J Autom Reasoning (2017) 59:331–344
DOI 10.1007/s10817-016-9395-z

Automated Reducible Geometric Theorem Proving and Discovery by Gröbner Basis Method

Jie Zhou[1] · Dingkang Wang[2] · Yao Sun[3]

Received: 16 April 2015 / Accepted: 17 November 2016 / Published online: 22 November 2016

Abstract In this paper, we investigate the problem that the conclusion is true on some components of the hypotheses for a geometric statement. In that case, the affine variety associated with the hypotheses is reducible. A polynomial vanishes on some but not all the components of a variety if and only if it is a zero divisor in a quotient ring with respect to the radical ideal defined by the variety. Based on this fact, we present an algorithm to decide if a geometric statement is generally true or generally true on components by the Gröbner basis method. This method can also be used in geometric theorem discovery, which can give the complementary conditions such that the geometric statement becomes true or true on components. Some reducible geometric statements are given to illustrate our method.

Keywords Zero divisor · True on components · Gröbner basis · Geometric theorem proving · Geometric theorem discovery

This research was partially supported by National Nature Science Foundation of China (Nos. 11301523 and 11371356).

✉ Jie Zhou
jiezhou@amss.ac.cn

Dingkang Wang
dwang@mmrc.iss.ac.cn

Yao Sun
sunyao@iie.ac.cn

1 Science and Technology on Communication Security Laboratory, Chengdu 610041, China

2 Academy of Mathematics and Systems Science, CAS, Beijing 100190, China

3 Institute of Information Engineering, CAS, Beijing 100093, China

1 Introduction

Automated theorem proving is the deriving of mathematical theorems by a computer program, which has been studied for several decades. It can be traced back to the excellent work of Gelernter et.al. [1], Tarski [2], Seidenberg [3], Collins [4] and so on. After that, there are two main algebraic methods to prove geometric theorem automatically: the Wu's method and the Gröbner basis method. In 1977, Wu Wen-Tsün [5,6] used the characteristic sets and pseudo division to prove the geometric statement mechanically. Many complicated geometry theorems have been proved by Wu's method [7]. The work of Ritt [8] provided excellent algebraic tools and algorithms for Wu's method. The notion of Gröbner basis was introduced by Buchberger in 1965. It is a powerful tool for solving multivariable polynomial systems. Chou [9], Kapur [10], Kutzler and Stifter [11] have done some works to prove geometry theorem by Gröbner basis method.

A geometric statement contains some hypotheses and a conclusion. Let $K[U, X]$ be a polynomial ring over an algebraic closed field K in U and X, where $U = \{u_1, \ldots, u_m\}$ are parameters which can be arbitrarily chosen, and $X = \{x_1, \ldots, x_n\}$ are variables. For a statement from elementary geometry, the hypotheses can be expressed by the following parametric polynomial equations

$$\begin{cases} f_1(u_1, \ldots, u_m, x_1, \ldots, x_n) = 0, \\ \cdots \\ f_s(u_1, \ldots, u_m, x_1, \ldots, x_n) = 0, \end{cases}$$

and the conclusion is

$$f(u_1, \ldots, u_m, x_1, \ldots, x_n) = 0,$$

where $f_1, \ldots, f_s, f$ are polynomials in $K[U, X]$.

In Wu's method, it firstly reduces the hypotheses $f_1, \ldots, f_s$ to a triangle form with respect to the variables X, and then successive pseudo divides the conclusion f by the triangle form. If the remainder is zero, then the geometric theorem is generally true. In Gröbner basis method, a Gröbner basis G of the ideal $\langle f_1, \ldots, f_s, fy - 1\rangle$ in $K(U)[X, y]$ is computed, where $K(U)[X, y]$ is a polynomial ring over field $K(U)$ in variables X and y. If 1 is in G, then the geometric theorem is generally true.

If the variety V defined by the ideal $I = \langle f_1, \ldots, f_s\rangle$ is reducible, the automated geometric theorem proving will become more difficult. Since the conclusion f may vanish on some but not all components of V, the Morley's trisector theorem [12] and V.Thèbault's conjecture [13] are the case in point. This problem can be solved by combining Ritt's decomposition algorithm with Wu's method. It decomposes the variety V by using Ritt's decomposition algorithm, and then checks the conclusion f on every components of V by Wu's method. Chou pointed out that the Gröbner basis method alone can not solve such problem unless factorization is used (cited from [7], page 88). However, the procedure of factorization is difficult and time-consuming. In this paper, we will prove the reducible geometric theorem by Gröbner method without factorization.

The automated theorem discovery is finding complementary conditions for a given geometric statement to become true [14]. Many related works have been done on these topics, which include [7,15–20]. In these works, the notion of comprehensive Gröbner system (CGS) has been used. The CGS was introduced by Weispfenning in 1992 [21]. Since then, many algorithms have been developed for computing CGS efficiently, including [22–28]. In 2004, Chen et al. [16] applied CGS to prove and discover geometric theorem automatically. Their method can not only prove whether the geometric theorem is generally true, but also provide complementary hypotheses such that the statement becomes true. Instead of computing a Gröbner basis in geometric theorem proving, a CGS of the parametric polynomial system $\{f_1, \ldots, f_s, fy - 1\}$ is computed in geometric theorem discovery. Manubens and Montes [29] proposed the minimal canonical CGS, which can also be used in mechanical proving. In Chen or Manubens' method, only the complementary conditions which make the geometric statement become true can be found. However, the problem of reducible geometric statement discovery can not be solved completely by their methods.

In this paper, we will present two algorithms to prove or discover the reducible geometry theorem automatically by Gröbner basis method without factorization. For a geometric statement, the conclusion vanishes on some but not all the components of a variety associated with the hypotheses if and only if it is a zero divisor in a quotient ring with respect to the radical ideal defined by the variety. Based on the observation, we prove and discover geometric theorems automatically regardless whether it is irreducible or reducible.

This paper is organized as follows. Some concepts and results about geometric theorem are given in Sect. 2. In Sect. 3, an algorithm is presented to prove geometric theorem automatically. Some reducible geometric theorems are proved by the algorithm. In Sect. 4, we extend geometric statement to parametric case and give an algorithm to find complementary conditions such that the statement becomes true or true on components. Finally, some conclusions are presented in Sect. 5.

2 Some Results About Geometric Theorem

From the view of algebra, we review some previous results about geometric theorem in this section.

Let K be an algebraically closed field, $K[U, X]$ be a polynomial ring over K in U and X, where $U = \{u_1, \ldots, u_m\}$ are parameters and $X = \{x_1, \ldots, x_n\}$ are variables. Given a geometric statement, it is assumed that the hypotheses can be expressed as $f_1(U, X) = 0, \ldots, f_s(U, X) = 0$, and the conclusion can be expressed as $f(U, X) = 0$, where $f_1, \ldots, f_s, f$ are polynomials in $K[U, X]$. Let $I = \langle f_1, \ldots, f_n \rangle$ be an ideal in $K[U, X]$, $V = \mathbb{V}(I) = \{(a_1, \ldots, a_{m+n}) \in K^{m+n} \mid f_i(a_1, \ldots, a_{m+n}) = 0, 1 \leq i \leq s\}$, $\mathbb{I}(V) = \{f \in K[U, X] \mid f(p) = 0, \forall\, p \in V\}$. We call $\mathbb{V}(I)$ the *affine variety* defined by I, and $\mathbb{I}(V)$ the ideal of V.

