On Checking Linear Dependence of Parametric Vectors

Xiaodong Ma[1], Yao Sun[2], Dingkang Wang[3], and Yushan Xue[4(✉)]

[1] College of Science, China Agricultural University, Beijing 100083, China
maxiaodong@cau.edu.cn
[2] SKLOIS, IEE, Chinese Academy of Sciences, Beijing 100093, China
sunyao@iie.ac.cn
[3] KLMM, AMSS, Chinese Academy of Sciences, Beijing 100190, China
dwang@mmrc.iss.ac.cn
[4] School of Mathematics and Statistics,
Central University of Finance and Economics, Beijing 100081, China
cnxueyushan@cufe.edu.cn

Abstract. Checking linear dependence of a finite number of vectors is a basic problem in linear algebra. We aim to extend the theory of linear dependence to parametric vectors where the entries are polynomials. This dependency depends on the specifications of the parameters or values of the variables in the polynomials. We propose a new method to check if parametric vectors are linearly dependent. Furthermore, this new method can also give the maximal linearly independent subset, and by which the remaining vectors are expressed in a linear combination. The new method is based on the computation of comprehensive Gröbner system for a finite set of parametric polynomials.

1 Introduction

One basic problem in linear algebra is to check linear dependence of a finite number of vectors in a vector space over some field [1]. If the entries of the vectors are elements of a field, a classical way to solve this dependency problem is using Gaussian elimination. What if the entries are polynomials? There is a natural question: how to define the linear dependence of a finite number of parametric vectors whose entries are polynomials. The vectors whose entries are polynomials are called polynomial vectors, or parametric vectors without confusion in this paper.

To answer this, we first introduce the definition of **specialization**. Let R be a polynomial ring with variables $u_1, \cdots, u_m$ over the field k, i.e. $R = k[u_1, \cdots, u_m]$. Given a field L, a specialization of R is a homomorphism $\sigma : R \longrightarrow L$. In this paper, we always assume that L is an algebraically closed field containing k, and we only consider the specializations induced by the elements in L^m. That is, for $\bar{a} \in L^m$, the induced specialization $\sigma_{\bar{a}}$ is defined as follows:

$$\sigma_{\bar{a}} : f \longrightarrow f(\bar{a}),$$

D.-S. Huang et al. (Eds.): ICIC 2017, Part II, LNCS 10362, pp. 188–196, 2017.
DOI: 10.1007/978-3-319-63312-1_17

where $f \in R$.

Given a polynomial vector $\mathbf{f} = (f_1, \cdots, f_l) \in R^l$, we can extend a specialization $\sigma_{\bar{a}}$ to R^l:

$$\sigma_{\bar{a}}(\mathbf{f}) = (\sigma_{\bar{a}}(f_1), \cdots, \sigma_{\bar{a}}(f_l)) \in L^l.$$

Note that $\sigma_{\bar{a}}(\mathbf{f})$ does not contain any variable after the above specialization. For a simple example, $f = u_1u_2 + u_3 \in \mathbb{Q}[u_1, u_2, u_3]$ and $\bar{a} = (1, -1, 2) \in \mathbb{C}^3$, where $\mathbb{Q}$ and $\mathbb{C}$ are the field of rational numbers and the field of complex numbers respectively. Then the specialization $\sigma_{\bar{a}}(f) = f(1, -1, 2) = 1$.

One main goal of this paper is to solve the following interesting problem. **The dependency problem**: given a set of parametric vectors $\mathbf{f}_1, \cdots, \mathbf{f}_s \in R^l$, we would like to know for which point $\bar{a} \in L^m$, the vectors $\sigma_{\bar{a}}(\mathbf{f}_1), \cdots, \sigma_{\bar{a}}(\mathbf{f}_s)$ are *linearly independent* over L in the vector space L^l; and for which point $\bar{a} \in L^m$, the vectors $\sigma_{\bar{a}}(\mathbf{f}_1), \cdots, \sigma_{\bar{a}}(\mathbf{f}_s)$ are *linearly dependent* over L.

We will divide the parametric space into finitely many partitions, such that the maximal linearly independent subset are fixed for every specification of the parameters in each partition, and the remaining vectors will be expressed in the linear combinations of the maximal linearly independent subset after specialization.

