The card of $\{x \in \mathbb{R} \mid f(x) = 0\}$ is denoted by $\mathrm{Card}_{\mathbb{R}}(f)$, the card of $\{x \in \mathbb{R} \mid f(x) = 0, g(x) > 0\}$ is denoted by $\mathrm{Card}_{\mathbb{R}}(f, g > 0)$, and the card of $\{x \in \mathbb{R} \mid f(x) = 0, g(x) > 0, h(x) > 0\}$ is denoted by $\mathrm{Card}_{\mathbb{R}}(f, g > 0, h > 0)$. The following result is similar to the equation (7) in [13].

Proposition 3.7 *Let $f(x)$ and $g(x)$ be two polynomials in $\mathbb{R}[x]$. If* $\mathrm{Res}(f, x, x) \neq 0$ *and* $\mathrm{Res}(f, g, x) \neq 0$*, then*

$$\begin{aligned}&\mathrm{Card}_{\mathbb{R}}(f, g > 0, x > 0)\\ =&\frac{1}{2}(\mathrm{Card}_{\mathbb{R}}(f, x > 0) + \mathrm{Card}_{\mathbb{R}}(f, g > 0) + \mathrm{Card}_{\mathbb{R}}(f, xg > 0) - \mathrm{Card}_{\mathbb{R}}(f)).\end{aligned} \tag{4}$$

Proof Since

$$\begin{cases}\mathrm{Card}_{\mathbb{R}}(f) = \mathrm{Card}_{\mathbb{R}}(f, x > 0) + \mathrm{Card}_{\mathbb{R}}(f, x = 0) + \mathrm{Card}_{\mathbb{R}}(f, x < 0),\\ \mathrm{Card}_{\mathbb{R}}(f, g > 0) = \mathrm{Card}_{\mathbb{R}}(f, g > 0, x > 0) + \mathrm{Card}_{\mathbb{R}}(f, g > 0, x = 0) + \mathrm{Card}_{\mathbb{R}}(f, g > 0, x < 0),\\ \mathrm{Card}_{\mathbb{R}}(f, x < 0) = \mathrm{Card}_{\mathbb{R}}(f, g > 0, x < 0) + \mathrm{Card}_{\mathbb{R}}(f, g = 0, x < 0) + \mathrm{Card}_{\mathbb{R}}(f, g < 0, x < 0),\\ \mathrm{Card}_{\mathbb{R}}(f, xg > 0) = \mathrm{Card}_{\mathbb{R}}(f, g > 0, x > 0) + \mathrm{Card}_{\mathbb{R}}(f, g < 0, x < 0),\end{cases}$$

we have

$$\begin{aligned}&\mathrm{Card}_{\mathbb{R}}(f, g > 0, x > 0)\\ =&\frac{1}{2}(\mathrm{Card}_{\mathbb{R}}(f, xg > 0) + \mathrm{Card}_{\mathbb{R}}(f, x > 0) + \mathrm{Card}_{\mathbb{R}}(f, g > 0) - \mathrm{Card}_{\mathbb{R}}(f)\\ &+\mathrm{Card}_{\mathbb{R}}(f, x = 0) + \mathrm{Card}_{\mathbb{R}}(f, g = 0, x < 0) - \mathrm{Card}_{\mathbb{R}}(f, g > 0, x = 0)).\end{aligned} \tag{5}$$

If $\mathrm{Res}(f, x, x) \neq 0$ and $\mathrm{Res}(f, g, x) \neq 0$, then $\mathrm{Card}_{\mathbb{R}}(f, x = 0) = 0$, $\mathrm{Card}_{\mathbb{R}}(f, g > 0, x = 0) = 0$ and $\mathrm{Card}_{\mathbb{R}}(f, g = 0, x < 0) = 0$. Thus, we have

$$\mathrm{Card}_{\mathbb{R}}(f, g > 0, x > 0) = \frac{1}{2}(\mathrm{Card}_{\mathbb{R}}(f, x > 0) + \mathrm{Card}_{\mathbb{R}}(f, g > 0) + \mathrm{Card}_{\mathbb{R}}(f, xg > 0) - \mathrm{Card}_{\mathbb{R}}(f)).$$

The proof is completed. ∎

Using the same notation as in [13], $\mathcal{C}_f^{(n,j)}(g > 0)$ denotes the conditions that make $f(x)$ having j real solutions such that $g > 0$ if $f(x) = 0$ has n real solutions.

4 The CGS of P3P Problem

In this section, we first compute a minimal CGS of the parametric polynomial systems (3), and then analyze the number of complex solutions of the P3P problem from the CGS.

Many algorithms have been provided for computing the CGS, such as CGB[17], ACGB[23], and BUILDTREE[24]. In this article, we use the implementation† given by Kapur, et al.[20] in the computer algebra system Singular.

We computed a minimal CGS $\mathcal{G}$ on $S = \mathbb{V}(\emptyset)\backslash\mathbb{V}(a, b, p^2 + q^2 + r^2 - pqr - 4)$ for $\{p_1, p_2\}$ in (3) w.r.t. the lexicographic term ordering $y \succ x$. There are total 13 branches in $\mathcal{G}$, which have complex solutions. We list them in the following.

Branch 1 $(\mathbb{V}(E_1)\backslash\mathbb{V}(N_1), G_1)$

$E_1 = \{e_{1,1}, e_{1,2}, \cdots, e_{1,43}\}$,

$N_1 = \{abq(-pqr + p^2 + q^2 + r^2 - 4)(-pqr + ar^2 + q^2 + r^2 - 4a)(-pq + r)\}$,

$G_1 = \{f_1, g_1\}$,

$$\begin{cases} f_1(x) = (pqr - ar^2 - q^2 - qr^2 + 4a)(qx - 1), \\ g_1(x, y) = b_{1,1}y + b_{1,2}x^4 + b_{1,3}x^2 + b_{1,4}x + b_{1,5}. \end{cases}$$

Branch 2 $(\mathbb{V}(E_2)\backslash\mathbb{V}(N_2), G_2)$

$E_2 = \{b - 1, a - 3, r^2 - 3, qr - 3p, pr - q, 3p^2 - q^2\}$,

$N_2 = \{abq(-ar^2 + p^2 + 4a - 4)(q^2 - 3)\}$,

$G_2 = \{f_2, g_2\}$,

$$\begin{cases} f_2(x) = (q^3 - 3q)x - q^2 + 3, \\ g_2(x, y) = (q^2 - 3)y + (3r - 3pq)x. \end{cases}$$

Branch 3 $(\mathbb{V}(E_3)\backslash\mathbb{V}(N_3), G_3)$

$E_3 = \{e_{3,1}, e_{3,2}, \cdots, e_{3,7}\}$,

$N_3 = \{abpr(-pqr + p^2 + q^2 + r^2 - 4)(-bpqr + bp^2 + bq^2 + ar^2 + br^2 - p^2 - 4a - 4b + 4)(ar^2 - p^2 - 4a + 4)\}$,

$G_3 = \{f_3, g_3\}$,

$$\begin{cases} f_3(x) = a_{3,1}x^2 + a_{3,2}x + a_{3,3}, \\ g_3(x, y) = b_{3,1}y + b_{3,2}x^3 + b_{3,3}x^2 + b_{3,4}x. \end{cases}$$

Branch 4 $(\mathbb{V}(E_4)\backslash\mathbb{V}(N_4), G_4)$

$E_4 = \{q, p, abr^2 - (a + b - 1)^2\}$,

$N_4 = \{abr(r^2 - 4)(ar^2 + br^2 - 4a - 4b + 4)(a^2 - 2ab + b^2 - a - b)(a + b - 1)(-a + b + 1)\}$,

$G_4 = \{f_4, g_4\}$,

$$\begin{cases} f_4(x) = (a + b - 1)(a^2 - 2ab + b^2 - a - b)x^2 + a^2br^2 - 4a^2b + 4ab, \\ g_4(x, y) = (a - b - 1)ry + (br^2 - 2a - 2b + 2)x. \end{cases}$$

†The implementation can be found at http://www.mmrc.iss.ac.cn/ dwang/software.html.

Branch 5 $(\mathbb{V}(E_5)\backslash\mathbb{V}(N_5), G_5)$

$E_5 = \{r, a+b-1, p^2-4b\}$,

$N_5 = \{bp(q^2+4b-4)(-b+1)(bq^2+4b^2-4b)(q^2+4b-4)\}$,

$G_5 = \{f_5, g_5\}$,

$$\begin{cases} f_5(x) = (q^2+4b-4)x^2, \\ g_5(x,y) = 2py + (q^2+4b-4)x^2 + 2xq - 4. \end{cases}$$

Branch 6 $(\mathbb{V}(E_6)\backslash\mathbb{V}(N_6), G_6)$

$E_6 = \{r, a+b-1\}$,

$N_6 = \{bp(p^2+q^2-4)(-b+1)(bp^2+bq^2-p^2)(-p^2+4b)\}$,

$G_6 = \{f_6, g_6\}$,

$$\begin{cases} f_6(x) = (bp^2+bq^2-p^2)x^2 + (p^2q-4bq)x - p^2 + 4b, \\ g_6(x,y) = py + xq - 2. \end{cases}$$

Branch 7 $(\mathbb{V}(E_7)\backslash\mathbb{V}(N_7), G_7)$

$E_7 = \{abr^2 - (a+b-1)^2\}$,

$N_7 = \{ab(-pqr+p^2+q^2+r^2-4)(aqr^2-apr+bpr-2aq-2bq-pr+2q)((a-1)pqr$
$\quad -(a+b-1)p^2-(a-b-1)r^2)\}$,

$G_7 = \{f_7, g_7\}$,

$$\begin{cases} f_7(x) = a_{7,1}x^3 + a_{7,2}x^2 + a_{7,3}x + a_{7,4}, \\ g_7(x,y) = b_{7,1}y + b_{7,2}x^3 + b_{7,3}x^2 + b_{7,4}x + b_{7,5}. \end{cases}$$

Branch 8 $(\mathbb{V}(E_8)\backslash\mathbb{V}(N_8), G_8)$

$E_8 = \emptyset$,

$N_8 = \{ab(-pqr+p^2+q^2+r^2-4)(abr^2-(a+b-1)^2)((a-1)pqr-(a+b-1)p^2$
$\quad -(a-b-1)r^2)\}$,

$G_8 = \{f_8, g_8\}$,

$$\begin{cases} f_8(x) = a_{8,1}x^4 + a_{8,2}x^3 + a_{8,3}x^2 + a_{8,4}x + a_{8,5}, \\ g_8(x,y) = b_{8,1}y + b_{8,2}x^3 + b_{8,3}x^2 + b_{8,4}x + b_{8,5}. \end{cases}$$

Branch 9 $(\mathbb{V}(E_9)\backslash\mathbb{V}(N_9), G_9)$

$E_9 = \{p, a-b-1, br^2+r^2-4b\}$,

$N_9 = \{b(q^2+r^2-4)(b+1)(bq^2+2br^2+r^2-8b)\}$,

$G_9 = \{f_9, g_9\}$,

$$\begin{cases} f_9(x) = (q^2+r^2-4)x, \\ g_9(x,y) = by^2 + (-br-r)xy + (b+2)x^2 - 1. \end{cases}$$

Branch 10 $(\mathbb{V}(E_{10})\backslash\mathbb{V}(N_{10}), G_{10})$

$E_{10} = \{r, p, a+b-1\}$,

$N_{10} = \{bq(q^2-4)(-b+1)\}$,

$G_{10} = \{f_{10}, g_{10}\}$,

$$\begin{cases} f_{10}(x) = xq - 2, \\ g_{10}(x,y) = by^2 + (b-1)x^2 + 1. \end{cases}$$

Branch 11 $(\mathbb{V}(E_{11})\backslash\mathbb{V}(N_{11}), G_{11})$

$E_{11} = \{r, p\}$,

$N_{11} = \{ab(q^2-4)(a+b-1)\}$,

$G_{11} = \{f_{11}, g_{11}\}$,

$$\begin{cases} f_{11}(x) = (a+b-1)x^2 + (q-aq)x + a - b - 1, \\ g_{11}(x,y) = by^2 + (b-1)x^2 + xq - 1. \end{cases}$$

Branch 12 $(\mathbb{V}(E_{12})\backslash\mathbb{V}(N_{12}), G_{12})$

$E_{12} = \{abr^2 - (a+b-1)^2, (a-1)pqr - (a+b-1)p^2 - (a-b-1)r^2, bp^2r + pqr^2 - br^3$
$-apq - bpq - p^2r - r^3 + pq + 2ar + 2br - 2r, bp^3 - apq^2 + p^2qr - bpr^2 - p^3 + pq^2$
$+aqr - bqr - pr^2 + 2ap + 2bp - qr - 2p\}$,