Definition 2.1 Let W be an irreducible variety in the affine space K^{m+n} with coordinates $u_1, \ldots, u_m, x_1, \ldots, x_n$. We say that $u_1, \ldots, u_m$ are algebraically independent on W if no nonzero polynomial in the u_i alone vanishes identically on W, i.e. $\mathbb{I}(W) \cap K[u_1, \ldots, u_m] = \{0\}$.

From Chapter 6 of the book [30], V has a minimal decomposition

$$V = V_1 \cup \cdots \cup V_p \cup V_{p+1} \cdots \cup V_{p+q}, \tag{1}$$

where each V_k is an irreducible subvariety of V, $V_i \not\subset V_j$ for $i \neq j$, and the parameters U are algebraically independent on the subvariety V_i for $1 \leq i \leq p$ and U are not algebraically independent on the subvariety V_j for $p + 1 \leq j \leq p + q$. Each V_k is called a *component* of V, $1 \leq k \leq p + q$. Moreover, the components $V_1, \ldots, V_p$ are called the *non-degenerate components* of V, which are corresponding to the non-degenerate cases of the hypotheses. The components $V_{p+1}, \ldots, V_{p+q}$ are called the *degenerate components* of V, which are corresponding to the degenerate cases of the hypotheses. If $p > 1$, the geometric statement is called *reducible*.

Definition 2.2 Given a geometric statement, it is assumed that the hypotheses are expressed as $f_1(U, X) = 0, \ldots, f_n(U, X) = 0$ and the conclusion is expressed as $f(U, X) = 0$. Let V be the variety defined by $\{f_1, \ldots, f_n\}$ and have a minimal decomposition as in (1). The geometric statement is called *true* if f vanishes on every point of V. The geometric statement is called *generally true* if f vanishes on all non-degenerate components of V, i.e. f vanishes on $V_1 \cup \cdots \cup V_p$. The geometric statement is called *generally true on components* if f vanishes on some but not all non-degenerate components of V. Otherwise, the geometric statement is called *generally false*.

From the Hilbert's Nullstellensatz, a geometric statement is true if and only if f is a member of the radical of I in $K[U, X]$. So we can decide whether a geometric statement is true by solving an ideal membership problem.

The following proposition gives methods to decide whether a geometric statement is generally true.

Proposition 2.3 *Let all the notations be the same as in Definition 2.2 and y be a new variable different from U and X. The following assertions are equivalent:*

1. *The geometric statement is generally true, i.e. the conclusion f vanishes on $V_1 \cup \cdots \cup V_p$.*
2. *The conclusion f is in the radical ideal of $\langle f_1, \ldots, f_n \rangle$ in $K(U)[X]$.*
3. *$\{1\}$ is the reduced Gröbner basis of the ideal $\langle f_1, \ldots, f_n, fy - 1 \rangle$ in $K(U)[X, y]$.*
4. *The Gröbner basis of $\langle f_1, \ldots, f_n, fy - 1 \rangle$ in $K[U, X, y]$ (in lexicographic ordering $U < X$ and $U < y$) contains at least a polynomial $g(U)$ in $K[U]$.*

Proof The detailed proofs of ($1 \Leftrightarrow 2$) and ($2 \Leftrightarrow 3$) can refer to the Chapter 5 of [9]. Now we proof ($3 \Leftrightarrow 4$).
$'' \Rightarrow ''$ From (3), there exits $p_1, \ldots, p_s, p_{s+1}$ in $K(U)[X, y]$ such that

$$1 = p_1 f_1 + p_2 f_2 + \cdots + p_s f_s + p_{s+1}(fy - 1).$$

There must exit $h(U) \in K[U]$ such that $h(U)p_i \in K[U][X, y]$ for $1 \le i \le s + 1$. So

$$\begin{aligned} h(U) = h(U)p_1 f_1 + h(U)p_2 f_2 + \cdots + h(U)p_s f_s \\ + h(U)p_{s+1}(fy - 1) \in \langle f_1, \ldots, f_n, fy - 1 \rangle \subset K[U, X, y]. \end{aligned}$$

For any Gröbner basis G of $\langle f_1, \ldots, f_n, fy - 1 \rangle$ in $K[U, X, y]$ (in lexicographic ordering $U < X$ and $U < y$), there exists a polynomial $g(U)$ in G such that the leading term of $h(U)$ is divided by the leading term of $g(U)$. According to term ordering, $g(U)$ is in $K[U]$.
"$\Leftarrow$" It is obvious through a similar analysis. □

3 Reducible Geometric Theorem Proving

The problem of deciding whether a geometric statement is generally true can be solved by Proposition 2.3. In this section, we focus on whether a geometric statement is generally true on components when it is not true or generally true.

3.1 Generally True on Components

Let J be an ideal in $K(U)[X]$, a polynomial f is a *zero divisor* in $K(U)[X]/J$ if $f \notin J$ and there exists a polynomial h in $K(U)[X]$ such that $h \notin J$ and $fh \in J$. The following theorem gives necessary and sufficient conditions to decide whether a geometric statement is generally true on components from an algebraic view.

Theorem 3.1 *Let all the notations be the same as in Definition 2.2, J be the ideal generated by $f_1, \ldots, f_n$ in $K(U)[X]$, and $\sqrt{J}$ be the radical ideal of J. Then the geometric statement is generally true on components if and only if f is a zero divisor in $K(U)[X]/\sqrt{J}$.*

Proof Let

$$V = \mathbb{V}(f_1, \ldots, f_n) = V_1 \cup \cdots \cup V_p \cup V_{p+1} \cdots \cup V_{p+q}, \tag{2}$$

be a minimal decomposition of V, where the parameters U are algebraically independent on the components V_i for $1 \le i \le p$ and algebraically dependent on the components V_j for $p+1 \le j \le p+q$. Let $W = V_1 \cup \cdots \cup V_p$.

("$\Rightarrow$") If the geometric statement is generally true on components, then f vanishes on some V_i but not all, $1 \le i \le p$. Without loss of generality, we assume f vanishes on $V_1 \cup \cdots \cup V_{i_0}$ but does not vanish on $V_{i_0+1}, \ldots, V_p$, where $1 \le i_0 < p$. Since $V_{i_0+1} \cup \ldots \cup V_p \subsetneqq W$, $\mathbb{I}(W) \subsetneqq \mathbb{I}(V_{i_0+1} \cup \cdots \cup V_p)$. There exists a polynomial $h \in \mathbb{I}(V_{i_0+1} \cup \cdots \cup V_p) \backslash \mathbb{I}(W)$. So h vanishes on $V_{i_0+1} \cup \cdots \cup V_p$ but does not vanish on W. Hence, fh vanishes on W. From Proposition 2.3, fh is in the $\sqrt{J}$. Since f and h are not in $\sqrt{J}$, f is a zero divisor in $K(U)[X]/\sqrt{J}$.