There is a natural way to solve the dependency problem, and it is based on a generalization of the Gaussian elimination from linear algebra. To some extent it is a generalization of the classical way of checking linear dependence of vectors with entries in a field. The difficulty is that we need to discuss every polynomial entry of the parametric vectors carefully, and the process is very complicated.

In this paper, we propose a new method to solve the dependency problem. It is based on computing a minimal comprehensive Gröbner system for a specific parametric polynomials. The new method is easy to implement, and the advantage is that we can use all existing efficient algorithms for computing comprehensive Grönber systems. In addition, we use a trick to record the computation process in order to find the maximal linearly independent subset, and the remaining vectors will be expressed in the linear combinations of the maximal linearly independent subset after specialization.

The paper is organized as follows. Some preliminaries are given in Sect. 2. In Sect. 3, we propose a new method to solve the dependency problem, which is based on the minimal comprehensive Gröbner system. A complete example is given to illustrate the new method in Sect. 4.

2 Preliminaries

Let k be a field, R be the polynomial ring $k[u]$ in the variables $u = \{u_1, \cdots, u_m\}$, and $R[x]$ be the polynomial ring in the variables $x = \{x_1, \cdots, x_n\}$ over the ring R, where the variables x and u are disjoint.

Lexicographic order and graded reverse lexicographic order are two classic term orders, and are also used in the paper. Let $\succ$ be a term order on x. For a nonzero $f \in k[x]$, the leading monomial, leading term and leading coefficient of f are denoted by $\mathrm{lm}(f)$, $\mathrm{lt}(f)$ and $\mathrm{lc}(f)$ respectively. For a nonzero

$f \in R[x] = k[u][x]$, the leading monomial, leading term and leading coefficient of f w.r.t. $\succ_x$ are denoted by $\mathrm{lm}_x(f)$, $\mathrm{lt}_x(f)$ and $\mathrm{lc}_x(f)$ respectively. Note that $\mathrm{lc}_x(f) \in k[u]$ and $\mathrm{lt}_x(f) = \mathrm{lc}_x(f)\mathrm{lm}_x(f)$.

For a specialization $\sigma : R \longrightarrow L$, it can be extended canonically to a specialization $\sigma : R[x] \longrightarrow L[x]$ by applying σ coefficient-wise. Note that the field L is always assumed to be an algebraically closed field containing k.

Let F be a subset of $R[x]$ and $\bar{a}$ be a given point in L^m. Then we set $\sigma_{\bar{a}}(F) = \{\sigma_{\bar{a}}(f) \mid f \in F\} \subseteq L[x]$, where $\sigma_{\bar{a}}$ is the specialization induced by $\bar{a}$. Let $\langle \sigma_{\bar{a}}(F) \rangle$ be an ideal generated by the set $\sigma_{\bar{a}}(F)$ in $L[x]$.

Let I be an ideal in $k[x]$. The concept of Groebner basis of I w.r.t $\succ$ was proposed by Buchberger, and he also gave an algorithm to compute it. The Groebner basis and minimal Gröbner basis are introduced as follows. For more information, please refer to [2,3].

Definition 1. *Let I be an ideal. A finite set $G = \{g_1, g_2, \cdots, g_t\} \subseteq I$ is called a Gröbner basis for I w.r.t $\succ$, if for any nonzero $f \in I$, $< (f)$ is divisible by $< (g_i)$ for some i.*

Definition 2. *Let G be a Gröbner basis for I. Then G is called a minimal Gröbner basis for I if for any $f \in G$, there does not exist $g \in G \setminus \{f\}$ such that $< (f)$ is divisible by $< (g)$.*

There are many efficient algorithms to compute Gröbner bases and minimal Gröbner bases. If I is an ideal generated by a set of linear polynomials, a minimal basis G for I w.r.t. any given term order $\succ$ can also be obtained by Gaussian elimination of the corresponding coefficient matrix, and all the polynomials in G are linear polynomials.

The concept of a comprehensive Gröbner system (CGS) was introduced by Weispfenning [4]. There are many efficient algorithms to compute CGS, such as [5–11]. It is a powerful tool and widely used in computer science, algebraic geometry, engineering problems, automated geometry theorem proving and automated geometry theorem discovery [12–14]. For a parametric polynomial system $F \subseteq R[x]$, the CGS and minimal CGS are definition below.