$N_{12} = \{abpr(-pqr + p^2 + q^2 + r^2 - 4)(aqr^2 - apr + bpr - 2aq - 2bq - pr + 2q)(-pqr$
$+ar^2 + br^2 + r^2 - 4a - 4b + 4)\}$,

$G_{12} = \{f_{12}, g_{12}, h_{12}\}$,

$$\begin{cases} f_{12}(x) = a_{12,1}x^2 + a_{12,2}x + a_{12,3}, \\ g_{12}(x,y) = by^2 - bpy - ax^2 + aqx - a + b, \\ h_{12}(x,y) = brxy - ybp + (1-a-b)x^2 + (aq-q)x - a + b + 1. \end{cases}$$

Branch 13 $(\mathbb{V}(E_{13})\backslash\mathbb{V}(N_{13}), G_{13})$

$E_{13} = \{(a-1)pqr - (a+b-1)p^2 - (a-b-1)r^2\}$,

$N_{13} = \{abr(abr^2 - (a+b-1)^2)(-pqr + p^2 + q^2 + r^2 - 4)\}$,

$G_{13} = \{f_{13}, g_{13}, h_{13}\}$,

$$\begin{cases} f_{13}(x) = a_{13,1}x^3 + a_{13,2}x^2 + a_{13,3}x + a_{13,4}, \\ g_{13}(x,y) = by^2 - bpy - ax^2 + aqx - a + b, \\ h_{13}(x,y) = brxy - bpy + (1-a-b)x^2 + (aq-q)x - a + b + 1. \end{cases}$$

In the above, $b_{1,1}, \cdots, b_{1,5}$; $a_{3,1}, a_{3,2}, a_{3,3}$; $b_{3,1}, b_{3,2}, b_{3,3}$; $a_{7,1}, \cdots, a_{7,4}$; $b_{7,1}, \cdots, b_{7,4}$; $a_{8,1}, \cdots, a_{8,5}$; $b_{8,1}, \cdots, b_{8,5}$; $a_{12,1}, a_{12,2}, a_{12,3}$; $a_{13,1}, a_{13,2}, a_{13,3}$ and $e_{1,1}, \cdots, e_{1,45}$; $e_{3,1}, \cdots, e_{3,7}$ are polynomials in $k[a, b, p, q, r]$, which can be found in the appendix.

In every branch (A_i, G_i), since G_i is a Gröbner basis of $\{p_1, p_2\}$ under the parametric constraint A_i, the number of complex solutions counted with multiplicities of polynomials in G_i is decided. From Lemma 3.3, we have the following results.

Theorem 4.1 *The P3P problem has one complex solution under the parametric constraint A_1 or A_2; has two complex solutions under the parametric constraint A_3, A_4, A_5, A_6, A_9 or A_{10}; has three complex solutions under the parametric constraint A_7 or A_{12}; and has four complex solutions under the parametric constraint A_8, A_{11} or A_{13}.*

Note that the number of complex solutions may be different from the number of real solutions.

5 The Number of Real Solutions of P3P Problem

In this section, combing the minimal CGS $\mathcal{G}$ with the discriminant sequence, we give explicit conditions to determine the number of distinct real positive solutions of the P3P problem in every branch of $\mathcal{G}$.

In Branch 5, the solution of $\{f_5 = 0, g_5 = 0\}$ is $\{x = 0, y = \frac{2}{p}\}$. In Branch 9, the solutions of $\{f_9 = 0, g_9 = 0\}$ are $\left\{x = 0, y = \sqrt{\frac{1}{b}}\right\}$ and $\left\{x = 0, y = -\sqrt{\frac{1}{b}}\right\}$. There is no positive solution in these two branches, so we do not discuss them below. We divide the rest 11 branches of $\mathcal{G}$ into six cases to analyze the number of real solutions of the P3P problem.

Case 1 Branch 1, Branch 2 and Branch 10;

Case 2 Branch 3, Branch 4, Branch 6, and Branch 11;

Case 3 Branch 7;

Case 4 Branch 8;

Case 5 Branch 12;

Case 6 Branch 13.

In each case, some values of parameters are given such that the P3P problem has different number of real positive solutions.

5.1 The Number of Real Solutions in Case 1

In Case 1, $f_i(x)$ is linear in x, and $g_i(x, y)$ is linear or quadratic in y, for $i = 1, 2, 10$. Under the parametric constraint A_i, the P3P problem has at most one positive solution by Theorem 4.1, where $i = 1, 2, 10$.

In Branch 1, the solution of $\{f_1 = 0, g_1 = 0\}$ is $\{x = \frac{1}{q}, y = -\frac{1}{q^4 b_{1,1}}(b_{1,2} + b_{1,3}q^2 + b_{1,4}q^3 + b_{1,5}q^4)\}$. Hence, the conditions under which the P3P problem has one positive solution are $q > 0$ and $-b_{1,1}(b_{1,2} + b_{1,3}q^2 + b_{1,4}q^3 + b_{1,5}q^4) > 0$, under the parametric constraint A_1.

In Branch 2, the solution of $\{f_2 = 0, g_2 = 0\}$ is $\{x = \frac{1}{q}, y = \frac{1}{p}\}$. Hence, the conditions under which the P3P problem has one positive solution are $p > 0$ and $q > 0$, under the parametric constraint A_2.

In Branch 10, the solutions of $\{f_{10} = 0, g_{10} = 0\}$ are $\left\{x = \frac{2}{q}, y = \sqrt{\frac{4(1-b)-q^2}{bq^2}}\right\}$ and $\left\{x = \frac{2}{q}, y = -\sqrt{\frac{4(1-b)-q^2}{bq^2}}\right\}$. Hence, the conditions under which the P3P problem has one positive solution are $q > 0$ and $4(1-b) - q^2 > 0$, under the parametric constraint A_{10}.

5.2 The Number of Real Solutions in Case 2

In Case 2, $f_i(x)$ is quadratic in x and $g_i(x,y)$ is linear or quadratic in y, $i=3,4,6,11$. Under the parametric constraint A_i, the P3P problem has at most two distinct positive solutions by Theorem 4.1, where $i=3,4,6,11$.

Since $g_i(x,y)$ is linear for $i=3,4,6$, we can express y as a polynomial $\widetilde{g_i}(x)$ with the variable x. In Branch 11, we can express $y^2=-\frac{1}{b}((b-1)x^2+xq-1)$ from $g_{11}(x,y)=0$. Let $\widetilde{g_{11}}(x)=-(b-1)x^2-xq+1$. The card of $\{(x,y)\in(\mathbb{R}^+)^2\mid f_i(x)=0,g_i(x,y)=0\}$ is equal to the card of $\{x\in\mathbb{R}^+\mid f_i(x)=0,\widetilde{g_i}(x)>0\}$, for $i=3,4,6,11$.

All these branches can be treated similarly. We take Branch 11 as an example.

$$\begin{cases} f_{11}(x)=(a+b-1)x^2+(q-aq)x+a-b-1=0, \\ \widetilde{g_{11}}(x)=-(b-1)x^2-xq+1>0. \end{cases} \tag{6}$$

Firstly assume that $\mathrm{Res}(f_{11},x,x)\neq 0$ and $\mathrm{Res}(f_{11},\widetilde{g_{11}},x)\neq 0$. We have the following corollary from Proposition 3.7.

Corollary 5.1 *Under the parametric constraint* A_{11}, *if* $\mathrm{Res}(f_{11},x,x)=a-b-1\neq 0$ *and* $\mathrm{Res}(f_{11},\widetilde{g_{11}},x)=b^2((a-b+1)^2-aq^2)\neq 0$, *then*

(i) *P3P problem has two positive solutions if and only if* $\mathrm{Card}_{\mathbb{R}}(f_{11})=2$, $\mathcal{C}_{f_{11}}^{(2,2)}(x>0)$, $\mathcal{C}_{f_{11}}^{(2,2)}(\widetilde{g_{11}}>0)$ *and* $\mathcal{C}_{f_{11}}^{(2,2)}(x\widetilde{g_{11}}>0)$.

(ii) *P3P problem has one positive solutions if and only if one of the following conditions hold:*

1) $\mathrm{Card}_{\mathbb{R}}(f_{11})=2$, $\mathcal{C}_{f_{11}}^{(2,2)}(x>0)$, $\mathcal{C}_{f_{11}}^{(2,1)}(\widetilde{g_{11}}>0)$ *and* $\mathcal{C}_{f_{11}}^{(2,1)}(x\widetilde{g_{11}}>0)$;

2) $\mathrm{Card}_{\mathbb{R}}(f_{11})=2$, $\mathcal{C}_{f_{11}}^{(2,1)}(x>0)$, $\mathcal{C}_{f_{11}}^{(2,2)}(\widetilde{g_{11}}>0)$ *and* $\mathcal{C}_{f_{11}}^{(2,1)}(x\widetilde{g_{11}}>0)$;

3) $\mathrm{Card}_{\mathbb{R}}(f_{11})=2$, $\mathcal{C}_{f_{11}}^{(2,1)}(x>0)$, $\mathcal{C}_{f_{11}}^{(2,1)}(\widetilde{g_{11}}>0)$ *and* $\mathcal{C}_{f_{11}}^{(2,2)}(x\widetilde{g_{11}}>0)$;

4) $\mathrm{Card}_{\mathbb{R}}(f_{11})=1$, $\mathcal{C}_{f}^{(1,1)}(x>0)$, $\mathcal{C}_{f}^{(1,1)}(\widetilde{g_{11}}>0)$ *and* $\mathcal{C}_{f}^{(1,1)}(x\widetilde{g_{11}}>0)$.

According to Lemmas 3.5 and 3.6, we have the following results.

Corollary 5.2 *Let* $f(x)$ *and* $g(x)$ *be two polynomials having no common zero, where* $f(x)$ *is quadratic. Assume* $\mathrm{GDL}(f,1)=[1,B_1,B_2]$, $\mathrm{GDL}(f,g)=[1,C_1,C_2]$, *and let* l_1,l_2,v_1,v_2 *be the numbers of the non-vanishing members and the sign changes of* $\mathrm{RSGDL}(f,1)$, $\mathrm{RSGDL}(f,g)$ *respectively. Then the following assertions hold:*

1) $\mathrm{Card}_{\mathbb{R}}(f)=2$ *if and only if* $l_1=3,v_1=0$;

1.1) $\mathcal{C}_f^{(2,2)}(g>0)$ *if and only if* $l_2=3,v_2=0$;

1.2) $\mathcal{C}_f^{(2,1)}(g>0)$ *if and only if* $l_2=3,v_2=1$, *or* $l_2=1,v_2=0$;

2) $\mathrm{Card}_{\mathbb{R}}(f)=1$ *if and only if* $l_1=2,v_1=0$;

2.1) $\mathcal{C}_f^{(1,1)}(g>0)$ *if and only if* $l_2=2,v_2=0$.

If $\mathrm{Res}(f_{11},x,x)=a-b-1=0$, we have following results.

Theorem 5.3 *Under the parametric constraint* A_{11}, *if* $\mathrm{Res}(f_{11},x,x)=0$, *then P3P problem has at most one positive solution. Furthermore, it has one positive solution if and only if* $q>0$ *and* $(b+1)q^2-4<0$.

Proof If $\mathrm{Res}(f_{11},x,x) = a-b-1 = 0$, then the number of real positive solution of System (6) is equal to the following system:

$$\begin{cases} \widetilde{f_{11}}(x) = 2x - q = 0, \\ \widetilde{g_{11}}(x) = -(b-1)x^2 - xq + 1 > 0. \end{cases} \tag{7}$$

The solution of $\widetilde{f_{11}}(x)$ is $x = \frac{q}{2}$. Hence, the system (7) has at most one real positive solution. Furthermore, it has one real positive solution if and only if $q > 0$ and $(b+1)q^2 - 4 < 0$. The proof is completed. ∎

If $\mathrm{Res}(f_{11},\widetilde{g_{11}},x) = b^2((a-b+1)^2 - aq^2) = 0$, we have following results.

Theorem 5.4 *Under the parametric constraint A_{11}, if $\mathrm{Res}(f_{11},\widetilde{g_{11}},x) = 0$, then*

1) *If $q = \frac{a-b+1}{\sqrt{a}}$, P3P problem has one positive solutions if and only if $a-b+1 \neq 0$, $x_2 > 0$ and $(1-b)x_2^2 - qx_2 + 1 > 0$, where $x_2 = \frac{\sqrt{a}(a-b-1)}{a+b-1}$.*

2) *If $q = -\frac{a-b+1}{\sqrt{a}}$, P3P problem has one positive solutions if and only if $a-b+1 \neq 0$, $\overline{x_2} > 0$ and $(1-b)\overline{x_2}^2 - q\overline{x_2} + 1 > 0$, where $\overline{x_2} = -\frac{\sqrt{a}(a-b-1)}{a+b-1}$.*

Proof If $\mathrm{Res}(f_{11},\widetilde{g_{11}},x) = 0$, then $f_{11}(x)$ and $\widetilde{g_{11}}(x)$ have common zeros.