("$\Leftarrow$") If f is a zero divisor in $K(U)[X]/\sqrt{J}$, then f is not in $\sqrt{J}$ and there exists a polynomial $h \notin \sqrt{J}$ such that $fh \in \sqrt{J}$. From Proposition 2.3, fh vanishes on W. Since h is not in $\sqrt{J}$, h does not vanish on some V_i for $1 \le i \le p$, and then f must vanish on these components where h does not vanish. Moreover, f is not in $\sqrt{J}$, then f does not vanish on W. Hence, the geometric statement is generally true on components. □

Remark 3.2 Assume f vanishes and only vanishes on the components $V_{i_0}, \ldots, V_{i_k}$ of V, it is obvious that $I + \langle f \rangle$ vanishes on $V_{i_0} \cup \cdots \cup V_{i_k}$. So $V_{i_0} \cup \cdots \cup V_{i_k} \subset \mathbb{V}(I, f)$. That is to say, the conclusion f only vanishes on those components of V where $I + \langle f \rangle$ vanishes.

From Theorem 3.1, we can judge whether a geometric statement is generally true on components by checking whether the conclusion polynomial f is a zero divisor in $K(U)[X]/\sqrt{J}$. By decomposing the radical ideal $\sqrt{J}$, we can decide whether f is a zero divisor in $K(U)[X]/\sqrt{J}$. However, it is difficult and time-consuming procedure to decompose an ideal. In the following, we give another method to check whether f is a zero divisor in $K(U)[X]/\sqrt{J}$ without decomposing the radical ideal $\sqrt{J}$.

Let $J : f^s = \{g \mid gf^s \in J\}$, $J : f^\infty = \{g \mid gf^m \in J, \exists\, m \in \mathbb{N}^+\}$, $J : f^\infty = J : f^m$ means $J : f^{m-1} \subsetneqq J : f^m = J : f^{m+1}$.

Theorem 3.3 *Let J be an ideal and f be a polynomial in $K(U)[X]$, $\sqrt{J}$ be the radical ideal of J. Then f is a zero divisor in $K(U)[X]/\sqrt{J}$ if and only if $f \notin \sqrt{J}$ and there exists a polynomial $h \in J : f^\infty$ such that $h \notin \sqrt{J}$.*

Proof ("$\Rightarrow$"): If f is a zero divisor in $K(U)[X]/\sqrt{J}$, then $f \notin \sqrt{J}$ and there exists a polynomial $g \notin \sqrt{J}$ such that $fg \in \sqrt{J}$. Hence, there exists an integer s such that $f^s g^s \in J$, and $g^s \in J : f^s \subset J : f^\infty$. Let $h = g^s$, we have $h \in J : f^\infty$ and $h \notin \sqrt{J}$.

("$\Leftarrow$"): if $f \notin \sqrt{J}$ and there exists a polynomial $h \in J : f^\infty$ such that $h \notin \sqrt{J}$. Assume $J : f^\infty = J : f^m$, we have $hf^m \in J$. Clearly, $h^m f^m \in J$ and $hf \in \sqrt{J}$. Thus, f is a zero divisor in $K(U)[X]/\sqrt{J}$ since $f \notin \sqrt{J}$ and $h \notin \sqrt{J}$. □

If a Gröbner basis of $J : f^\infty$ is computed, then we can decide whether f is a zero divisor in $K(U)[X]/\sqrt{J}$ by checking elements in the Gröbner basis. Consequently, whether a geometric statement is generally true on components can be determined by Theorem 3.1.

Example 3.4 Consider a geometric statement whose hypotheses is $f_1 = (x_1 - u)(x_2 + u)^2 = 0$ and the conclusion is $f = x_2 + u = 0$.

Since $\sqrt{J} = \sqrt{\langle f_1 \rangle} = \langle (x_1 - u)(x_2 + u) \rangle \subset K(U)[X]$, it is obvious that f is not in the radical ideal $\sqrt{J}$ but it is a zero divisor in $K(U)[X]/\sqrt{J}$. Hence, the geometric statement is generally true on components by Theorem 3.1.

Example 3.5 Consider a geometric statement whose hypotheses are $f_1 = xy = 0$, $f_2 = x^2 = 0$, and the conclusion is $f = y = 0$.

Since $\sqrt{J} = \sqrt{\langle f_1, f_2 \rangle} = \langle x \rangle$, it is obvious that f is not in the radical ideal $\sqrt{J}$ and it is not a zero divisor of $K[X]/\sqrt{J}$. Hence, the geometric statement is generally false.

Remark 3.6 When the geometric statement is not generally true, Montes and Recio [17] also give a method to decide whether a geometric statement is generally true on components by deciding whether the ideal $\langle f_1, \ldots, f_n, f \rangle$ is $\langle 1 \rangle$ in $K(U)[X]$. It concludes that if $\langle f_1, \ldots, f_n, f \rangle \neq \langle 1 \rangle$, then the geometric statement is generally true on components. The ideal $\langle f_1, f_2, f \rangle \neq \langle 1 \rangle$ in Example 3.5,, and thus the geometric statement should generally true on components by [17]. However, it is obvious that the variety $\mathbb{V}(f_1, f_2, f) = \{(0, 0)\}$ is not a subvariety of $\mathbb{V}(f_1, f_2)$. Hence the conclusion f does not vanish on any non-degenerated component of V by Remark 3.2. The geometric statement is not generally true on components by our method.

3.2 The Algorithm for Proving Reducible Geometric Theorem

Given a geometric statement, assume $f_1 = 0, \ldots, f_n = 0$ are the hypotheses equations and $f = 0$ is the conclusion equation, where $f_1, \ldots, f_n, f$ are polynomials in $K[U, X]$. Let J be the ideal generated by $\{f_1, \ldots, f_n\}$ in $K(U)[X]$. We compute a Gröbner basis G of the ideal $H = J + \langle fy - 1 \rangle$ in $K(U)[X, y]$ with respect to any monomial ordering in $K(U)[X, y]$, where y is a new variable different from X. By Proposition 2.3, if $1 \in G$, then $f \in \sqrt{J}$ and the geometric statement is generally true. If the geometric statement is not generally true, let $G_\infty = G \cap K(U)[X]$. Note that G_∞ is a Gröbner basis of $J : f^\infty \subset K(U)[X]$. Combining Theorems 3.1 and 3.3, if there is a polynomial h in G_∞ but not in $\sqrt{J}$, then f is a zero divisor in $K(U)[X]/\sqrt{J}$ and the geometric statement is generally true on components. Otherwise, the geometric statement is generally false. In summary, we give the Algorithm 1 to prove geometric theorems automatically.

Algorithm 1 Proving Reducible Geometric Theorem Automatically

Require: Given a geometric statement, assume $f_1 = 0, \ldots, f_n = 0$ are the hypotheses equations, and $f = 0$ is the conclusion equation, where $f_1, \ldots, f_n, f \in K[U, X]$;

Ensure: Decide whether the geometric statement is generally true or generally true on components.