Definition 3 (CGS). *For $F \subseteq R[x]$, a finite set $\mathcal{G} = \{(A_1, G_1), \cdots, (A_l, G_t)\}$ is called a comprehensive Gröbner system for F, if for each $1 \leq i \leq t$, $\sigma_{\bar{a}}(G_i)$ is a Gröbner basis for $\langle \sigma_{\bar{a}}(F) \rangle$ in $L[x]$, and for each $g \in G_i$, $\sigma_{\bar{a}}(\mathrm{lc}_x(g)) \neq 0$ for any $\bar{a} \in A_i$, where each A_i is an algebraically constructible set such that $L^m = A_1 \cup \cdots \cup A_t$ and $A_i \cap A_j \neq \emptyset$ for $i \neq j$, and $G_i \subseteq R[x]$.*

A comprehensive Gröbner system $\mathcal{G} = \{(A_1, G_1), \cdots, (A_l, G_l)\}$ for F is said to be **minimal** if for each $1 \leq i \leq t$, $\sigma_{\bar{a}}(G_i)$ is a minimal Gröbner basi s of the ideal $\langle \sigma_{\bar{a}}(F) \rangle \subseteq L[X]$ for $\bar{a} \in A_i$.

For a subset E of $R = k[u]$, the variety definition by E in L^m is the set of all common zeros of the polynomials in E, denoted by $V(E)$. Here, the algebraically constructible set A_i always has the following form:

$$A_i = V(E_i) \setminus V(N_i) = \{\bar{a} \in L^m \mid \bar{a} \in V(E_i), \bar{a} \notin V(N_i)\}.$$

We also call A_i the constraint.

3 Checking Linear Dependence of Parametric Vectors

In this section, we propose a new method to solve the dependency problem, which is based on a minimal comprehensive Gröbner system.

In the following, we will propose a new method to solve the dependency problem for a finite set of parametric vectors $\mathbf{F} = \{\mathbf{f}_1, \cdots, \mathbf{f}_s\} \subseteq R^l$.

Consider the map $\varphi : k^l \to k[z]$, for any for any $\mathbf{f} = (a_1, \cdots, a_l) \in k^l$,

$$\varphi(\mathbf{f}) = a_1 z_1 + \cdots + a_l z_l,$$

where k is a field, and $z = \{z_1, \cdots, z_l\}$ are distinct variables.

In this paper, we consider the parametric vectors. The map can naturally extend to $k[u]$. That is, $\varphi : R^l = (k[u])^l \to k[u][z]$, for any $\mathbf{f} = (a_1, \cdots, a_l) \in R^l$,

$$\varphi(\mathbf{f}) = a_1 z_1 + \cdots + a_l z_l,$$

where $z = \{z_1, \cdots, z_l\}$ are new variables different from $u = \{u_1, \cdots, u_m\}$.

Note that the degree of $\varphi(\mathbf{f})$ w.r.t each variable z_i is one, and $\sigma_{\bar{a}}(\varphi(\mathbf{f}))$ is a linear polynomial in $L[z_1, \cdots, z_l]$ for each $\bar{a} \in L^l$.

Let $\varphi(\mathbf{F}) = \{\varphi(\mathbf{f}_1), \cdots, \varphi(\mathbf{f}_s)\}$. Note that $\varphi(\mathbf{F})$ is a parametric linear system. For the parametric linear systems, Sit W Y has given an algorithm to solve them. For more, please see [15].

Let $\{(A_1, G_1), \cdots, (A_t, G_t)\}$ be a minimal comprehensive Gröbner system for $\varphi(\mathbf{F})$ w.r.t. any given term order on z. It is easy to check that each G_i is a set of liner polynomials in $z_1, \cdots, z_l$ with coefficients in R, and the number of polynomials in G_i is less or equal to s for $1 \le i \le t$.

In the following, we will give a result for the minimal comprehensive Gröbner system without proof.