When $q = \frac{a-b+1}{\sqrt{a}}$, the solutions of $f_{11} = 0$ are $\{x_1 = \frac{1}{\sqrt{a}}, x_2 = \frac{\sqrt{a}(a-b-1)}{a+b-1}\}$, and the solutions of $\widetilde{g_{11}} = 0$ are $\{x_1' = \frac{1}{\sqrt{a}}, x_2' = \frac{\sqrt{a}}{1-b}\}$. If $x_2 = x_2'$, i.e. $a+b-1 = 0$, then $\{f_{11} = 0, \widetilde{g_{11}} > 0\}$ has no real positive solution. If $x_2 \neq x_2'$, $x_2 > 0$ and $(1-b)x_2^2 - qx_2 + 1 > 0$, then system $\{f_{11} = 0, \widetilde{g_{11}} > 0\}$ has one real positive solution.

When $q = -\frac{a-b+1}{\sqrt{a}}$, the solutions of $f_{11} = 0$ are $\{\overline{x_1} = -\frac{1}{\sqrt{a}}, \overline{x_2} = -\frac{\sqrt{a}(a-b-1)}{a+b-1}\}$, and the solutions of $\widetilde{g_{11}} = 0$ are $\{\overline{x_1}' = -\frac{1}{\sqrt{a}}, \overline{x_2}' = -\frac{\sqrt{a}}{1-b}\}$. If $\overline{x_2} = \overline{x_2}'$, i.e., $a+b-1 = 0$, then $\{f_{11} = 0, \widetilde{g_{11}} > 0\}$ has no real positive solution. If $\overline{x_2} \neq \overline{x_2}'$, $\overline{x_2} > 0$ and $(1-b)\overline{x_2}^2 - q\overline{x_2} + 1 > 0$, then system $\{f_{11} = 0, \widetilde{g_{11}} > 0\}$ has one real positive solution. The proof is completed. ∎

The following example shows that there are values of parameters in A_{11} such that the P3P problem has one or two real positive solutions.

Example 5.5 Let $P_1 = \{a = 1, b = \frac{1}{2}, r = 0, p = 0, q = -\frac{7}{4}\}$ and $P_2 = \{a = \frac{5}{2}, b = 1, r = 0, p = 0, q = \frac{14}{9}\}$ in $A_{11} = \mathbb{V}(E_{11}) \setminus \mathbb{V}(N_{11})$. When the parameters a, b, r, p, q are assigned the value P_1, the P3P problem has one real positive solution. When the parameters are assigned the value P_2, the P3P problem has two real positive solutions.

When the parameters are assigned the value P_1, we have $\mathrm{Res}(f_{11},x,x)|_{P_1} \neq 0$, $\mathrm{Res}(f_{11},\widetilde{g_{11}},x)|_{P_1} \neq 0$, $\mathrm{RSGDL}(f_{11},1)|_{P_1} = [1,1,1]$, $\mathrm{RSGDL}(f_{11},x)|_{P_1} = [1,-1,-1]$, $\mathrm{RSGDL}(f_{11},g_{11})|_{P_1} = [1,1,-1]$ and $\mathrm{RSGDL}(f_{11},xg_{11})|_{P_1} = [1,1,1]$. By Corollary 5.2, $\mathrm{Card}_{\mathbb{R}}(f_{11}) = 2$, $\mathcal{C}^{(2,1)}_{f_{11}}(x > 0)$, $\mathcal{C}^{(2,1)}_{f_{11}}(\widetilde{g_{11}} > 0)$, $\mathcal{C}^{(2,2)}_{f_{11}}(x\widetilde{g_{11}} > 0)$. According to Corollary 5.1, the P3P problem has only one real positive solution when $\{a = 1, b = \frac{1}{2}, r = 0, p = 0, q = -\frac{7}{4}\}$.

When the parameters are assigned the value P_2, we have $\mathrm{Res}(f_{11},x,x)|_{P_2} \neq 0$, $\mathrm{Res}(f_{11},\widetilde{g_{11}},x)|_{P_2} \neq 0$, $\mathrm{RSGDL}(f_{11},1)|_{P_2} = [1,1,1]$, $\mathrm{RSGDL}(f_{11},x)|_{P_2} = [1,1,1]$, $\mathrm{RSGDL}(f_{11},g_{11})|_{P_2} = [1,1,1]$ and $\mathrm{RSGDL}(f_{11},xg_{11})|_{P_2} = [1,1,1]$. By Corollary 5.2, $\mathrm{Card}_{\mathbb{R}}(f_{11}) = 2$, $\mathcal{C}^{(2,2)}_{f_{11}}(x > 0)$, $\mathcal{C}^{(2,2)}_{f_{11}}(\widetilde{g_{11}} > 0)$, $\mathcal{C}^{(2,2)}_{f_{11}}(x\widetilde{g_{11}} > 0)$. According to Corollary 5.1, the P3P problem has two real positive solutions when $\{a = \frac{5}{2}, b = 1, r = 0, p = 0, q = \frac{14}{9}\}$.

5.3 The Number of Real Solutions in Case 3

In Case 3, $f_7(x)$ is cubic in x and $g_7(x,y)$ is linear in y, where

$$\begin{cases} f_7(x) = a_{7,1}x^3 + a_{7,2}x^2 + a_{7,3}x + a_{7,4} = 0, \\ g_7(x,y) = b_{7,1}y + b_{7,2}x^3 + b_{7,3}x^2 + b_{7,4}x + b_{7,5} = 0. \end{cases} \tag{8}$$

Under the parametric constraint A_7, the P3P problem has at most three different positive solutions by Theorem 4.1.

The number of positive solutions of the system (8) equals that of the system:

$$\begin{cases} f_7(x) = a_{7,1}x^3 + a_{7,2}x^2 + a_{7,3}x + a_{7,4} = 0, \\ \widetilde{g_7}(x) = -b_{7,1}(b_{7,2}x^3 + b_{7,3}x^2 + b_{7,4}x + b_{7,5}) > 0. \end{cases} \tag{9}$$

We first assume $\text{Res}(f_7, x, x) = a_{7,4} \neq 0$ and $\text{Res}(f_7, \widetilde{g_7}, x) \neq 0$. As in Case 2, from Proposition 3.7, we have the following results.

Corollary 5.6 *Under the parametric constraint A_7, if $\text{Res}(f_7, x, x) \neq 0$ and $\text{Res}(f_7, \widetilde{g_7}, x) \neq 0$, then*

(i) *P3P problem has three positive solutions if and only if*
$\text{Card}_{\mathbb{R}}(f_7) = 3$, $\mathcal{C}_{f_7}^{(3,3)}(x > 0)$, $\mathcal{C}_{f_7}^{(3,3)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(3,3)}(x\widetilde{g_7} > 0)$*;*

(ii) *P3P problem has two positive solutions if and only if*
1) $\text{Card}_{\mathbb{R}}(f_7) = 3$, $\mathcal{C}_{f_7}^{(3,3)}(x > 0)$, $\mathcal{C}_{f_7}^{(3,2)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(3,2)}(x\widetilde{g_7} > 0)$*;*
2) $\text{Card}_{\mathbb{R}}(f_7) = 3$, $\mathcal{C}_{f_7}^{(3,2)}(x > 0)$, $\mathcal{C}_{f_7}^{(3,3)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(3,2)}(x\widetilde{g_7} > 0)$*;*
3) $\text{Card}_{\mathbb{R}}(f_7) = 3$, $\mathcal{C}_{f_7}^{(3,2)}(x > 0)$, $\mathcal{C}_{f_7}^{(3,2)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(3,3)}(x\widetilde{g_7} > 0)$*;*
4) $\text{Card}_{\mathbb{R}}(f_7) = 2$, $\mathcal{C}_{f_7}^{(2,2)}(x > 0)$, $\mathcal{C}_{f_7}^{(2,2)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(2,2)}(x\widetilde{g_7} > 0)$*;*

(iii) *P3P problem has one positive solutions if and only if one of the following conditions hold:*
1) $\text{Card}_{\mathbb{R}}(f_7) = 3$, $\mathcal{C}_{f_7}^{(3,3)}(x > 0)$, $\mathcal{C}_{f_7}^{(3,1)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(3,1)}(x\widetilde{g_7} > 0)$*;*
2) $\text{Card}_{\mathbb{R}}(f_7) = 3$, $\mathcal{C}_{f_7}^{(3,1)}(x > 0)$, $\mathcal{C}_{f_7}^{(3,3)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(3,1)}(x\widetilde{g_7} > 0)$*;*
3) $\text{Card}_{\mathbb{R}}(f_7) = 3$, $\mathcal{C}_{f_7}^{(3,1)}(x > 0)$, $\mathcal{C}_{f_7}^{(3,1)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(3,3)}(x\widetilde{g_7} > 0)$*;*
4) $\text{Card}_{\mathbb{R}}(f_7) = 3$, $\mathcal{C}_{f_7}^{(3,1)}(x > 0)$, $\mathcal{C}_{f_7}^{(3,2)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(3,2)}(x\widetilde{g_7} > 0)$*;*
5) $\text{Card}_{\mathbb{R}}(f_7) = 3$, $\mathcal{C}_{f_7}^{(3,2)}(x > 0)$, $\mathcal{C}_{f_7}^{(3,1)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(3,2)}(x\widetilde{g_7} > 0)$*;*
6) $\text{Card}_{\mathbb{R}}(f_7) = 3$, $\mathcal{C}_{f_7}^{(3,2)}(x > 0)$, $\mathcal{C}_{f_7}^{(3,2)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(3,1)}(x\widetilde{g_7} > 0)$*;*
7) $\text{Card}_{\mathbb{R}}(f_7) = 2$, $\mathcal{C}_{f_7}^{(2,2)}(x > 0)$, $\mathcal{C}_{f_7}^{(2,1)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(2,1)}(x\widetilde{g_7} > 0)$*;*
8) $\text{Card}_{\mathbb{R}}(f_7) = 2$, $\mathcal{C}_{f_7}^{(2,1)}(x > 0)$, $\mathcal{C}_{f_7}^{(2,2)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(2,1)}(x\widetilde{g_7} > 0)$*;*
9) $\text{Card}_{\mathbb{R}}(f_7) = 2$, $\mathcal{C}_{f_7}^{(2,1)}(x > 0)$, $\mathcal{C}_{f_7}^{(2,1)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(2,2)}(x\widetilde{g_7} > 0)$*;*
10) $\text{Card}_{\mathbb{R}}(f_7) = 1$, $\mathcal{C}_{f_7}^{(1,1)}(x > 0)$, $\mathcal{C}_{f_7}^{(1,1)}(\widetilde{g_7} > 0)$ *and* $\mathcal{C}_{f_7}^{(1,1)}(x\widetilde{g_7} > 0)$.

According to Lemmas 3.5 and 3.6, we have the following results.

Corollary 5.7 *Let $f(x)$ and $g(x)$ be two polynomials having no common zero, where $f(x)$ is cubic. Assume* $\text{GDL}(f, 1) = [1, B_1, B_2, B_3]$, $\text{GDL}(f, g) = [1, C_1, C_2, C_3]$, *and let l_1, l_2, v_1, v_2 be* *the numbers of the non-vanishing members and the sign changes of* RSGDL$(f,1)$, RSGDL(f,g) *respectively. Then the following assertions hold:*

1) $\mathrm{Card}_{\mathbb{R}}(f)=3$ *if and only if* $l_1=4, v_1=0$;
1.1) $\mathcal{C}_f^{(3,3)}(g>0)$ *if and only if* $l_2=4, v_2=0$;
1.2) $\mathcal{C}_f^{(3,2)}(g>0)$ *if and only if* $l_2=4, v_2=1$, *or* $l_2=2, v_2=0$;
1.3) $\mathcal{C}_f^{(3,1)}(g>0)$ *if and only if* $l_2=4, v_2=2$, *or* $l_2=2, v_2=1$;
2) $\mathrm{Card}_{\mathbb{R}}(f)=2$ *if and only if* $l_1=3, v_1=0$;
2.1) $\mathcal{C}_f^{(2,2)}(g>0)$ *if and only if* $l_2=3, v_2=0$;
2.2) $\mathcal{C}_f^{(2,1)}(g>0)$ *if and only if* $l_2=3, v_2=1$, *or* $l_2=1, v_2=0$;
3) $\mathrm{Card}_{\mathbb{R}}(f)=1$ *if and only if* $l_1=4, v_1=1$ *or* $l_1=2, v_1=0$;
3.1) $\mathcal{C}_f^{(1,1)}(g>0)$ *if and only if* $l_1=4, v_1=1$ *or* $l_2=2, v_2=0$.