Step 1: Let $J = \langle f_1, \ldots, f_n \rangle \subset K(U)[X]$. Computing a minimal Gröbner basis G of ideal $H = J + \langle fy - 1 \rangle \subset K(U)[X, y]$ w.r.t. any monomial ordering.

Step 2: If $1 \in G$, then the geometric statement is generally true and the algorithm is terminated. Else, go to Step 3.

Step 3: Let $G_\infty = G \cap K(U)[X]$. If there is a polynomial h in G_∞ but not in $\sqrt{J}$, then the geometric statement is generally true on components. Otherwise, the geometric statement is generally false.

3.3 Application to Reducible Geometric Theorem Proving

There are two reducible geometric theorems proved by Algorithm 1. The first example comes from [7].

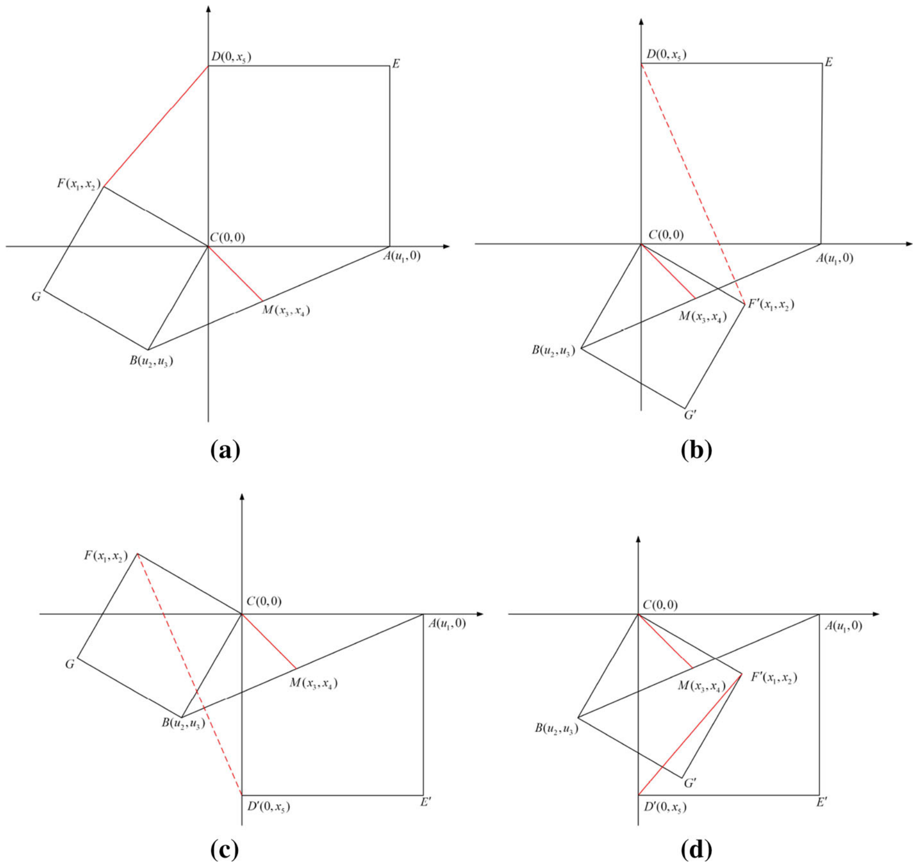

Fig. 1 Example 3.7

Example 3.7 Let $\triangle ABC$ be a triangle, $ACDE$ and $BCFG$ be two squares drawn on the two sides AC and BC respectively, and M be the midpoint of the line segment AB. Checking whether the conclusion $|DF| = 2|CM|$ is true or not.

Without loss of generality, fix a triangle $\triangle ABC$ with $A = (u_1, 0)$, $B = (u_2, u_3)$, $C = (0, 0)$, where $U = \{u_1, u_2, u_3\}$ are parameters. Since the positions $ACDE$ and $BCFG$ w.r.t. triangle $\triangle ABC$ is not unique, there are totally four possible cases based on the hypotheses (see Fig. 1a–d). We want to prove whether the geometric statement is generally true or generally true on components in the following analysis.

Let $F = (x_1, x_2)$, $M = (x_3, x_4)$ and $D = (0, x_5)$, where $X = \{x_1, x_2, x_3, x_4, x_5\}$ are variables. Then the hypotheses of the statement can be expressed as:

$$\begin{aligned}
f_1 &= x_2^2 + x_1^2 - u_3^2 - u_2^2 = 0, && (|CF| = |BC|)\\
f_2 &= x_5^2 - u_1^2 = 0, && (|DC| = |CA|)\\
f_3 &= u_3 x_2 + u_2 x_1 = 0, && (CF \perp BC)\\
f_4 &= 2x_3 - u_2 - u_1 = 0, && (M \text{is the midpoint of} AB)\\
f_5 &= 2x_4 - u_3 = 0.
\end{aligned}$$

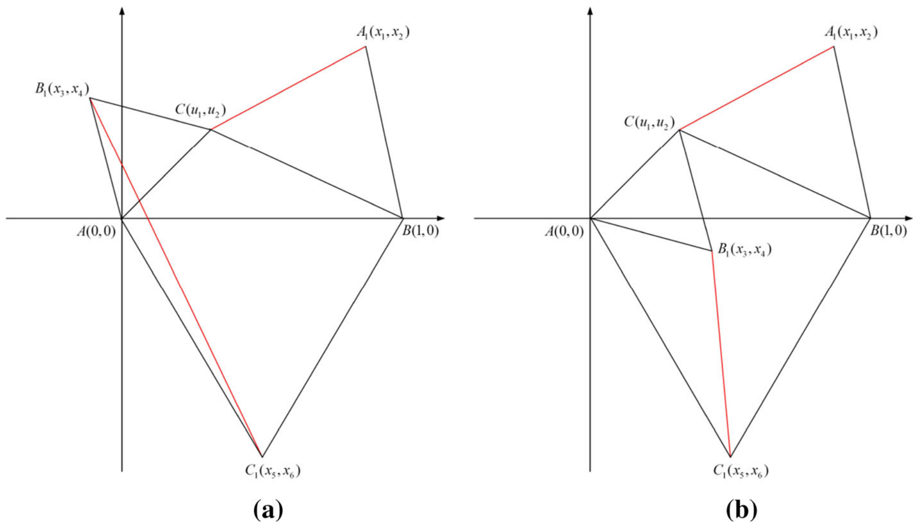

Fig. 2 Example 3.8

The conclusion is

$$f = x_1^2 + (x_2 - x_5)^2 - 4(x_3^2 + x_4^2) = 0, \quad (|DF| = 2|CM|).$$

Step 1 : Let $J = \langle f_1, f_2, f_3, f_4, f_5 \rangle \subset K(U)[X]$. Computing a Gröbner basis G of $J + \langle fy - 1 \rangle$ in $K(U)[X, y]$ w.r.t. a lexicographical ordering with $y > x_5 > \cdots > x_1$, we get $G = \{x_1^2 - u_3^2, u_3x_2 + u_2x_1, 2x_3 - u_2 - u_1, 2x_4 - u_3, u_3x_5 + u_1x_1, 4u_1u_2y + 1\}$.
Step 2 : Since $1 \notin G$, the geometric statement is not generally true.
Step 3 : Let $G_\infty = G \cap K(U)[X] = \{-u_3^2 + x_1^2, u_3x_2 + u_2x_1, 2x_3 - u_2 - u_1, 2x_4 - u_3, u_3x_5 + u_1x_1\}$. Checking polynomials in G_∞, we get $h = u_1x_1 + u_3x_5 \in G_\infty$ but h not in $\sqrt{J}$. Hence, the geometric statement is generally true on components by Theorem 3.1 and 3.3.