Theorem 1. *Let $\mathbf{F} = \{\mathbf{f}_1, \cdots, \mathbf{f}_s\}$ be a subset of R^l, and $\mathcal{G} = \{(A_1, G_1), \cdots, (A_t, G_t)\}$ be a minimal comprehensive Gröbner system for $\varphi(\mathbf{F}) \subset k[u][z]$ w.r.t. any term order in z, where $\varphi(\mathbf{F}) = \{\varphi(\mathbf{f}_1), \cdots, \varphi(\mathbf{f}_s)\}$. Then for each $\bar{a} \in A_i$, $\sigma_{\bar{a}}(\mathbf{f}_1), \cdots, \sigma_{\bar{a}}(\mathbf{f}_s)$ are linearly independent over L if and only if the number of polynomials in G_i is exactly s.*

Theorem 1 provides a simple way to compute two subsets A and B such that $L^m = A \cup B$, and for each $\bar{a} \in A$, $\sigma_{\bar{a}}(\mathbf{f}_1), \cdots, \sigma_{\bar{a}}(\mathbf{f}_s)$ are linearly independent over L; for each $\bar{a} \in B$, $\sigma_{\bar{a}}(\mathbf{f}_1), \cdots, \sigma_{\bar{a}}(\mathbf{f}_s)$ are linearly dependent over L. If $\mathcal{G} = \{(A_1, G_1), \cdots, (A_t, G_t)\}$ is a minimal comprehensive Groebner system for $\varphi(\mathbf{F}) \subset k[u][z]$, where $\mathcal{G}$ can be obtained by any existing algorithms for computing comprehensive Groebner systems. Then we have

$$A = \bigcup_{|G_i|=s} A_i, \text{ and } B = \bigcup_{|G_i|<s} A_i,$$

where $|G_i|$ is the number of polynomials in G_i.

To solve the dependency problem for $\mathbf{F}$ completely, we also need to compute the maximal linearly independent subset for the dependent part B. We should compute a finite of disjoint subsets $B_1, \cdots, B_t$ and $\mathbf{M}_1, \cdots, \mathbf{M}_t \subseteq \mathbf{F}$ such that $B = B_1 \cup B_2 \cup \cdots \cup B_t$, and for each dependent case B_i, $\mathbf{M}_i$ is the maximal linearly dependent subset for $\mathbf{F}$, and every vector $\mathbf{f}$ in $\mathbf{F} \setminus \mathbf{M}_i$ can be expressed in a linear combination of $\mathbf{M}_i$ after specialization.

For this purpose, we need to compute a minimal Gröbner system for

$$\{\varphi(\mathbf{f}_1) + e_1, \cdots, \varphi(\mathbf{f}_s) + e_s\} \subseteq R[z, e],$$

w.r.t. a block order $\succ := (\succ_z, \succ_e)$, where $e = \{e_1, \cdots, e_s\}$ are new variables different from u and z, $\succ_z$ and $\succ_e$ are two term orders with $z_1 > z_2 > \cdots > z_l$ and $e_1 > e_2 > \cdots > e_s$ respectively.

Here, we say a term order $\succ := (\succ_z, \succ_e)$ is a block order, if $z^{\alpha_1} e^{\beta_1} \succ z^{\alpha_2} e^{\beta_2}$ if and only if $z^{\alpha_1} \succ_z z^{\alpha_2}$, or $z^{\alpha_1} = z^{\alpha_2}$ and $e^{\beta_1} \succ_e e^{\beta_2}$.

The variables $e_1, \cdots, e_s$ are used to record the computation process of the minimal comprehensive Gröbner system. This trick has been used in many ways (such as computing syzygies) in most algebraic computer textbooks, for example [12]. The trick can also help us to compute the maximal linearly independent subset, and the expression of the linear combinations of the maximal linearly independent subset for the remaining vectors after specialization. Thus, we give the following theorem. We use the notation $|G|$ to be the number of the elements in G.