If $\mathrm{Res}(f_7,x,x)=0$ or $\mathrm{Res}(f_7,\widetilde{g_7},x)=0$, the problem degenerates to a quadratic case which can be treated similarly as in Case 2.

The following example shows that there are values of parameters in A_7 such that the P3P problem has one, two or three real positive solutions.

Example 5.8 Let $P_1=\{a=1,b=1,r=1,p=\frac{5}{6},q=-\frac{3}{2}\}$, $P_2=\{a=1,b=1,r=0,p=\frac{1}{5},q=\frac{1}{2}\}$, and $P_3=\{a=1,b=1,r=1,p=\frac{5}{4},q=\frac{6}{5}\}$ in $A_7=\mathbb{V}(E_7)\setminus\mathbb{V}(N_7)$. When the parameters a,b,r,p,q are assigned the value P_1, the P3P problem has one real positive solution. When the parameters are assigned the value P_2, the P3P problem has two real positive solutions. When the parameters are assigned the value P_3, the P3P problem has three real positive solutions.

When the parameters are assigned the value P_1, we have $\mathrm{Res}(f_7,x,x)|_{P_1}\neq 0$, $\mathrm{Res}(f_7,\widetilde{g_7},x)|_{P_1}\neq 0$, $\mathrm{RSGDL}(f_7,1)|_{P_1}=[1,1,-1,-1]$, $\mathrm{RSGDL}(f_7,x)|_{P_1}=[1,1,1,-1]$, $\mathrm{RSGDL}(f_7,\widetilde{g_7})|_{P_1}=[1,1,-1,-1]$ and $\mathrm{RSGDL}(f_7,x\widetilde{g_7})|_{P_1}=[1,-1,-1,1]$. By Corollary 5.7, $\mathrm{Card}_{\mathbb{R}}(f_7)=1$, $\mathcal{C}_{f_7}^{(1,1)}(x>0)$, $\mathcal{C}_{f_7}^{(1,1)}(\widetilde{g_7}>0)$, $\mathcal{C}_{f_7}^{(1,1)}(x\widetilde{g_7}>0)$. According to Corollary 5.6, the P3P problem has only one real positive solution when $\{a=1,b=1,r=1,p=\frac{5}{6},q=-\frac{3}{2}\}$.

When the parameters are assigned the value P_2, we have $\mathrm{Res}(f_7,x,x)|_{P_2}\neq 0$, $\mathrm{Res}(f_7,\widetilde{g_7},x)|_{P_2}\neq 0$, $\mathrm{RSGDL}(f_7,1)|_{P_2}=[1,1,1,1]$, $\mathrm{RSGDL}(f_7,x)|_{P_2}=[1,1,1,1]$, $\mathrm{RSGDL}(f_7,\widetilde{g_7})|_{P_2}=[1,-1,-1,-1]$ and $\mathrm{GDL}(f_7,x\widetilde{g_7})|_{P_2}=[1,-1,-1,-1]$. By Corollary 5.7, $\mathrm{Card}_{\mathbb{R}}(f_7)=3$, $\mathcal{C}_{f_7}^{(3,3)}(x>0)$, $\mathcal{C}_{f_7}^{(3,2)}(\widetilde{g_7}>0)$, $\mathcal{C}_{f_7}^{(3,2)}(x\widetilde{g_7}>0)$. According to Corollary 5.6, the P3P problem has two real positive solutions when $\{a=1,b=1,r=0,p=\frac{1}{5},q=\frac{1}{2}\}$.

When the parameters are assigned value P_3, we have $\mathrm{Res}(f_7,x,x)|_{P_3}\neq 0$, $\mathrm{Res}(f_7,\widetilde{g_7},x)|_{P_3}\neq 0$, $\mathrm{RSGDL}(f_7,1)|_{P_3}=[1,1,1,1]$, $\mathrm{RSGDL}(f_7,x)|_{P_3}=[1,1,1,1]$, $\mathrm{RSGDL}(f_7,\widetilde{g_7})|_{P_3}=[1,1,1,1]$ and $\mathrm{RSGDL}(f_7,x\widetilde{g_7})|_{P_3}=[1,1,1,1]$. By Corollary 5.7, $\mathrm{Card}_{\mathbb{R}}(f_7)=3$, $\mathcal{C}_{f_7}^{(3,3)}(x>0)$, $\mathcal{C}_{f_7}^{(3,3)}(\widetilde{g_7}>0)$, $\mathcal{C}_{f_7}^{(3,3)}(x\widetilde{g_7}>0)$. According to Corollary 5.6, the P3P problem has three real positive solutions when $\{a=1,b=1,r=1,p=\frac{5}{4},q=\frac{6}{5}\}$.

5.4 The Number of Real Solutions in Case 4

In Case 4, $f_8(x)$ is quartic in x and $g_8(x,y)$ is linear in y, where

$$\begin{cases} f_8(x) = a_{8,1}x^4 + a_{8,2}x^3 + a_{8,3}x^2 + a_{8,4}x + a_{8,5} = 0, \\ g_8(x,y) = b_{8,1}y + b_{8,2}x^3 + b_{8,3}x^2 + b_{8,4}x + b_{8,5} = 0. \end{cases} \tag{10}$$

Under the constraint of A_8, the P3P problem has at most four different positive solutions by Theorem 4.1.

The number of positive solutions of the system (10) equals that of the system:

$$\begin{cases} f_8(x) = a_{8,1}x^4 + a_{8,2}x^3 + a_{8,3}x^2 + a_{8,4}x + a_{8,5} = 0, \\ \widetilde{g_8}(x) = -b_{8,1}(b_{8,2}x^3 + b_{8,3}x^2 + b_{8,4}x + b_{8,5}) > 0. \end{cases} \tag{11}$$

We first assume $\mathrm{Res}(f_8, x, x) = a_{8,5} \neq 0$ and $\mathrm{Res}(f_8, \widetilde{g_8}, x) \neq 0$. Similarly, from Proposition 3.7, we have the following results.

Corollary 5.9 *Under the parametric constraint* A_8*, if* $\mathrm{Res}(f_8, \widetilde{g_8}, x) \neq 0$*, and* $\mathrm{Res}(f_8, x, x) \neq 0$*, then*

(i) *P3P problem has four positive solutions if and only if* $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,4)}(x > 0)$, $\mathcal{C}_{f_8}^{(4,4)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,4)}(x\widetilde{g_8} > 0)$*;*

(ii) *P3P problem has three positive solutions if and only if one of the following conditions hold:*

1) $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,4)}(x > 0), \mathcal{C}_{f_8}^{(4,3)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,3)}(x\widetilde{g_8} > 0)$*;*

2) $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,3)}(x > 0), \mathcal{C}_{f_8}^{(4,4)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,3)}(x\widetilde{g_8} > 0)$*;*

3) $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,3)}(x > 0), \mathcal{C}_{f_8}^{(4,3)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,4)}(x\widetilde{g_8} > 0)$*;*

4) $\mathrm{Card}_{\mathbb{R}}(f_8) = 3$, $\mathcal{C}_{f_8}^{(4,3)}(x > 0), \mathcal{C}_{f_8}^{(4,3)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,3)}(x\widetilde{g_8} > 0)$*;*

(iii) *P3P problem has two positive solutions if and only if one of the following conditions hold:*

1) $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,4)}(x > 0), \mathcal{C}_{f_8}^{(4,2)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,2)}(x\widetilde{g_8} > 0)$*;*

2) $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,2)}(x > 0), \mathcal{C}_{f_8}^{(4,4)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,2)}(x\widetilde{g_8} > 0)$*;*

3) $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,2)}(x > 0), \mathcal{C}_{f_8}^{(4,2)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,4)}(x\widetilde{g_8} > 0)$*;*

4) $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,2)}(x > 0), \mathcal{C}_{f_8}^{(4,3)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,3)}(x\widetilde{g_8} > 0)$*;*

5) $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,3)}(x > 0), \mathcal{C}_{f_8}^{(4,2)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,3)}(x\widetilde{g_8} > 0)$*;*

6) $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,3)}(x > 0), \mathcal{C}_{f_8}^{(4,3)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,2)}(x\widetilde{g_8} > 0)$*;*

7) $\mathrm{Card}_{\mathbb{R}}(f_8) = 3$, $\mathcal{C}_{f_8}^{(3,3)}(x > 0), \mathcal{C}_{f_8}^{(3,2)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(3,2)}(x\widetilde{g_8} > 0)$*;*

8) $\mathrm{Card}_{\mathbb{R}}(f_8) = 3$, $\mathcal{C}_{f_8}^{(3,2)}(x > 0), \mathcal{C}_{f_8}^{(3,3)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(3,2)}(x\widetilde{g_8} > 0)$*;*

9) $\mathrm{Card}_{\mathbb{R}}(f_8) = 3$, $\mathcal{C}_{f_8}^{(3,2)}(x > 0), \mathcal{C}_{f_8}^{(3,2)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(3,3)}(x\widetilde{g_8} > 0)$*;*

10) $\mathrm{Card}_{\mathbb{R}}(f_8) = 2$, $\mathcal{C}_{f_8}^{(2,2)}(x > 0), \mathcal{C}_{f_8}^{(2,2)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(2,2)}(x\widetilde{g_8} > 0)$*;*

(iv) *P3P problem has one positive solution if and only if one of the following conditions hold:*

1) $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}_{f_8}^{(4,1)}(x > 0), \mathcal{C}_{f_8}^{(4,1)}(\widetilde{g_8} > 0)$ *and* $\mathcal{C}_{f_8}^{(4,4)}(x\widetilde{g_8} > 0)$*;*

2) $\mathrm{Card}_{\mathbb{R}}(f_8)=4$, $\mathcal{C}_{f_8}^{(4,1)}(x>0), \mathcal{C}_{f_8}^{(4,4)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(4,1)}(x\widetilde{g}_8>0)$*;*
3) $\mathrm{Card}_{\mathbb{R}}(f_8)=4$, $\mathcal{C}_{f_8}^{(4,4)}(x>0), \mathcal{C}_{f_8}^{(4,1)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(4,1)}(x\widetilde{g}_8>0)$*;*
4) $\mathrm{Card}_{\mathbb{R}}(f_8)=4$, $\mathcal{C}_{f_8}^{(4,1)}(x>0), \mathcal{C}_{f_8}^{(4,2)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(4,3)}(x\widetilde{g}_8>0)$*;*
5) $\mathrm{Card}_{\mathbb{R}}(f_8)=4$, $\mathcal{C}_{f_8}^{(4,1)}(x>0), \mathcal{C}_{f_8}^{(4,3)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(4,2)}(x\widetilde{g}_8>0)$*;*
6) $\mathrm{Card}_{\mathbb{R}}(f_8)=4$, $\mathcal{C}_{f_8}^{(4,2)}(x>0), \mathcal{C}_{f_8}^{(4,1)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(4,3)}(x\widetilde{g}_8>0)$*;*
7) $\mathrm{Card}_{\mathbb{R}}(f_8)=4$, $\mathcal{C}_{f_8}^{(4,2)}(x>0), \mathcal{C}_{f_8}^{(4,3)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(4,1)}(x\widetilde{g}_8>0)$*;*
8) $\mathrm{Card}_{\mathbb{R}}(f_8)=4$, $\mathcal{C}_{f_8}^{(4,3)}(x>0), \mathcal{C}_{f_8}^{(4,1)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(4,2)}(x\widetilde{g}_8>0)$*;*
9) $\mathrm{Card}_{\mathbb{R}}(f_8)=4$, $\mathcal{C}_{f_8}^{(4,3)}(x>0), \mathcal{C}_{f_8}^{(4,2)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(4,1)}(x\widetilde{g}_8>0)$*;*
10) $\mathrm{Card}_{\mathbb{R}}(f_8)=4$, $\mathcal{C}_{f_8}^{(4,2)}(x>0), \mathcal{C}_{f_8}^{(4,2)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(4,2)}(x\widetilde{g}_8>0)$*;*
11) $\mathrm{Card}_{\mathbb{R}}(f_8)=3$, $\mathcal{C}_{f_8}^{(3,1)}(x>0), \mathcal{C}_{f_8}^{(3,2)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(3,2)}(x\widetilde{g}_8>0)$*;*
12) $\mathrm{Card}_{\mathbb{R}}(f_8)=3$, $\mathcal{C}_{f_8}^{(3,2)}(x>0), \mathcal{C}_{f_8}^{(3,1)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(3,2)}(x\widetilde{g}_8>0)$*;*
13) $\mathrm{Card}_{\mathbb{R}}(f_8)=3$, $\mathcal{C}_{f_8}^{(3,2)}(x>0), \mathcal{C}_{f_8}^{(3,2)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(3,1)}(x\widetilde{g}_8>0)$*;*
14) $\mathrm{Card}_{\mathbb{R}}(f_8)=3$, $\mathcal{C}_{f_8}^{(3,1)}(x>0), \mathcal{C}_{f_8}^{(3,1)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(3,3)}(x\widetilde{g}_8>0)$*;*
15) $\mathrm{Card}_{\mathbb{R}}(f_8)=3$, $\mathcal{C}_{f_8}^{(3,1)}(x>0), \mathcal{C}_{f_8}^{(3,3)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(3,1)}(x\widetilde{g}_8>0)$*;*
16) $\mathrm{Card}_{\mathbb{R}}(f_8)=3$, $\mathcal{C}_{f_8}^{(3,3)}(x>0), \mathcal{C}_{f_8}^{(3,1)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(3,1)}(x\widetilde{g}_8>0)$*;*
17) $\mathrm{Card}_{\mathbb{R}}(f_8)=2$, $\mathcal{C}_{f_8}^{(2,2)}(x>0), \mathcal{C}_{f_8}^{(2,1)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(2,1)}(x\widetilde{g}_8>0)$*;*
18) $\mathrm{Card}_{\mathbb{R}}(f_8)=2$, $\mathcal{C}_{f_8}^{(2,1)}(x>0), \mathcal{C}_{f_8}^{(2,2)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(2,1)}(x\widetilde{g}_8>0)$*;*
19) $\mathrm{Card}_{\mathbb{R}}(f_8)=2$, $\mathcal{C}_{f_8}^{(2,1)}(x>0), \mathcal{C}_{f_8}^{(2,1)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(2,2)}(x\widetilde{g}_8>0)$*;*
20) $\mathrm{Card}_{\mathbb{R}}(f_8)=1$, $\mathcal{C}_{f_8}^{(1,1)}(x>0), \mathcal{C}_{f_8}^{(1,1)}(\widetilde{g}_8>0)$ *and* $\mathcal{C}_{f_8}^{(1,1)}(x\widetilde{g}_8>0)$.