The following is another geometric statement which is generally true on components.

Example 3.8 Three equilateral triangles A_1BC, AB_1C, ABC_1 are erected on the three sides of triangle ABC. Checking whether the conclusion $|B_1C_1| = |A_1C|$ is true or not (see Fig. 2a, b).

Without loss of generality, fix a triangle $\triangle ABC$ with $A = (0, 0)$, $B = (1, 0)$ and $C = (u_1, u_2)$, where $U = \{u_1, u_2\}$ are parameters. Since the positions of triangles A_1BC, AB_1C and ABC_1 w.r.t. this triangle $\triangle ABC$ are not unique, there are totally eight possible cases based on the hypotheses. Two representative cases are showed in Fig. 2a, b. We need to check whether the conclusion $|B_1C_1| = |A_1C|$ is true.

From Fig. 2a, b, we know intuitively the conclusion $|B_1C_1| = |A_1C|$ does not always hold. Now we prove the reducible geometric statement automatically.

Let $A_1 = (x_1, x_2)$, $B_1 = (x_3, x_4)$, $C_1 = (x_5, x_6)$, where $X = \{x_1, \ldots, x_6\}$ are variables. The hypotheses of the geometric statement can be expressed as:

$$\begin{aligned}
f_1 &= x_5^2 + x_6^2 - 1, && (|AC_1| = |AB|)\\
f_2 &= (x_5 - 1)^2 + x_6^2 - 1, && (|BC_1| = |AB|)\\
f_3 &= (x_1 - u_1)^2 + (x_2 - u_2)^2 - (u_1 - 1)^2 - u_2^2, && (|CA_1| = |BC|)\\
f_4 &= (x_1 - 1)^2 + x_2^2 - (u_1 - 1)^2 - u_2^2, && (|BA_1| = |BC|)\\
f_5 &= (x_3^2 + x_4^2) - (u_1^2 + u_2^2), && (|AB_1| = |AC|)\\
f_6 &= (x_3 - u_1)^2 + (x_4 - u_2)^2 - (u_1^2 + u_2^2). && (|CB_1| = |AC|)
\end{aligned}$$

The conclusion can be expressed as:

$$f = (x_5 - x_3)^2 + (x_6 - x_4)^2 - (x_1 - u_1)^2 - (x_2 - u_2)^2, \quad (|B_1C_1| = |A_1C|).$$

Step 1 : Let $J = \langle f_1, f_2, f_3, f_4, f_5, f_6 \rangle \subset K(U)[X]$. Computing a Gröbner basis G of $J + \langle fy - 1 \rangle$ in $K(U)[X, y]$ w.r.t. a lexicographical ordering $y > x_6 > \cdots > x_2 > x_1$, we get $G = \{4x_1^2 + (-4u_1 - 4)x_1 + u_1^2 + 2u_1 - 3u_2^2 + 1, 2u_2x_2 + (2u_1 - 2)x_1 - u_1^2 - u_2^2 + 1, 4x_3^2 - 4u_1x_3 - 3u_2^2 + u_1^2, 2u_2x_4 + 2u_1x_3 - u_1^2 - u_2^2, 2x_5 - 1, 2u_2x_6 - 2x_3 + u_1, (9u_1^2 - 3u_2^2)y - 2x_3 - 2u_1\}$.
Step 2 : Since $1 \notin G$, the geometric statement is not generally true.
Step 3 : Let $G_\infty = \{4x_1^2 + (-4u_1 - 4)x_1 + u_1^2 + 2u_1 - 3u_2^2 + 1, 2u_2x_2 + (2u_1 - 2)x_1 - u_1^2 - u_2^2 + 1, 4x_3^2 - 4u_1x_3 - 3u_2^2 + u_1^2, 2u_2x_4 + 2u_1x_3 - u_1^2 - u_2^2, 2x_5 - 1, 2u_2x_6 - 2x_3 + u_1\}$. Checking polynomials in G_∞, we get $h = 2u_2x_6 - 2x_3 + u_1 \in G_\infty$ is not in $\sqrt{J}$. Hence, the geometric statement is generally true on components by Theorem 3.1 and 3.3.

4 Geometric Theorem Discovery

In this section, we aim to find the complementary conditions such that the geometric statement becomes true or true on components. Instead of the Gröbner basis, the CGS is used.

4.1 Comprehensive Gröbner Systems

Let K be an algebraically closed field, R be a polynomial ring $K[U]$ in parameters $U = \{u_1, \ldots, u_m\}$, and $R[X]$ be a polynomial ring over R in variables $X = \{x_1, \ldots, x_n\}$ where X and U are disjoint. For a polynomial $f \in R[X] = K[U][X]$, the leading coefficient and leading term of f w.r.t. the ordering $<$ are denoted by $\mathrm{lc}_X(f)$ and $\mathrm{lt}_X(f)$ respectively. Note that $\mathrm{lc}_X(f) \in K[U]$ and $\mathrm{lt}_X(f)$ are monomials in $K[X]$.

A specialization of R is a homomorphism $\sigma : R \longrightarrow K$. We only consider the specializations induced by the elements in K^m. That is, for $\overline{a} \in K^m$, the induced specialization $\sigma_{\overline{a}}$ is defined as follows:

$$\sigma_{\overline{a}} : f \longrightarrow f(\overline{a}),$$

where $f \in R$. Every specialization $\sigma : R \longrightarrow K$ extends canonically to a specialization $\sigma : R[X] \longrightarrow K[X]$ by applying σ coefficient-wise.

Following [23], an algebraically constructible set A is defined to be of the form: $A = \mathbb{V}(E) \setminus \mathbb{V}(N)$, where E, N are subsets of $K[U]$ and $\mathbb{V}(E)$ (or $\mathbb{V}(N)$) is the affine variety defined by E (or N). The E is called the equation constraint of A and N is called the non-equation constraint of A.

For a parametric polynomial system, a CGS is defined below.