Theorem 2. *Let* $\mathbf{F} = \{\mathbf{f}_1, \cdots, \mathbf{f}_s\}$ *be a subset of* R^l, *and* $\mathcal{G} = \{(A_1, G_1), \cdots, (A_t, G_t)\}$ *be a minimal comprehensive Gröbner system for*

$$\{\varphi(\mathbf{f}_1) + e_1, \cdots, \varphi(\mathbf{f}_s) + e_s\} \subseteq R[z, e]$$

w.r.t. a block order $\succ := (\succ_z, \succ_e)$, *where* $\succ_z$ *and* $\succ_e$ *are two any term orders with* $z_1 > z_2 > \cdots > z_l$ *and* $e_1 > e_2 > \cdots > e_s$ *respectively,* $R = k[u]$, $u = \{u_1, \cdots, u_m\}$, $e = \{e_1, \cdots, e_s\}$ *and* $z = \{z_1, \cdots, z_l\}$.

For each $1 \le i \le t$, *suppose that* $G_i' = G_i \cap R[e]$ *and* $G_i{}'' = G_i \setminus G_i'$.

Then $|G_i| = |G_i'| + |G_i''| = s$, *and if* G_i' *is not empty, the polynomials of* G_i' *have the following form:*

$$g_1 = a_{1i_1} e_{i_1} + \sum_{j=i_1+1}^{s} a_{1j} e_j,$$

$$g_2 = a_{2i_2} e_{i_2} + \sum_{j=i_2+1}^{s} a_{2j} e_j,$$

$$\cdots$$

$$g_r = a_{ri_r} e_{i_r} + \sum_{j=i_r+1}^{s} a_{rj} e_j,$$

with $1 \le i_1 < i_2 < \cdots < i_r \le s$, *where* $r = |G_i'|$ *and* $a_{ij} \in R$. *Furthermore,* *(i) if $|G_i''| = s$, then for each $\bar{a} \in A_i$, $\sigma_{\bar{a}}(\mathbf{f}_1), \cdots, \sigma_{\bar{a}}(\mathbf{f}_s)$ are linearly independent over L.*

(ii) if $|G_i''| < s$, then for each $\bar{a} \in A_i$, $\sigma_{\bar{a}}(\mathbf{f}_1), \cdots, \sigma_{\bar{a}}(\mathbf{f}_s)$ are linearly dependent over L, and for each $g_k \in G_i'$, $p_1, \cdots, p_s$ are the corresponding coefficients, where p_i is the coefficient of e_i in g_k. Moreover, $\mathbf{M} = \{\mathbf{f}_i \mid \mathbf{f}_i \in \mathbf{F}, i \notin \{i_1, \cdots, i_r\}\}$ the maximal linearly independent subset for $\mathbf{F}$, and every parametric vector of $\mathbf{F} \setminus \mathbf{M}$ can be expressed in the linear combination of $\mathbf{M}$ over L after specialization.

Theorem 2 can be proved by Theorem 1 and the properties of the minimal Groebner system, and here we omit the proof.

4 A Complete Example

In this section, we will use a complete example to show how to apply Theorem 2 to solve the dependency problem.

Example 1. Let $\mathbf{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} \subset \mathbb{Q}[a, b]^3$, where

$$\mathbf{f}_1 = (a, b, 0), \mathbf{f}_2 = (0, b+1, b), \text{ and } \mathbf{f}_3 = (a+1, 0, -1)$$

are three parametric vectors with the parameters a and b, and $\succ_u$ be the graded reverse lexicographic orders with $a > b$. Here the coefficient field $R = \mathbb{Q}[a, b]$. Next, We use the new method provided by Theorem 2 to solve the dependency problem for $\mathbf{F}$.

To solve the dependency problem for $\mathbf{F}$, we first construct a specific parametric polynomial system using the map φ. Let $f_i = \varphi(\mathbf{f}_i) + e_i$ for $i = 1, 2, 3$. Then we obtain a parametric polynomial system:

$$F = \{f_1 = az_1 + bz_2 + e_1, f_2 = (b+1)z_2 + bz_3 + e_2, f_3 = (a+1)z_1 - z_3 + e_3\} \subseteq \mathbb{Q}[a, b][z, e].$$