According to Lemmas 3.5 and 3.6, we have the following results.

Corollary 5.10 *Let $f(x)$ and $g(x)$ be two polynomials having no common zero, where $f(x)$ is quartic. Assume $\mathrm{GDL}(f,1)=[1,B_1,B_2,B_3,B_4]$, $\mathrm{GDL}(f,g)=[1,C_1,C_2,C_3,C_4]$, and let l_1,l_2,v_1,v_2 be the numbers of the non-vanishing members and the sign changes of* $\mathrm{RSGDL}(f,1)$, $\mathrm{RSGDL}(f,g)$, *respectively. Then the following assertions hold:*

1) $\mathrm{Card}_{\mathbb{R}}(f)=4$ *if and only if $l_1=5, v_1=0$, that is, $B_i>0$, for $i=1,2,3,4$;*
1.1) $\mathcal{C}_f^{(4,4)}(g>0)$ *if and only if $l_2=5, v_2=0$, that is, $C_i>0$, for $i=1,2,3,4$;*
1.2) $\mathcal{C}_f^{(4,3)}(g>0)$ *if and only if $l_2=5, v_2=1$ or $l_2=3, v_2=0$;*
1.3) $\mathcal{C}_f^{(4,2)}(g>0)$ *if and only if $l_2=5, v_2=2$ or $l_2=3, v_2=1$ or $l_2=1, v_2=0$;*
1.4) $\mathcal{C}_f^{(4,1)}(g>0)$ *if and only if $l_2=5, v_2=3$ or $l_2=3, v_2=2$;*
2) $\mathrm{Card}_{\mathbb{R}}(f)=3$ *if and only if $l_1=4, v_1=0$, that is, $B_4=0$, $B_i>0$, for $i=1,2,3$;*
2.1) $\mathcal{C}_f^{(3,3)}(g>0)$ *if and only if $l_2=4, v_2=0$, that is, $C_4=0$, $C_i>0$, for $i=1,2,3$;*
2.2) $\mathcal{C}_f^{(3,2)}(g>0)$ *if and only if $l_2=4, v_2=1$, or $l_2=2, v_2=0$;*
2.3) $\mathcal{C}_f^{(3,1)}(g>0)$ *if and only if $l_2=4, v_2=2$, or $l_2=2, v_2=1$;*
3) $\mathrm{Card}_{\mathbb{R}}(f)=2$ *if and only if $l_1=5, v_1=1$, or $l_1=3, v_2=0$;*
3.1) $\mathcal{C}_f^{(2,2)}(g>0)$ *if and only if $l_2=5, v_2=1$, or $l_2=3, v_2=0$;*
3.2) $\mathcal{C}_f^{(2,1)}(g>0)$ *if and only if $l_2=5, v_2=2$, or $l_2=3, v_2=1$, or $l_2=1, v_2=0$;*
4) $\mathrm{Card}_{\mathbb{R}}(f)=1$ *if and only if $l_1=4, v_1=1$, or $l_1=2, v_1=0$;*
4.1) $\mathcal{C}_f^{(1,1)}(g>0)$ *if and only if $l_2=4, v_2=1$, or $l_2=2, v_2=0$.*

If $\mathrm{Res}(f_8, x, x) = 0$, then the system (11) becomes

$$\begin{cases} f_8(x) = a'_{8,1}x^3 + a'_{8,2}x^2 + a'_{8,3}x + a'_{8,4} = 0, \\ \overline{g_8}(x) = -b_{8,1}(b_{8,2}x^3 + b_{8,3}x^2 + b_{8,4}x + b_{8,5}) > 0, \end{cases} \tag{12}$$

which can be treated with the same method as in Case 3. It is similar when $\mathrm{Res}(f_8, \widetilde{g_8}, x) = 0$.

The following example shows that there are values of parameters in A_8 such that the P3P problem has one, two, three or four real positive solutions.

Example 5.11 Let $P_1 = \{a = 1, b = 2, r = \frac{1}{2}, p = \frac{3}{2}, q = 1\}$, $P_2 = \{a = 1, b = 2, r = \frac{1}{2}, p = \frac{7}{5}, q = \frac{1}{2}\}$, $P_3 = \{a = 1, b = \frac{1}{2}, r = \frac{1}{2}, p = 1, q = \frac{31}{20}\}$ and $P_4 = \{a = 1, b = 1, r = \frac{4}{3}, p = \frac{3}{2}, q = \frac{3}{2}\}$ in $A_8 = \mathbb{V}(E_8) \setminus \mathbb{V}(N_8)$. When the parameters a, b, r, p, q are assigned the value P_1, the P3P problem has one real positive solution. When the parameters are assigned the value P_2, the P3P problem has two real positive solutions. When the parameters are assigned the value P_3, the P3P problem has three real positive solutions. When the parameters are assigned the value P_4, the P3P problem has four real positive solutions.

When the parameters are assigned the value P_1, we have $\mathrm{Res}(f_8, x, x)|_{P_1} \neq 0$, $\mathrm{Res}(f_8, \widetilde{g_8}, x)|_{P_1} \neq 0$, $\mathrm{RSGDL}(f_8, 1)|_{P_1} = [1, 1, 1, -1, -1]$, $\mathrm{RSGDL}(f_8, x)|_{P_1} = [1, -1, -1, 1, 1]$, $\mathrm{RSGDL}(f_8, \widetilde{g_8})|_{P_1} = [1, 1, -1, 1, 1]$ and $\mathrm{RSGDL}(f_8, x\widetilde{g_8})|_{P_1} = [1, 1, 1, -1, -1]$. By Corollary 5.10, $\mathrm{Card}_{\mathbb{R}}(f_8) = 2$, $\mathcal{C}^{(1,1)}_{f_8}(x > 0)$, $\mathcal{C}^{(1,1)}_{f_8}(\widetilde{g_8} > 0)$, $\mathcal{C}^{(1,2)}_{f_8}(x\widetilde{g_8} > 0)$. According to Corollary 5.9, the P3P problem has only one real positive solution when $\{a = 1, b = 2, r = \frac{1}{2}, p = \frac{3}{2}, q = 1\}$.

When the parameters are assigned the value P_2, we have $\mathrm{Res}(f_8, x, x)|_{P_2} \neq 0$, $\mathrm{Res}(f_8, \widetilde{g_8}, x)|_{P_2} \neq 0$, $\mathrm{RSGDL}(f_8, 1)|_{P_2} = [1, 1, 1, 1, 1]$, $\mathrm{RSGDL}(f_8, x)|_{P_2} = [1, -1, -1, -1, 1]$, $\mathrm{RSGDL}(f_8, \widetilde{g_8})|_{P_2} = [1, 1, -1, -1, -1]$ and $\mathrm{RSGDL}(f_8, x\widetilde{g_8})|_{P_2} = [1, 1, 1, 1, -1]$. By Corollary 5.10, $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}^{(4,2)}_{f_8}(x > 0)$, $\mathcal{C}^{(4,3)}_{f_8}(\widetilde{g_8} > 0)$, $\mathcal{C}^{(4,3)}_{f_8}(x\widetilde{g_8} > 0)$. According to Corollary 5.9, the P3P problem has two real positive solutions when $\{a = 1, b = 2, r = \frac{1}{2}, p = \frac{7}{5}, q = \frac{1}{2}\}$.

When the parameters are assigned the value P_3, we have $\mathrm{Res}(f_8, x, x)|_{P_3} \neq 0$, $\mathrm{Res}(f_8, \widetilde{g_8}, x)|_{P_3} \neq 0$, $\mathrm{RSGDL}(f_8, 1)|_{P_3} = [1, 1, 1, 1, 1]$, $\mathrm{RSGDL}(f_8, x)|_{P_3} = [1, -1, -1, -1, -1]$, $\mathrm{RSGDL}(f_8, \widetilde{g_8})|_{P_3} = [1, -1, -1, -1, -1]$ and $\mathrm{RSGDL}(f_8, x\widetilde{g_8})|_{P_3} = [1, 1, 1, 1, 1]$. By Corollary 5.10, $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}^{(4,3)}_{f_8}(x > 0)$, $\mathcal{C}^{(4,3)}_{f_8}(\widetilde{g_8} > 0)$, $\mathcal{C}^{(4,4)}_{f_8}(x\widetilde{g_8} > 0)$. According to Corollary 5.9, the P3P problem has three real positive solutions when $\{a = 1, b = \frac{1}{2}, r = \frac{1}{2}, p = 1, q = \frac{31}{20}\}$.

When the parameters are assigned the value P_4, we have $\mathrm{Res}(f_8, x, x)|_{P_4} \neq 0$, $\mathrm{Res}(f_8, \widetilde{g_8}, x)|_{P_4} \neq 0$, $\mathrm{RSGDL}(f_8, 1)|_{P_4} = [1, 1, 1, 1, 1]$, $\mathrm{RSGDL}(f_8, x)|_{P_4} = [1, 1, 1, 1, 1]$, $\mathrm{RSGDL}(f_8, \widetilde{g_8})|_{P_4} = [1, 1, 1, 1, 1]$ and $\mathrm{RSGDL}(f_8, x\widetilde{g_8})|_{P_4} = [1, 1, 1, 1, 1]$. By Corollary 5.10, $\mathrm{Card}_{\mathbb{R}}(f_8) = 4$, $\mathcal{C}^{(4,4)}_{f_8}(x > 0)$, $\mathcal{C}^{(4,4)}_{f_8}(\widetilde{g_8} > 0)$, $\mathcal{C}^{(4,4)}_{f_8}(x\widetilde{g_8} > 0)$. According to Corollary 5.9, the P3P problem has four real positive solutions when $\{a = 1, b = 1, r = \frac{4}{3}, p = \frac{3}{2}, q = \frac{3}{2}\}$.

5.5 The Number of Real Solutions in Case 5

In Case 5, $f_{12}(x)$ is quadratic in x, $g_{12}(x, y)$ is quadratic in y, and there is a third polynomial h_{12}, where

$$\begin{cases} f_{12}(x) = a_{12,1}x^2 + a_{12,2}x + a_{12,3} = 0, \\ g_{12}(x, y) = by^2 - bpy - ax^2 + aqx - a + b = 0, \\ h_{12}(x, y) = (brx - bp)y + (1 - a - b)x^2 + (aq - q)x - a + b + 1 = 0. \end{cases}$$

Under the constraint of A_{12}, the P3P problem has at most three different positive solutions by Theorem 4.1.