Definition 4.1 Let F be a subset of $R[X]$, $A_1, \ldots, A_l$ be algebraically constructible subsets of K^m, S be a subset of K^m such that $S \subset A_1 \cup \cdots \cup A_l$, and $G_1, \ldots, G_l$ be subsets of $R[X]$. A finite set $\mathcal{G} = \{(A_1, G_1), \ldots, (A_l, G_l)\}$ is called a *comprehensive Gröbner system*(CGS) on S for F, if $\sigma_{\overline{a}}(G_i)$ is a Gröbner basis for the ideal $\langle \sigma_{\overline{a}}(F) \rangle$ in $K[X]$ for any $\overline{a} \in A_i, i = 1, \ldots, l$. Each (A_i, G_i) is called a branch of $\mathcal{G}$ and A_i is called the parametric constraint of this branch. If $S = K^m$, $\mathcal{G}$ is called a comprehensive Gröbner system for F.

Definition 4.2 A comprehensive Gröbner system $\mathcal{G} = \{(A_1, G_1), \ldots, (A_l, G_l)\}$ for F is said to be *minimal*,[1] if for each $i = 1, \ldots, l$,

1. $A_i \neq \emptyset$, and furthermore, for each $i, j = 1, \ldots, l$, $A_i \cap A_j = \emptyset$ whenever $i \neq j$;
2. for each $g \in G_i$, $\sigma_{\overline{a}}(\text{lc}_X(g)) \neq 0$ for any $\overline{a} \in A_i$;
3. for all $g \in G_i$, $\text{lt}_X(g)$ is not divisible by any leading term of $G_i \setminus \{g\}$.

4.2 The Algorithm for Discovering Reducible Geometric Theorem

Given a geometric statement, assume the hypotheses are expressed as $f_1(U, X) = 0, \ldots, f_n(U, X) = 0$ and the conclusion is expressed as $f(U, X) = 0$, where $f_1, \ldots, f_n, f$ are polynomials in $K[U][X]$. Let $\mathcal{G} = \{(A_1, G_1), \ldots, (A_l, G_l)\}$ be a minimal CGS for $F = \{f_1, \ldots, f_n\}$. For any branch $(A_i, G_i) \in \mathcal{G}$ and any $\overline{a} \in A_i$, the variety $V_{\overline{a}} = \mathbb{V}(\sigma_{\overline{a}}(f_1), \ldots, \sigma_{\overline{a}}(f_n)) \subset K^n$ has a minimal decomposition

$$V_{\overline{a}} = V_{\overline{a},1} \cup \cdots \cup V_{\overline{a},s}, \tag{3}$$

each $V_{\overline{a},i}$ is called a component of $V_{\overline{a}}$.

Assume (A_i, G_i) is a branch of $\mathcal{G}$, the geometric statement is called *true under* A_i if the conclusion $\sigma_{\overline{a}}(f)$ vanishes on every point of $V_{\overline{a}}$ for any $\overline{a} \in A_i$. The geometric statement is called *true on components under* A_i if the conclusion $\sigma_{\overline{a}}(f)$ vanishes on some but not all components of $V_{\overline{a}}$ for any $\overline{a} \in A_i$.

Theorem 4.3 *Given a geometric statement, assume the hypotheses are expressed as* $f_1(U, X) = 0, \ldots, f_n(U, X) = 0$ *and the conclusion is expressed as* $f(U, X) = 0$*, where* $f_1, \ldots, f_n, f$ *are polynomials in* $K[U][X]$*. Let* $\mathcal{G} = \{(A_1, G_1), \ldots, (A_l, G_l)\}$ *be a minimal CGS for* $H = \{f_1, \ldots, f_n, fy - 1\}$ *w.r.t. a blocking ordering such that* $y > X > U$*,*[2] *where* y *is a new variable different from* U *and* X*. For any branch* $(A, G) \in \mathcal{G}$*, assume* $A = \mathbb{V}(E) \backslash \mathbb{V}(N)$*, we have the following assertions:*

1. *If there is a nonzero polynomial* $g(U) \in K[U]$ *in* G*, then the geometric statement is true under* A*. What's more,* A *is the complementary condition such that the geometric statement becomes true.*
2. *If* f *is a zero divisor in* $K[U, X]/\sqrt{\widetilde{I}}$*, then the geometric statement is true on components under* A*, where* $\widetilde{I} = \langle f_1, \ldots, f_n, E \rangle \subset K[U, X]$*. What's more,* A *is the complementary condition such that the geometric statement becomes true on components.*

Proof (1) If there is a nonzero polynomial $g(U) \in K[U]$ in G. From the definition of minimal CGS, $\sigma_{\overline{a}}(g)$ is a nonzero constant for any $\overline{a} \in A$. Then $\langle \sigma_{\overline{a}}(f_1), \ldots, \sigma_{\overline{a}}(f_n), \sigma_{\overline{a}}(f)y - 1 \rangle = \langle 1 \rangle$. Hence, $\sigma_{\overline{a}}(f) \in \sqrt{\langle \sigma_{\overline{a}}(f_1), \ldots, \sigma_{\overline{a}}(f_n) \rangle}$, and $\sigma_{\overline{a}}(f)$ vanishes on the variety $\mathbb{V}(\sigma_{\overline{a}}(f_1), \ldots, \sigma_{\overline{a}}(f_n)) \subset K^n$. Therefore, the geometric statement is true under A, and A is the complementary condition such that the geometric statement becomes true.

[1] Programs for computing minimal comprehensive Gröbner systems, which is based on the algorithm in [23], are available at http://mmrc.iss.ac.cn/~dwang/software.html.

[2] The variables are divided in three blocks y, X, and U. A monomial ordering is chosen for each block. To compare two monomials, we firstly compare their y part. If the y part is equal, then we compare their X part. If the y and X parts are equal, then we compare their U part.

(2) If f is a zero divisor in $K[U, X]/\sqrt{\widetilde{I}}$, where $\widetilde{I} = \langle f_1, \ldots, f_n, E\rangle \subset K[U, X]$. Then $\sigma_{\overline{a}}(f)$ is a zero divisor in $K[X]/\sqrt{\langle\sigma_{\overline{a}}(f_1), \ldots, \sigma_{\overline{a}}(f_n)\rangle}$ for any $\overline{a} \in A$. From Theorem 3.1, $\sigma_{\overline{a}}(f)$ vanishes on some but not all components of $\mathbb{V}(\sigma_{\overline{a}}(f_1), \ldots, \sigma_{\overline{a}}(f_n)) \subset K^n$. Hence, the geometric statement is true on components under A, and A is the complementary condition such that the geometric statement becomes true on components. □

Based on the Theorem 4.3, we give an Algorithm 2 to discover reducible geometric statement.

Algorithm 2 Discovering Reducible Geometric Theorem Automatically

Require: Given a geometric statement, assume $f_1 = 0, \ldots, f_s = 0$ are the hypotheses equations, and $f = 0$ is the conclusion equation, where $f_1, \ldots, f_s, f \in K[U][X]$;

Ensure: Discover the complementary conditions such that the geometric statement becomes true or true on component.

Step 1: Computing a minimal comprehensive Gröbner systems $\mathcal{G}$ for $H = F \cup \{fy-1\}$ w.r.t. an admissible block term order such that $y > X > U$, where $F = \{f_1, \ldots, f_n\}$. For any branch (A, G) of $\mathcal{G}$, assume $A = \mathbb{V}(E)\backslash\mathbb{V}(N)$.