Using the algorithm proposed in [6], we can get a minimal comprehensive Gröbner system $\mathcal{G}$ for F w.r.t. a block order $(\succ_z, \succ_e)$, where $\succ_z$ and $\succ_e$ are two lexicographical order with $z_1 > z_2 > z_3$ and $e_1 > e_2 > e_3$ respectively. Here, $\mathcal{G} = \{(A_1, G_1), (A_2, G_2), (A_3, G_3), (A_4, G_4)\}$, where

$$A_1 = L^3 \setminus V((ab^2 - ab + b^2 - a)(b+1)),$$

$$G_1 = \{z_1 + z_2 + (b-1)z_3 - e_1 + e_2 + e_3, (b+1)z_2 + bz_3 + e_2,$$
$$(ab^2 - ab + b^2 - a)z_3 - (ab + a + b + 1)e_1 + (ab + b)e_2 + (ab + a)e_3\},$$

$$A_2 = V(ab^2 - ab + b^2 - a) \setminus V(b+1),$$

$$G_2 = \{z_1 - bz_2 - z_3 - e_1 + e_3, (b+1)z_2 + bz_3 + e_2, (b+1)e_1$$
$$- (ab^2 - ab + b^2 - a + b)e_2 + (ab^2 - ab - a)e_3\},$$

$$A_3 = V(b+1) \setminus V(a+1),$$

$$G_3 = \{z_1 + z_2 - 2z_3 - e_1 + e_2 + e_3, (a+1)z_2 - (a+1)e_1 - ae_2 + ae_3, z_3 - e_2\}.$$

$$A_4 = V(a+1, b+1),$$

$$G_4 = \{z_1 + z_2 - z_3 - e_1 + e_3, z_3 - e_2, e_2 - e_3\},$$

For the minimal comprehensive Gröbner system $\mathcal{G}$, the polynomials in G_i are all linear in z and e with coefficient in $\mathbb{Q}[a, b]$, and the number of polynomials in G_i is exactly 3 for $i = 1, 2, 3$.

By Theorem 2, we get the following results.

(1) For A_1 and A_3, note that

$$G_1' = G_1 \cap R[e] = \emptyset, G_1'' = G_1 \setminus G_1' = G_1,$$

and

$$G_3' = G_3 \cap R[e] = \emptyset, G_3'' = G_1 \setminus G_3' = G_3.$$

We have $|G_1''| = |G_3''| = 3$. Thus, by Theorem 2, $\sigma_{\bar{a}}(\mathbf{f}_1), \sigma_{\bar{a}}(\mathbf{f}_2), \sigma_{\bar{a}}(\mathbf{f}_3)$ are linearly independent over L for each $\bar{a} \in A_1 \cup A_3$.

(2) For case A_2, we have

$$G_2' = G_2 \cap R[e] = \{z_1 - bz_2 - z_3 - e_1 + e_3, (b+1)z_2 + bz_3 + e_2\}$$

and

$$G_2'' = G_2 \setminus G_2' = \{(b+1)e_1 - (ab^2 - ab + b^2 - a + b)e_2 + (ab^2 - ab - a)e_3\}.$$

Note that $|G_2'| < 3$. Then for each $\bar{a} \in A_2$, $\sigma_{\bar{a}}(\mathbf{f}_1), \sigma_{\bar{a}}(\mathbf{f}_2), \sigma_{\bar{a}}(\mathbf{f}_3)$ are linearly dependent over L. Moreover, from the polynomial G_2'', the maximal linearly independent subset is $\mathbf{M}_2 = \{\mathbf{f}_2, \mathbf{f}_3\}$, and we get a linear combination for $\mathbf{f}_1$ after specialization:

$$\sigma_{\bar{a}}(\mathbf{f}_1) = \sigma_{\bar{a}}(b+1)^{-1}(\sigma_{\bar{a}}(ab^2 - ab + b^2 - a + b)\sigma_{\bar{a}}(\mathbf{f}_2) - \sigma_{\bar{a}}(ab^2 - ab - a)\sigma_{\bar{a}}(\mathbf{f}_3)).$$

And $(b+1)$, $-(ab^2 - ab + b^2 - a + b)$ and $(ab^2 - ab - a)$ are the corresponding coefficients for $\mathbf{F}$, such that

$$\sigma_{\bar{a}}((b+1)\mathbf{f}_1 - (ab^2 - ab + b^2 - a + b)\mathbf{f}_2 + (ab^2 - ab - a)\mathbf{f}_3) = (0, 0, 0).$$

(3) For case A_4, we have

$$G'_4 = G_4 \cap R[e] = \{z_1 + z_2 - z_3 - e_1 + e_3, z_3 - e_2\},$$

and $G''_4 = G_4 \setminus G'_4 = \{e_2 - e_3\}$.