Since h_{12} is linear in y, if $brx - bp \neq 0$, we can express y as

$$y = -\frac{(1 - a - b)x^2 + (aq - q)x - a + b + 1}{brx - bp}. \tag{13}$$

If $brx-bp = 0$, then $x = p/r$. Let $x_1 = p/r$. It is easy to check $f_{12}(x_1) = 0$ and $h_{12}(x_1, y) = 0$ under the constraint A_{12}. So $x_1 = p/r$ is a zero of $f_{12}(x)$. Let x_2 be another zero of $f_{12}(x)$. According to the Viete's theorem, we have $x_1x_2 = a_{12,3}/a_{12,1}$, so $x_2 = \frac{ra_{12,3}}{pa_{12,1}}$.

Substituting $x = x_1$ into $g_{12}(x, y)$ and eliminating the denominator, we have

$$\widetilde{g_{12}}(y) = br^2y^2 - bpr^2y + apqr - ap^2 - ar^2 + br^2.$$

If $\mathrm{Res}(\widetilde{g_{12}}, y, y) = apqr - ap^2 - ar^2 + br^2 \neq 0$, then $y = 0$ is not a solution of $\widetilde{g_{12}}$. We let $\mathrm{GDL}(\widetilde{g}_{12}, 1) = [1, D_1, D_2]$ and $\mathrm{GDL}(\widetilde{g}_{12}, y) = [1, F_1, F_2]$, where $D_1 = 2b^2r^4$, $D_2 = b^3r^6(-4apqr + 4ap^2 + 4ar^2 - 4br^2 + br^2p^2)$, $F_1 = pb^2r^4$, $F_2 = -b^2r^4(-apqr + ap^2 + ar^2 - br^2)(-4apqr + 4ap^2 + 4ar^2 - 4br^2 + br^2p^2)]$. By Lemma 3.6, Corollaries 5.1 and 5.2, $\widetilde{g_{12}}(y)$ has two real positive solutions if and only if $D_1 > 0, D_2 > 0, F_1 > 0, F_2 > 0$, and $\widetilde{g_{12}}(y)$ has one real positive solution if and only if one of following conditions hold:

1) $D_1 > 0, D_2 = 0, F_1 > 0, F_2 = 0$.

2) $D_1 > 0, D_2 > 0, F_1 = F_2 = 0$;

3) $D_1 > 0, D_2 > 0, F_2 < 0$;

If $apqr - ap^2 - ar^2 + br^2 = 0$, then $y_1 = 0$ and $y_2 = p$ are solutions of $\widetilde{g_{12}}(y)$.

Substituting $x = x_2$ into $h_{12}(x, y) = 0$, we have

$$y_3 = \frac{(1 - a - b)r^2a_{12,3}^2 + pq(a - 1)ra_{12,1}a_{12,3} + p^2a_{12,1}^2(-a + b + 1)}{bpa_{12,1}(p^2a_{12,1} - r^2a_{12,3})}. \tag{14}$$

By simple computations, we have $g_{12}(x_2, y_3) = 0$ under the constraint A_{12}. Hence, $\{x = x_2, y = y_3\}$ is a solution of $\{f_{12} = 0, g_{12} = 0, h_{12} = 0\}$. By the aforementioned analysis, we have the following results.

Theorem 5.12 *Under the parametric constraint A_{12}, let $x_1 = p/r$, $x_2 = \frac{ra_{12,3}}{pa_{12,1}}$, and y_3 be the same as in* (14)*. Then:*

(i) *P3P has three positive solutions if and only if $x_1 > 0, apqr - ap^2 - ar^2 + br^2 \neq 0$, $D_1 > 0$, $D_2 > 0$, $F_1 > 0$, $F_2 > 0$, $x_2 > 0$ and $y_3 > 0$;*

(ii) *P3P has two positive solutions if and only if one of the following conditions hold:*
1) $x_1 > 0, apqr - ap^2 - ar^2 + br^2 \neq 0,\ D_1 > 0, D_2 > 0, F_1 > 0, F_2 > 0$ *and* $x_2 \leq 0$;
2) $x_1 > 0, apqr - ap^2 - ar^2 + br^2 \neq 0,\ D_1 > 0, D_2 > 0, F_1 > 0, F_2 > 0$ *and* $y_3 \leq 0$;
3) $x_1 > 0, apqr - ap^2 - ar^2 + br^2 = 0, p > 0,\ x_2 > 0$ *and* $y_3 > 0$;
4) $x_1 > 0, apqr - ap^2 - ar^2 + br^2 \neq 0, D_1 > 0, D_2 = 0, F_1 > 0, F_2 = 0,\ x_2 > 0$ *and* $y_3 > 0$;
5) $x_1 > 0, apqr - ap^2 - ar^2 + br^2 \neq 0, D_1 > 0, D_2 > 0, F_1 = F_2 = 0,\ x_2 > 0$ *and* $y_3 > 0$;
6) $x_1 > 0, apqr - ap^2 - ar^2 + br^2 \neq 0, D_1 > 0, D_2 > 0, F_2 < 0,\ x_2 > 0$ *and* $y_3 > 0$;
(iii) *P3P has one positive solution if and only if one of the following conditions hold:*
1) $x_2 > 0, y_3 > 0$ *and* $x_1 \leq 0$;
2) $x_2 > 0, y_3 > 0,\ x_1 > 0, apqr - ap^2 - ar^2 + br^2 = 0$ *and* $p < 0$;
3) $x_2 > 0, y_3 > 0,\ x_1 > 0, apqr - ap^2 - ar^2 + br^2 \neq 0,$ *and none of the following conditions hold* (a) $D_1 > 0, D_2 > 0, F_1 > 0, F_2 > 0,$ (b) $D_1 > 0, D_2 = 0, F_1 > 0, F_2 = 0,$ (c) $D_1 > 0, D_2 > 0, F_1 = F_2 = 0,$ (d) $D_1 > 0, D_2 > 0, F_2 < 0$;
4) $x_2 \leq 0,\ x_1 > 0,\ apqr - ap^2 - ar^2 + br^2 = 0$ *and* $p > 0$;
5) $x_2 \leq 0,\ x_1 > 0,\ apqr - ap^2 - ar^2 + br^2 \neq 0,\ D_1 > 0, D_2 = 0, F_1 > 0$ *and* $F_2 = 0$;
6) $x_2 \leq 0,\ x_1 > 0,\ apqr - ap^2 - ar^2 + br^2 \neq 0,\ D_1 > 0, D_2 > 0$ *and* $F_1 = F_2 = 0$;
7) $x_2 \leq 0,\ x_1 > 0,\ apqr - ap^2 - ar^2 + br^2 \neq 0,\ D_1 > 0, D_2 > 0$ *and* $F_2 < 0$;
8) $y_3 \leq 0,\ x_1 > 0,\ apqr - ap^2 - ar^2 + br^2 = 0$ *and* $p > 0$;
9) $y_3 \leq 0,\ x_1 > 0,\ apqr - ap^2 - ar^2 + br^2 \neq 0,\ D_1 > 0, D_2 = 0, F_1 > 0$ *and* $F_2 = 0$;
10) $y_3 \leq 0,\ x_1 > 0,\ apqr - ap^2 - ar^2 + br^2 \neq 0,\ D_1 > 0, D_2 > 0$ *and* $F_1 = F_2 = 0$;
11) $y_3 \leq 0,\ x_1 > 0,\ apqr - ap^2 - ar^2 + br^2 \neq 0,\ D_1 > 0, D_2 > 0$ *and* $F_2 < 0$.

The following example shows that there are values of parameters in A_{12} such that the P3P problem has one, two or three real positive solutions.

Example 5.13 Let $P_1 = \{a = 1, b = 1, r = 1, p = 1, q = 1\}$, $P_2 = \{a = 1, b = \frac{1}{4}, r = \frac{1}{2}, p = \frac{1}{2}, q = \frac{225}{128}\}$, and $P_3 = \{a = \frac{3}{2}, b = \frac{2}{3}, r = \frac{7}{6}, p = 1, q = \frac{29}{18}\}$ in $A_{12} = \mathbb{V}(E_{12}) \setminus \mathbb{V}(N_{12})$. When the parameters a, b, r, p, q are assigned the value P_1, the P3P problem has one real positive solution. When the parameters are assigned the value P_2, the P3P problem has two real positive solutions. When the parameters are assigned the value P_3, the P3P problem has three real positive solutions.

When the parameters are assigned the value P_1, we have $x_1 = 1$, $(apqr - ap^2 - ar^2 + br^2)|_{P_1} = 0$, $p > 0$, and $x_2 = 0$. According to Theorem 5.12, the P3P problem has one real positive solution when $\{a = 1, b = 1, r = 1, p = 1, q = 1\}$.

When the parameters are assigned the value P_2, we have $x_1 = 1$, $(apqr - ap^2 - ar^2 + br^2)|_{P_2} = 1/512$, $[1, D_1, D_2]|_{P_2} = [1, 1/128, 1/524288]$, $[1, F_1, F_2]|_{P_2} = [1, 1/512, 1/16777216]$, and $x_2 = 0$. According to Theorem 5.12, the P3P problem has two real positive solutions when $\{a = 1, b = \frac{1}{4}, r = \frac{1}{2}, p = \frac{1}{2}, q = \frac{225}{128}\}$.

When the parameters are assigned the value P_3, we have $x_1 = 6/7$, $(apqr - ap^2 - ar^2 + br^2)|_{P_3} = 5/27$, $[1, D_1, D_2]|_{P_3} = [1, 2401/1458, 117649/944784]$, $[1, F_1, F_2]|_{P_3} = [1, 2401/2916, 12005/472392]$, $x_2 = 1$, and $y_3 = 1$. According to Theorem 5.12, the P3P problem has three real positive solutions when $\{a = \frac{3}{2}, b = \frac{2}{3}, r = \frac{7}{6}, p = 1, q = \frac{29}{18}\}$.

5.6 The Number of Real Solutions in Case 6

In Case 6, $f_{13}(x)$ is cubic in x, $g_{13}(x,y)$ is quadratic in y, and there is a third polynomial h_{13}, where

$$\begin{cases} f_{13}(x) = a_{13,1}x^3 + a_{13,2}x^2 + a_{13,3}x + a_{13,4} = 0, \\ g_{13}(x,y) = by^2 - bpy - ax^2 + aqx - a + b = 0, \\ h_{13}(x,y) = brxy - bpy + (1-a-b)x^2 + (aq-q)x - a + b + 1 = 0. \end{cases}$$

Under the constraint of A_{13}, the P3P problem has at most four different positive solutions by Theorem 4.1.

By the similar analysis of Case 5, $x_1 = p/r$ is a solution of $f_{13}(x) = 0$. So $f_{13}(x)$ can be expressed as $f_{13}(x) = (rx-p)(c_1x^2 + c_2x + c_3) = (rx-p)\widetilde{f_{13}}(x)$, where $c_1 = a_{13,1}/r, c_2 = (a_{13,2}r + pa_{13,1})/r^2, c_3 = -a_{13,4}/p$. The polynomial $\widetilde{f_{13}}(x)$ is quadratic in x, thus it is easy to obtain the two solutions of $\widetilde{f_{13}}(x)$. Assume them be x_2 and x_3.

Substituting $x = x_1$ into the polynomial $h_{13}(x,y) = 0$, we have $h_{13}(x_1,y) = 0$. Let y_1 and y_2 be the two solutions of $g_{13}(x_1,y) = 0$. Then $\{x = x_1, y = y_1\}$ and $\{x = x_1, y = y_2\}$ are the solutions of $\{f_{13}(x) = 0, g_{13}(x,y) = 0, h_{13}(x,y) = 0\}$. Assume y_3 and y_4 be the zeros of $h_{13}(x_2,y)$ and $h_{13}(x_3,y)$ respectively. It is easy to check $g_{13}(x_2,y_3) = 0$ and $g_{13}(x_3,y_4) = 0$. Hence, $(x_1,y_1),(x_1,y_2),(x_2,y_3),(x_3,y_4)$ are the solutions of $\{f_{13}(x) = 0, g_{13}(x,y) = 0, h_{13}(x,y) = 0\}$. The analysis of real positive solutions can be treated similarly as in Case 5.

The following example shows that there are values of parameters in A_{13} such that the P3P problem has one, two, three or four real positive solutions.

Example 5.14 Let $P_1 = \{a = 1, b = 1, r = -\frac{1}{2}, p = -\frac{1}{2}, q = \frac{1}{2}\}$, $P_2 = \{a = 1, b = 1, r = \frac{3}{2}, p = \frac{3}{2}, q = \frac{5}{3}\}$, $P_3 = \{a = 1, b = 1, r = \frac{4}{3}, p = \frac{4}{3}, q = 1\}$ and $P_4 = \{a = 1, b = 1, r = \frac{3}{2}, p = \frac{3}{2}, q = \frac{4}{3}\}$ in $A_{13} = \mathbb{V}(E_{13}) \setminus \mathbb{V}(N_{13})$. When the parameters a, b, r, p, q are assigned the value P_1, the P3P problem has one real positive solution. When the parameters are assigned the value P_2, the P3P problem has two real positive solutions. When the parameters are assigned the value P_3, the P3P problem has three real positive solutions. When the parameters are assigned the value P_4, the P3P problem has four real positive solutions.