Step 2: If there is a polynomial $g(U) \in K[U]$ in G, then the geometric statement is true under A, and the algorithm is terminated. What's more, A is the complementary condition such that the geometric statement becomes true. Else, go to Step 3.

Step 3: Let $G_\infty = G \cap K[U, X]$. If there is a polynomial h in G_∞ but not in $\sqrt{\widetilde{I}}$, where $\widetilde{I} = \langle f_1, \ldots, f_n, E\rangle \subset K[U, X]$, then the geometric statement is true on components under A, and A is the complementary condition such that the geometric statement becomes true on components.

4.3 Application to Reducible Geometric Theorem Discovery

In this part, we apply Algorithm 2 to discover a geometry theorem which is true on components.

Example 4.4 Let $\triangle ABC$ be a triangle. The line AE bisects the angle $\angle A$, and BF bisects the angle $\angle B$. Lines AE and BF intersect at a point O. In what cases, $|AO| = |BO|$ holds?

Let $A = (0, 0)$, $B = (u_1, 0)$, $C = (u_2, u_3)$, where $U = \{u_1, u_2, u_3\}$ are parameters. If u_1 or u_3 is zero, then the $\triangle ABC$ degenerates to a line segment. So we assume $u_1 u_3 \neq 0$ in this example. Since every angle has two angular bisectors (exterior angle bisector and interior angle bisector), there are four points where the angular bisectors of angles $\angle A$ and $\angle B$ intersect (see Fig. 3).

In Fig. 3, AE (or AE') is the interior (or exterior) angle bisector of angle $\angle A$, and BF (or BF') is the interior (or exterior) angle bisector of angle $\angle B$. The two interior angle bisector AE and BF intersect at the point O, and AE' and BF', AE and BF', BF and AE' intersect at the point O_1, O_2, O_3 respectively. Thus, we need to check whether $|AO| = |BO|$, $|AO_1| = |BO_1|$, $|AO_2| = |BO_2|$ and $|AO_3| = |BO_3|$ hold or not.

Let $O = (x_1, x_2)$, where $X = \{x_1, x_2\}$ are variables. The hypotheses of the statement can be expressed as:

$$f_1 = (u_2^2 + u_3^2)x_2^2 - (u_3 x_1 - u_2 x_2)^2, \quad (AE \text{ or } AE' \text{ bisects } \angle A).$$
$$f_2 = ((u_2^2 - u_1^2) + u_3^2)x_2^2 - (u_3 x_1 + (u_1 - u_2)x_2 - u_1 u_3)^2, \quad (BF \text{ or } BF' \text{ bisects } \angle B).$$

The conclusion to be proved is

$$f = (x_1^2 + x_2^2) - ((x_1 - u_1)^2 + x_2^2) = 2u_1 x_1 - u_1^2.$$

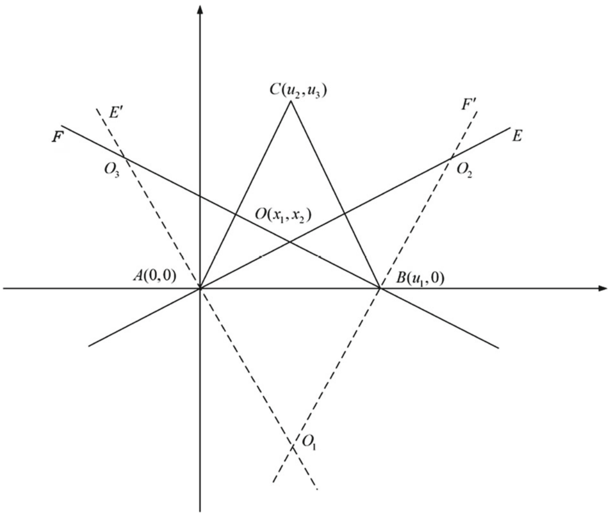

Fig. 3 Example 4.4

Step 1. Computing a minimal comprehensive Gröbner system $\mathcal{G}$ on $\mathbb{V}(\emptyset)\backslash\mathbb{V}(u_1u_3)$ for $\{f_1, f_2, fy-1\}$, we get $\mathcal{G} = \{(A_1, G_1), (A_2, G_2), (A_3, G_3)\}$, where A_i, G_i are as follows:

$$\begin{aligned}
A_1 &= \mathbb{C}^3 \setminus \mathbb{V}(u_1u_3(u_1-2u_2)),\\
G_1 &= \{4x_1^4u_3^2 - 8x_1^3u_1u_3^2 + 4x_1^2u_1^2u_3^2 + 4x_1^2u_1u_2u_3^2 - 4x_1^2u_2^2u_3^2 - 4x_1^2u_3^4\\
&\quad -4x_1u_1^2u_2u_3^2 + 4x_1u_1u_2^2u_3^2\\
&\quad +4x_1u_1u_3^4 - u_1^2u_3^4, x_2u_1u_3^3 - 2x_2u_2u_3^3 - 2x_1^3u_3^2 + 2x_1^2u_1u_3^2\\
&\quad +2x_1^2u_2u_3^2 - 2x_1u_1u_2u_3^2 + 2x_1u_3^4\\
&\quad -u_1u_3^4, yu_1^4u_3^2 - 2yu_1^3u_2u_3^2 + 4x_2u_3^3 - 4x_1^2u_3^2 + 2x_1u_1u_3^2\\
&\quad +4x_1u_2u_3^2 + u_1^2u_3^2 - 2u_1u_2u_3^2\},\\
A_2 &= \mathbb{V}(u_2^2 + u_3^2, u_1 - 2u_2) \setminus \mathbb{V}(u_2u_3),\\
G_2 &= \{1\},\\
A_3 &= \mathbb{V}(u_1 - 2u_2)\backslash\mathbb{V}(u_2u_3(u_2^2 + u_3^2)),\\
G_3 &= \{x_1^2u_3^2 - 2x_1u_2u_3^2 - u_3^4, x_2u_3 - u_3^2, 4yu_2^3u_3^2 + 4yu_2u_3^4 - x_1u_3^2 + u_2u_3^2\},
\end{aligned}$$

where $\mathbb{C}$ is the complex field.

In the branch (A_1, G_1), since there is no polynomial $g(U) \in K[U]$ in G_1, the geometric statement is not true under A_1. Let $G_1^\infty = G_1 \cap K[U, X] = \{4x_1^4u_3^2 - 8x_1^3u_1u_3^2 + 4x_1^2u_1^2u_3^2 + 4x_1^2u_1u_2u_3^2 - 4x_1^2u_2^2u_3^2 - 4x_1^2u_3^4 - 4x_1u_1^2u_2u_3^2 + 4x_1u_1u_2^2u_3^2 + 4x_1u_1u_3^4 - u_1^2u_3^4, x_2u_1u_3^3 - 2x_2u_2u_3^3 - 2x_1^3u_3^2 + 2x_1^2u_1u_3^2 + 2x_1^2u_2u_3^2 - 2x_1u_1u_2u_3^2 + 2x_1u_3^4 - u_1u_3^4\}$. Since there is no polynomial $g \in G_1^\infty$ such that $g \notin \sqrt{\widetilde{I_1}}$, where $\widetilde{I_1} = \langle f_1, \ldots, f_n\rangle \subset K[U, X]$, the geometric statement is not true on components under A_1.