Note that $|G'_4| = 2 < 3$. Then for each $\bar{a} \in A_4$, $\sigma_{\bar{a}}(\mathbf{f}_1), \sigma_{\bar{a}}(\mathbf{f}_2), \sigma_{\bar{a}}(\mathbf{f}_3)$ are linearly dependent over L. From G''_4, the maximal linearly independent subset is $\mathbf{M}_4 = \{\mathbf{f}_1, \mathbf{f}_3\}$, and we get a linear combination for $\mathbf{f}_1$ after specialization: $\sigma_{\bar{a}}(\mathbf{f}_2) = \sigma_{\bar{a}}(\mathbf{f}_3)$. And 0, 1 and -1 are corresponding coefficients such that $\sigma_{\bar{a}}(\mathbf{f}_2 - \mathbf{f}_3) = (0, 0, 0)$.

Note that $L^3 = A_1 \cup A_2 \cup A_3 \cup A_4$, we have solved the dependency problem for $\mathbf{F}$.

Acknowledgements. This work was supported in part by the Chinese Universities Scientific Fund (Grant No. 2017QC061), and the National Natural Science Foundation of China (Grant No. 11371356).

References

1. Greub, W.H.: Linear Algebra, 3rd edn. Springer, New York (1967)
2. Buchberger, B.: Gröbner-bases: an algorithmic method in polynomial ideal theory. In: Multidimensional Systems Theory - Progress, Directions and Open Problems in Multidimensional Systems, pp. 184–232. Reidel Publishing Company, Dodrecht-Boston-Lancaster 1985
3. Cox, D., Little, J., O'Shea, D.: Using Algebraic Geometry, 2nd edn. Springer, New York (2005)
4. Weispfenning, V.: Comprehensive Groebner bases. J. Symb. Comp. **14**, 1–29 (1992)
5. Kapur, D.: An approach for solving systems of parametric polynomial equations. In: Saraswat, V., Van Hentenryck, P. (eds.) Principles and Practice of Constraint Programming. MIT Press, Cambridge (1995)
6. Kapur, D., Sun, Y., Wang, D.K.: A new algorithm for computing comprehensive Groebner systems. In: Proceedings of ISSAC 2010. ACM Press, New York (2010)
7. Kapur, D., Sun, Y., Wang, D.K.: Computing comprehensive Groebner systems and comprehensive Groebner bases simultaneously. In: Proceedings of ISSAC 2011. ACM Press, New York (2011)
8. Manubens, M., Montes, A.: Minimal canonical comprehensive Groebner system. J. Symb. Comp. **44**, 463–478 (2009)
9. Nabeshima, K.: A speed-up of the algorithm for computing comprehensive Groebner systems. In: Proceedings of ISSAC 2007. ACM Press, New York (2007)
10. Suzuki, A., Sato, Y.: An alternative approach to comprehensive Groebner bases. J. Symb. Comp. **36**, 649–667 (2003)
11. Suzuki, A., Sato, Y.: A simple algorithm to compute comprehensive Groebner bases using Groebner bases. In: Proceedings of ISSAC 2006. ACM Press, New York (2006)
12. Chen, X., Li, P., Lin, L., Wang, D.: Proving geometric theorems by partitioned-parametric Gröbner bases. In: Hong, H., Wang, D. (eds.) ADG 2004. LNCS, vol. 3763, pp. 34–43. Springer, Heidelberg (2006). doi:10.1007/11615798_3

13. Montes, A.: A new algorithm for discussing Groebner basis with parameters. J. Symb. Comp. **33**, 183–208 (2002)
14. Montes, A., Recio, T.: Automatic discovery of geometry theorems using minimal canonical comprehensive Gröbner systems. In: Botana, F., Recio, T. (eds.) ADG 2006. LNCS, vol. 4869, pp. 113–138. Springer, Heidelberg (2007). doi:10.1007/978-3-540-77356-6_8
15. Sit, W.Y.: An algorithm for solving parametric linear systems. J. Symb. Comp. **13**, 353–394 (1992)