6 Conclusion

The solution classification of a parametric polynomial system can be obtained by combining the CGS with discriminant sequence. The minimal CGS can give a disjoint partition of the parametric space. In every partition, the number of complex solutions counted with multiplicities of the equation system is decided. We use this method to solve the P3P problem. As a result, we give a complete classification of solutions of the P3P problem, and the explicit conditions under which the P3P problem has one, two, three, or four different real positive solutions.

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Appendix

The following is some symbols using in the minimal CGS of P3P equation system in Section 4.

$b_{1,1} = 64b(pq - r)$,

$b_{1,2} = 8a^2qr^2 - 8abqr^2 - 8apr^3 - 8a^2pr + 8b^2pr + 8bqr^2 - 16bpr - 8qr^2 + 8pr$,

$$\begin{aligned} b_{1,3} = {} & 8abq^2r^2 - 40a^2q^2r^2 - 24apqr^3 + 8abr^4 + 104a^2pqr + 80abpqr - 8b^2pqr + 64ap^2r^2 - 8bp^2r^2 + 40aq^2r^2 \\ & + 24bq^2r^2 - 64a^2p^2 - 88abp^2 - 24b^2p^2 - 24a^2q^2 - 24abq^2 - 96apqr - 16bpqr + 32a^2r^2 + 48abr^2 + 32b^2r^2 \\ & + 8p^2r^2 - 16q^2r^2 + 88ap^2 + 48bp^2 + 80aq^2 + 56bq^2 + 24pqr - 24ar^2 - 24br^2 - 144a^2 - 288ab - 144b^2 \\ & - 24p^2 - 56q^2 - 8r^2 + 288a + 288b - 144, \end{aligned}$$

$$\begin{aligned} b_{1,4} = {} & 30bp^4r^2 - 42a^2pq^3r - 30ap^4r^2 + 32ap^2q^2r^2 - 32bp^2q^2r^2 + 22a^2pqr^3 - 60abpqr^3 + 42a^2p^2q^2 + 42a^2q^4 \\ & - 62ap^3qr + 38bp^3qr + 32bpq^3r - 60a^2p^2r^2 + 60abp^2r^2 - 30p^4r^2 - 148a^2q^2r^2 + 38abq^2r^2 + 96p^2q^2r^2 \\ & - 37apqr^3 + 7bpqr^3 - 8abr^4 + 22ap^2q^2 - 98bp^2q^2 + 6aq^4 + 198a^2pqr + 46abpqr - 124b^2pqr - 86p^3qr \\ & - 106pq^3r + 39ap^2r^2 - 35bp^2r^2 + 253aq^2r^2 + 21bq^2r^2 - 126pqr^3 + 36br^4 - 60a^2p^2 + 88abp^2 - 28b^2p^2 \\ & + 160a^2q^2 + 196abq^2 + 148b^2q^2 + 18p^2q^2 + 38q^4 - 108apqr + 310bpqr + 60a^2r^2 + 48abr^2 - 68b^2r^2 \\ & + 43p^2r^2 - 13q^2r^2 - 38ap^2 - 74bp^2 - 730aq^2 - 170bq^2 + 138pqr - 114ar^2 - 194br^2 - 264a^2 - 16ab \\ & + 248b^2 + 102p^2 + 226q^2 + 82r^2 + 536a + 24b - 272, \end{aligned}$$

$$\begin{aligned} b_{1,5} = {} & 30ap^3r^3 - 32apq^2r^3 + 42a^2pq^2r - 42abpq^2r + 38ap^2qr^2 - 6bp^2qr^2 + 8a^2pr^3 + 82abpr^3 - 60b^2pr^3 \\ & - 25aqr^4 - 42a^2p^2q + 42abp^2q - 42a^2q^3 + 42abq^3 - 36ap^3r - 54bp^3r + 64apq^2r + 20bpq^2r + 192a^2qr^2 \\ & - 334abqr^2 + 38b^2qr^2 - 64p^2qr^2 + 5apr^3 - bpr^3 - 140ap^2q + 40bp^2q - 6aq^3 + 18bq^3 - 230a^2pr + 100abpr \\ & + 10b^2pr + 102p^3r + 42pq^2r - 82aqr^2 + 76bqr^2 + 151pr^3 - 308a^2q + 352abq + 372b^2q + 40p^2q - 38q^3 \\ & + 106apr - 2bpr - 118qr^2 + 740aq - 732bq - 168pr - 88q, \end{aligned}$$

$a_{3,1} = bpqr - bp^2 - bq^2 - ar^2 - br^2 + p^2 + 4a + 4b - 4$,

$a_{3,2} = aqr^2 - p^2q - 4aq + 4q$,

$a_{3,3} = -ar^2 + p^2 + 4a - 4$,

$$b_{3,1}=apr^3-p^3r-4apr+4pr,$$
$$b_{3,2}=apr^2-aqr^3-bpr^2+2aqr+2bqr+pr^2-2qr,$$
$$b_{3,3}=aq^2r^3-ap^2r+bp^2r-3aq^2r-bq^2r-pqr^2-p^2r+3q^2r,$$
$$b_{3,4}=p^2qr-aqr^3+4aqr-4qr,$$
$$a_{7,1}=aqr^2-apr+bpr-2aq-2bq-pr+2q,$$
$$a_{7,2}=apqr-aq^2r^2+abr^2-bp^2+2aq^2+bq^2+pqr-ar^2-br^2-a^2-2ab-b^2+p^2-2q^2+6a+6b-5,$$
$$a_{7,3}=2aqr^2-p^2q-apr+bpr-6aq-2bq-pr+6q,$$
$$a_{7,4}=-ar^2+p^2+4a-4,$$
$$b_{7,1}=(a-1)pqr-(a+b-1)p^2-(a-b-1)r^2,$$
$$b_{7,2}=abr^3-a^2r-2abr-b^2r+2ar+2br-r,$$
$$b_{7,3}=abpr^2-abqr^3+aqr^3+a^2qr+2abqr+b^2qr-apr^2+bpr^2-a^2p-2abp-b^2p-4aqr-4bqr-pr^2 +2ap+2bp+3qr-p,$$
$$b_{7,4}=apqr^2-aq^2r^3+2aq^2r+bq^2r+pqr^2-br^3-apq-bpq-2q^2r+pq+2ar+2br-2r,$$
$$b_{7,5}=aqr^3-apr^2-3aqr-bqr-pr^2+2ap+2bp+3qr-2p,$$
$$a_{8,1}=abr^2-(a+b-1)^2,$$
$$a_{8,2}=b^2pr-abqr^2-abpr+2a^2q+2abq-bpr-4aq-2bq+2q,$$
$$a_{8,3}=abpqr-b^2p^2-a^2q^2+bpqr+abr^2-b^2r^2+bp^2+2aq^2-2a^2+2b^2-q^2+4a-2,$$
$$a_{8,4}=b^2pr-bp^2q-abpr+2a^2q-2abq-bpr-4aq+2bq+2q,$$
$$a_{8,5}=bp^2-a^2+2ab-b^2+2a-2b-1,$$
$$b_{8,1}=b((a-1)pqr-(a+b-1)p^2-(a-b-1)r^2),$$
$$b_{8,2}=abr^3-a^2r-2abr-b^2r+2ar+2br-r,$$
$$b_{8,3}=b^2pr^2-abqr^3+2a^2qr+2abqr-bpr^2-a^2p-2abp-b^2p-4aqr-2bqr+2ap+2bp+2qr-p,$$
$$b_{8,4}=bpqr^2-a^2q^2r+abr^3-b^2r^3+a^2pq+abpq+2aq^2r-2apq-bpq-a^2r+b^2r-q^2r+pq+2ar-r,$$
$$b_{8,5}=a^2qr-abqr-bpr^2-a^2p+b^2p-2aqr+bqr+2ap+qr-p,$$
$$a_{12,1}=(pqr-ar^2-br^2-r^2+4a+4b-4)r,$$
$$a_{12,2}=aqr^3-p^2qr+bpr^2-3aqr-bqr-2ap-2bp+3qr+2p,$$
$$a_{12,3}=-ar^3+p^2r+4ar-4r,$$
$$a_{13,1}=(abr^2-(a+b-1)^2)r,$$
$$a_{13,2}=b^2pr^2-abqr^3+2a^2qr+2abqr-bpr^2-a^2p-2abp-b^2p-4aqr-2bqr+2ap+2bp+2qr-p,$$
$$a_{13,3}=bpqr^2-a^2q^2r+abr^3-b^2r^3+a^2pq+abpq+2aq^2r-2apq-bpq-a^2r+b^2r-q^2r+pq+2ar-r,$$
$$a_{13,4}=a^2qr-abqr-bpr^2-a^2p+b^2p-2aqr+bqr+2ap+qr-p,$$
$$e_{1,1}=aqr^2-apr+bpr-2aq-2bq-pr+2q,$$
$$e_{1,2}=abr^2-(a+b-1)^2,$$
$$e_{1,3}=bpqr-bp^2-bq^2-ar^2-br^2+p^2+4a+4b-4,$$
$$e_{1,4}=abpr-b^2pr-a^2q+b^2q+bpr+2aq-q,$$
$$e_{1,5}=abp^2-b^2p^2-a^2q^2+abq^2+a^2r^2-b^2r^2-ap^2+2bp^2+2aq^2+bq^2+ar^2+br^2-4a^2+4b^2-p^2-q^2-8b+4,$$
$$e_{1,6}=ar^4+br^4-4ar^2-4br^2+4r^2,$$
$$e_{1,7}=bqr^3-apr^2-bpr^2+pr^2,$$
$$e_{1,8}=p^2r^3+q^2r^3-2bp^2r-2aq^2r-4pqr^2+2ar^3+2br^3+4apq+4bpq+2p^2r+2q^2r-4pq-8ar-8br+8r,$$
$$e_{1,9}=bpr^3-bqr^2-apr-3bpr-qr^2+2aq+2bq+3pr-2q,$$
$$e_{1,10}=apr^3+bpr^3-2bqr^2-2apr-2bpr+2pr,$$
$$e_{1,11}=a^2r^3+b^2r^3-ar^3-br^3-2a^2r-4abr-2b^2r+4ar+4br-2r,$$
$$e_{1,12}=b^2qr^2-a^2pr-3b^2pr-bqr^2-a^2q+2abq+3b^2q+2apr+4bpr+2aq-2bq-pr-q,$$
$$e_{1,13}=bp^2r^2-apqr-p^2r^2-q^2r^2-3bp^2+2aq^2-bq^2+3pqr-3ar^2-3br^2+3p^2-2q^2+12a+12b-12,$$