In the branch (A_2, G_2), since there is a polynomial $1 \in K[U]$ in G_2, the geometric statement is true under A_2. Note that there is no real solution of $\{u_2^2 + u_3^2 = 0, u_1 - 2u_2 = 0\}$ when $u_2u_3 \neq 0$. It has no geometric meaning in the real geometry.

The branch (A_3, G_3) is interesting, since the equation constraint $u_1 - 2u_2 = 0$ in A_3 means that the triangle $\triangle ABC$ is an isosceles triangle, i.e., $|AC| = |BC|$. In the following, we only focus on the branch (A_3, G_3).

Step 2. Since there is no polynomial $g \in G_3$ such that $g \in K[U]$, the geometric statement is not true under A_3.

Step 3. Let $G_3^{\infty} = G_3 \cap K[U, X] = \{x_1^2u_3^2 - 2x_1u_2u_3^2 - u_3^4, x_2u_3 - u_3^2\}$. Since there is a polynomial $h = x_2u_3 - u_3^2$ in G_3^{∞} but not in $\sqrt{\widetilde{I_3}}$, where $\widetilde{I_3} = \langle f_1, \ldots, f_n, u_1 - 2u_2\rangle \subset K[U, X]$, the geometric statement is true on components under A_3. Moreover, $u_1 - 2u_2 = 0$ and $u_2u_3(u_2^2 + u_3^3) \neq 0$ are the complementary conditions such that the geometric statement becomes true on components.

5 Conclusions

In this paper, we give an algorithm to prove the reducible geometric statement automatically. It can not only decide whether a geometric statement is generally true, but also decide whether the geometric statement is generally true on components. This method can be naturelly extended to parametric case for discovering geometric theorem automatically. The only difference is computing a minimal CGS instead of computing a Gröbner basis.

Comparing with Wu's method, our method does not need to decompose a variety. Although we need to check whether the conclusion is a zero divisor in a quotient ring, it is more efficient than decomposing a variety.

Acknowledgements The authors are grateful to Professors D. Kapur and B.C. Xia for their helpful discussions and suggestions.

References

1. Gelernter, H., Hanson, J.R., Loveland, D.W.: Empirical explorations of the geometry-theorem proving machine. In: Proceeding of the West Joint Computer Conference, pp. 143–147 (1960)
2. Tarski, A.: A decision method for elementary algebra and geometry. Texts Monogr. Symb. Comput. **35**, 24–84 (1951)
3. Seidenberg, A.: A new decision method for elememtary algebra. Ann. Math. **60**(2), 365–374 (1954)
4. Collins, G.E.: Quantifier elimination for real closed fields by cylindrical algebraic decomposition. Quantifier Elimination and Cylindrical Algebraic Decomposition, pp. 85–121. Springer, Vienna (1998)
5. Wu, W.T.: On the decision problem and the mechanization of theorem proving in elementary geometry. Sci. Sin. **21**, 159–172 (1978)
6. Wu, W.T.: Basic principles of mechanical theorem proving in elementary geometries. J. Autom. Reason. **2**(3), 221–252 (1986)
7. Chou, S.C.: Mechanical Geometry Theorem Proving, Mathematics and Its Applications. Reidel, Amsterdam (1988)
8. Ritt, J.F.: Differential equations from the algebraic standpoint. Am. Math. Soc. (1932)
9. Chou, S.C.: Proving geometry theorems with rewrite rules. J. Autom. Reason. **2**(3), 253–273 (1986)
10. Kapur, D.: Geometry theorem proving using Hilbert's Nullstellensatz. In: Proceedings of the ISSAC, pp. 202–208. ACM Press, New York (1986)
11. Kutzler, B., Stifter, S.: Automated geometry theorem proving using Buchberger's algorithm. In: Proceedings of ISSAC, pp. 209–214. ACM Press, New York (1986)
12. Coxeter, H.S.M.: Introduction to Geometry. Wiley, New York (1961)
13. Taylor, K.B.: Three circles with collinear centers. Am. Math. Mon. **90**, 484–486 (1983)
14. Dalzotto, G., Recio, T.: On protocols for the automated discovery of theorems in elementary geometry. J. Autom. Reason. **43**(2), 203–236 (2009)

15. Botana, F., Montes, A., Recio, T.: An algorithm for automatic discovery of algebraic loci. In: Proceedings of the ADG, pp. 53–59 (2012)
16. Chen, X.F., Li, P., Lin, L., Wang, D.K.: Proving geometric theorems by partitioned-parametric Gröbner bases. In: Proceedings of the ADG, pp. 34–43 (2004)
17. Montes, A., Recio, T.: Automatic discovery of geometry theorems using minimal canonical comprehensive Gröbner systems. In: Proceedings of the ADG, pp. 113–138 (2006)
18. Recio, T., Velez, M.P.: Automatic discovery of theorems in elementary geometry. J. Autom. Reason. **23**(1), 63–82 (1999)
19. Wang, D.K., Lin, L.: Automatic discovery of geometric theorem by computing Gröbner bases with parameters. In: Abstracts of Presentations of ACA, vol. 32, Japan (2005)
20. Winkler, F.: Gröbner bases in geometry theorem proving and simplest degeneracy conditions. Math. Pannon. **1**(1), 15–32 (1990)
21. Weispfenning, V.: Comprehensive Gröbner bases. J. Symb. Comput. **14**(1), 1–29 (1992)
22. Kapur, D.: An approach for solving systems of parametric polynomial equations. In: Saraswat, V., Van Hentenryck, (eds.) Principles and Practices of Constraints Programming, pp. 217–244. MIT Press, Cambridge (1995)
23. Kapur, D., Sun, Y., Wang, D.K.: A new algorithm for computing comprehensive Gröbner systems. In: Proceedings of the ISSAC, pp. 29–36. ACM Press, New York (2010)
24. Montes, A.: A new algorithm for discussing Gröbner bases with parameters. J. Symb. Comput. **33**(2), 183–208 (2002)
25. Montes, A., Wibmer, M.: Gröbner bases for polynomial systems with parameters. J. Symb. Comput. **45**(12), 1391–1425 (2010)
26. Nabeshima, K.: A speed-up of the algorithm for computing comprehensive Gröbner systems. In: Proceedings of the ISSAC, pp. 299–306. ACM Press, New York (2007)
27. Suzuki, A., Sato, Y.: An alternative approach to comprehensive Gröbner bases. J. Symb. Comput. **36**(3), 649–667 (2003)
28. Weispfenning, V.: Canonical comprehensive Gröbner bases. J. Symb. Comput. **36**(3), 669–683 (2003)
29. Manubens, M., Montes, A.: Minimal canonical comprehensive Groebner system. J. Symb. Comput. **44**(5), 463–478 (2009)
30. Cox, D., Little, J., O'Shea, D.: Ideals, Varieties, and Algorithms, 3rd edn. Springer, New York (2007)