$$e_{1,14} = ap^2r^2 - bq^2r^2,$$
$$e_{1,15} = b^2pr^2 + a^2qr - b^2qr - bpr^2 - a^2p - 2abp - b^2p - 2aqr + 2ap + 2bp + qr - p,$$
$$e_{1,16} = a^2pr^2 - a^2qr - 2abqr - b^2qr - apr^2 + a^2p + 2abp + b^2p + 2aqr + 2bqr - 2ap - 2bp - qr + p,$$
$$e_{1,17} = a^3r^2 + b^3r^2 - 2a^2r^2 - 2b^2r^2 - a^3 - 3a^2b - 3ab^2 - b^3 + ar^2 + br^2 + 2a^2 + 4ab + 2b^2 - a - b,$$
$$e_{1,18} = a^2q^2r - abq^2r - a^2r^3 + b^2r^3 - a^2pq + b^2pq + ap^2r - bp^2r - 2aq^2r - bq^2r - ar^3 - br^3 + 2apq + 4a^2r - 4b^2r + p^2r + q^2r - pq + 8br - 4r,$$
$$e_{1,19} = a^2pqr + bq^2r^2 - a^2p^2 - a^2q^2 - 2apqr + 2a^2r^2 + b^2r^2 - bp^2 + 2aq^2 + pqr - 2ar^2 - 2br^2 - 5a^2 - 6ab - b^2 + p^2 - q^2 + 10a + 6b - 5,$$
$$e_{1,20} = a^2bqr + 2ab^2qr + b^3qr - a^3p - 3a^2bp - 3ab^2p - b^3p - 2abqr - 2b^2qr + 3a^2p + 6abp + 3b^2p + bqr - 3ap - 3bp + p,$$
$$e_{1,21} = a^3qr - ab^2qr - a^3p - a^2bp + ab^2p + b^3p - 2a^2qr + a^2p - 2abp - 3b^2p + aqr + ap + 3bp - p,$$
$$e_{1,22} = bp^3r - apq^2r - p^2qr^2 - q^3r^2 - 2bp^2q + 2aq^3 - p^3r + 3pq^2r + 2p^2q - 2q^3 - 4apr + 4aq + 4bq - 4q,$$
$$e_{1,23} = b^2p^2r + abq^2r - b^2r^3 - 2abpq - 2b^2pq - ap^2r - bp^2r + ar^3 + br^3 + 2bpq - a^2r + 2abr + 3b^2r - 2ar - 6br + 3r,$$
$$e_{1,24} = a^2p^2r - 2abq^2r - b^2q^2r + b^2r^3 + a^2pq + 2abpq + b^2pq + bq^2r - ar^3 - br^3 - 2apq - 2bpq + a^2r - 2abr - 3b^2r + pq + 2ar + 6br - 3r,$$
$$e_{1,25} = a^3pr + 3b^3pr + a^3q - 5ab^2q - 4b^3q - 2a^2pr - 7b^2pr - 5a^2q + 7b^2q + apr + 4bpr + 7aq - 3q,$$
$$e_{1,26} = a^4r - 2a^2b^2r + b^4r - 3a^3r - a^2br - ab^2r - 3b^3r + 3a^2r + 2abr + 3b^2r - ar - br,$$
$$e_{1,27} = a^3p^2 + 3b^3p^2 + 2a^3q^2 - 5ab^2q^2 - b^3q^2 + 5b^3r^2 - 9b^2p^2 - 7a^2q^2 - 3abq^2 - b^2q^2 - 3a^2r^2 - 10b^2r^2 + 3a^3 + 2a^2b - 13ab^2 - 12b^3 + 9bp^2 + 8aq^2 + 5bq^2 + 5ar^2 + 5br^2 - 7a^2 + 16ab + 31b^2 - 3p^2 - 3q^2 - 3a - 26b + 7,$$
$$e_{1,28} = a^5 + a^4b - 2a^3b^2 - 2a^2b^3 + ab^4 + b^5 - 4a^4 - 4a^3b - 4ab^3 - 4b^4 + 6a^3 + 6a^2b + 6ab^2 + 6b^3 - 4a^2 - 4ab - 4b^2 + a + b,$$
$$e_{1,29} = b^2r^4 + a^2r^2 - 2abr^2 - 3b^2r^2 - 2ar^2 + 2br^2 + r^2,$$
$$e_{1,30} = b^3r^3 - abr^3 - b^2r^3 + a^3r - 3ab^2r - 2b^3r - 2a^2r + 2abr + 4b^2r + ar - 2br,$$
$$e_{1,31} = bq^3r^2 - ap^3r + bp^3r - 2ap^2q - 2bp^2q - p^3r + 2p^2q,$$
$$e_{1,32} = b^4r^2 - 2a^2br^2 - 2b^3r^2 + a^4 + a^3b - 2a^2b^2 - 3ab^3 - b^4 + abr^2 + b^2r^2 - 2a^3 + 4ab^2 + 2b^3 + a^2 - ab - b^2,$$
$$e_{1,33} = apq^3r + p^2q^2r^2 + q^4r^2 - bp^4 + bp^2q^2 - 2aq^4 + p^3qr - 3pq^3r - ap^2r^2 - bp^2r^2 + p^4 - 2p^2q^2 + 2q^4 + 4apqr + 4ap^2 + 4bp^2 - 4aq^2 - 4bq^2 - 4p^2 + 4q^2,$$
$$e_{1,34} = b^2q^3r + bpq^2r^2 + b^2qr^3 - a^2p^3 - 2b^2p^3 - 2a^2pq^2 + 2abpq^2 + b^2pq^2 + 2bp^2qr - bq^3r + 2a^2pr^2 - 2abpr^2 - b^2pr^2 - aqr^3 - bqr^3 + bp^3 + 4apq^2 - 2bpq^2 + a^2qr - 2abqr - 3b^2qr + p^2qr - 2apr^2 - 2bpr^2 - 5a^2p + 2abp + 7b^2p + p^3 - 2pq^2 + 2aqr + 6bqr + 10ap - 2bp - 3qr - 5p,$$
$$e_{1,35} = 2abq^3r + b^2q^3r + bpq^2r^2 - b^2qr^3 - a^2p^3 - 2a^2pq^2 - 2abpq^2 - b^2pq^2 - 2ap^2qr - bq^3r + 2a^2pr^2 + b^2pr^2 + aqr^3 + bqr^3 - bp^3 + 4apq^2 + 2bpq^2 - a^2qr + 2abqr + 3b^2qr + p^2qr - 2apr^2 - 2bpr^2 - 5a^2p - 6abp - b^2p + p^3 - 2pq^2 - 2aqr - 6bqr + 10ap + 6bp + 3qr - 5p,$$
$$e_{1,36} = 3ab^2q^2r + b^3q^2r + a^2br^3 - b^3r^3 - a^3pq - 2a^2bpq - 3ab^2pq - 2b^3pq - abp^2r + b^2p^2r - b^2q^2r + abr^3 + b^2r^3 + 3a^2pq + 4abpq + 3b^2pq - 4a^2br + 4b^3r - bp^2r - 3apq - 2bpq - 8b^2r + pq + 4br,$$
$$e_{1,37} = 2ab^3qr + 2b^4qr + a^4p - 4a^2b^2p - 4ab^3p - b^4p - 2ab^2qr - 4b^3qr - 3a^3p + a^2bp + 7ab^2p + 3b^3p + 2b^2qr + 3a^2p - 2abp - 3b^2p - ap + bp,$$
$$e_{1,38} = ap^4r - ap^2q^2r - bq^4r - p^3qr^2 - pq^3r^2 + 2ap^3q - bp^3q + 2apq^3 - bpq^3 + 4p^2q^2r + p^3q - 2pq^3 - 7ap^2r + 2bp^2r + 2aq^2r + bq^2r - ar^3 - br^3 + 2apq + 2bpq - 2p^2r - 2q^2r - 2pq + 4ar + 4br - 4r,$$
$$e_{1,39} = 4b^4pr + a^4q + 2a^3bq - 6ab^3q - 5b^4q - 3a^2bpr - ab^2pr - 8b^3pr - 2a^3q - 7a^2bq - 2ab^2q + 7b^3q + abpr + 4b^2pr + a^2q + 8abq + b^2q - 3bq,$$
$$e_{1,40} = b^2p^4 - 2b^2p^2q^2 - a^2q^4 + 2abq^4 - ap^3qr - 2bp^4 - 2ap^2q^2 + 3bp^2q^2 + 2aq^4 + bq^4 + bq^2r^2 - 8b^2p^2 + p^4 - 8a^2q^2 + 8abq^2 + 8b^2q^2 - p^2q^2 - q^4 + 5apqr + 4a^2r^2 - 4b^2r^2 + 15bp^2 + 10aq^2 - 7bq^2 + pqr$$

$$+ 3ar^2 + 3br^2 - 16a^2 + 16b^2 - 7p^2 - 2q^2 + 4a - 28b + 12,$$

$$e_{1,41} = a^2p^4 + 2a^2p^2q^2 - 2abq^4 - b^2q^4 + 2ap^3qr - 2ap^2q^2 - 2bp^2q^2 - 2bq^2r^2 + 4a^2p^2 + 4b^2p^2 + 4a^2q^2 - 8abq^2 - 4b^2q^2 + 2p^2q^2 - 2apqr - 4a^2r^2 + 4b^2r^2 - 6bp^2 - 4aq^2 + 6bq^2 - 2pqr - 2ar^2 - 2br^2 + 16a^2 - 16b^2 + 2p^2 - 8a + 24b - 8,$$

$$e_{1,42} = 4b^4p^2 + a^4q^2 + 2a^3bq^2 - 6ab^3q^2 - b^4q^2 - a^4r^2 - a^3br^2 + ab^3r^2 + 6b^4r^2 + a^3p^2 - a^2bp^2 - ab^2p^2 - 11b^3p^2 - 2a^3q^2 - 10a^2bq^2 - 6ab^2q^2 - 2b^3q^2 - a^3r^2 - 5a^2br^2 - 2ab^2r^2 - 11b^3r^2 + 4a^4 + 7a^3b + 2a^2b^2 - 17ab^3 - 16b^4 + a^2p^2 + abp^2 + 10b^2p^2 + a^2q^2 + 9abq^2 + 6b^2q^2 + 5abr^2 + 5b^2r^2 + a^2b + 24ab^2 + 39b^3 - 3bp^2 - 3bq^2 - 4a^2 - 7ab - 30b^2 + 7b,$$

$$e_{1,43} = 4b^4q^2r - 2a^2b^2r^3 + 2b^4r^3 + 3a^4pq + 2a^3bpq - 8a^2b^2pq - 6ab^3pq + b^4pq + 2ab^2p^2r - 2b^3p^2r - 6ab^2q^2r - 10b^3q^2r - 2ab^2r^3 - 2b^3r^3 - 9a^3pq - 3a^2bpq + 13ab^2pq + 3b^3pq + 8a^2b^2r - 8b^4r + 2b^2p^2r + 6b^2q^2r + 9a^2pq - 5b^2pq + 16b^3r - 3apq + bpq - 8b^2r,$$

$$e_{1,44} = bq^7r + p^5q^2r^2 + p^3q^4r^2 - bp^7 + 2bp^5q^2 - 4ap^3q^4 + bpq^6 + p^6qr - 4p^4q^3r - p^2q^5r - ap^5r^2 - bp^5r^2 + ap^3q^2r^2 + bp^3q^2r^2 + p^7 - 3p^5q^2 + 3p^3q^4 + 4ap^4qr + 3ap^2q^3r - 2bp^2q^3r - 2aq^5r - bq^5r + aq^3r^3 + bq^3r^3 + 4ap^5 + 4bp^5 - 8ap^3q^2 - 8bp^3q^2 + 2apq^4 + 2bpq^4 + 2p^2q^3r + 2q^5r - 4p^5 + 8p^3q^2 - 2pq^4 - 4aq^3r - 4bq^3r + 4q^3r,$$

$$e_{1,45} = p^6q^2r^2 + p^4q^4r^2 - bp^8 + 2bp^6q^2 - 4ap^4q^4 + 2bp^2q^6 + bq^8 + p^7qr - 4p^5q^3r - p^3q^5r - ap^6r^2 - bp^6r^2 + ap^4q^2r^2 + bp^4q^2r^2 + aq^6r^2 + bq^6r^2 + p^8 - 3p^6q^2 + 3p^4q^4 - p^2q^6 + 4ap^5qr + 3ap^3q^3r - 2bp^3q^3r - 2apq^5r - bpq^5r + apq^3r^3 + bpq^3r^3 + 4ap^6 + 4bp^6 - 8ap^4q^2 - 8bp^4q^2 + 2ap^2q^4 + 2bp^2q^4 - 4aq^6 - 4bq^6 + 2p^3q^3r + 2pq^5r - 4p^6 + 8p^4q^2 - 2p^2q^4 + 4q^6 - 4apq^3r - 4bpq^3r + 4pq^3.$$

$$e_{3,1} = aqr^2 - apr + bpr - 2aq - 2bq - pr + 2q,$$

$$e_{3,2} = abr^2 - (a + b - 1)^2,$$

$$e_{3,3} = abpr - b^2pr - a^2q + b^2q + bpr + 2aq - q,$$

$$e_{3,4} = bpr^3 - bqr^2 - apr - 3bpr - qr^2 + 2aq + 2bq + 3pr - 2q,$$

$$e_{3,5} = b^2pr^2 + a^2qr - b^2qr - bpr^2 - a^2p - 2abp - b^2p - 2aqr + 2ap + 2bp + qr - p,$$

$$e_{3,6} = a^3qr - ab^2qr - a^3p - a^2bp + ab^2p + b^3p - 2a^2qr + a^2p - 2abp - 3b^2p + aqr + ap + 3bp - p,$$

$$e_{3,7} = a^3bp^2 + a^2b^2p^2 - ab^3p^2 - b^4p^2 - a^4q^2 - a^3bq^2 + a^2b^2q^2 + ab^3q^2 + 4b^3pqr - a^2bp^2 + 2ab^2p^2 + 3b^3p^2 + 5a^3q^2 + 6a^2bq^2 - 3ab^2q^2 - 4b^3q^2 - 8b^2pqr - abp^2 - 3b^2p^2 - 11a^2q^2 - 9abq^2 + 4b^2q^2 + 4bpqr + bp^2 + 11aq^2 + 4bq^2 - 4q^2.$$