Extending the GVW Algorithm to Local Ring

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ABSTRACT

A new algorithm, which combines the GVW algorithm with the Mora normal form algorithm, is presented to compute the standard bases of ideals in a local ring. Since term orders in local ring are not well-orderings, there may not be a minimal signature in an infinite set, and we can not extend the GVW algorithm from a polynomial ring to a local ring directly. Nevertheless, when given an anti-graded order in R and a term-over-position order in R^m that are compatible, we can construct a special set such that it has a minimal signature, where R, R^m are a local ring and a R-module, respectively. That is, for any given polynomial $v_0 \in R$, the set consisting of signatures of pairs $(\mathbf{u}, v) \in R^m \times R$ has a minimal element, where the leading power products of v and v_0 are equal. In this case, we prove a cover theorem in R, and use three criteria (syzygy criterion, signature criterion and rewrite criterion) to discard useless J-pairs without any reductions. Mora normal form algorithm is also extended to do regular top-reductions in $R^m \times R$, and the correctness and termination of the algorithm are proved. The proposed algorithm has been implemented in the computer algebra system Maple, and experiment results show that most of J-pairs can be discarded by three criteria in the examples.

CCS CONCEPTS

• **Computing methodologies → Symbolic and algebraic algorithms**; **Algebraic algorithms**;

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1 INTRODUCTION

The Gröbner bases was first presented by Buchberger in 1965 [3]. It is useful for solving polynomial equations, the ideal membership problem and so on. The original algorithm of computing Gröbner bases was proposed by Buchberger, and it has been implemented in most computer algebra systems. Since then, many researchers have done some works to improve the efficiency of the algorithm, such as Buchberger[4], [5], Faugère [12], Gebauer and Möller [16], Giovini et.al. [18], Mora et.al. [21]. One important improvement is that Lazard pointed out the connection betwwen Gröbner basis and linear algebra [20], which will speed up the reduction step. In Buchberger original algorithm, there are many useless S-polynomials which are reduced to zero. The other improvement is deleting these useless S-polynomials without performing any reduction. In 2002, the notation of "signature" and rewriting rules, which can detect many useless S-polynomials, were proposed by Faugère in the F5 algorithm [13]. After that, several variants of F5 have been presented including Arri and Perry [1], Eder and Perry [9, 10], Hashemi and Ars [2], Sun and Wang [23],[24], Gerdt, Hashemi and M.-Alizadeh [17]. Eder et.al. [11] generalized signature-based Gröbner basis algorithms to Euclidean rings, in particular, the integers. They also shown how signature based computation can be efficiently used as a pre-reduction step for a classical Gröbner basis computation over Euclidean rings. There is by now a large literature on signature-based Gröbner basis computation; see [8] for a comprehensive survey.

Gao et. al. presented a new simple theory for computing Gröbner bases. Based on the theory, they proposed an incremental signature-based algorithm G^2V [14], and an extended version GVW algorithm [15]. The correctness and finite termination of the GVW algorithm have been proved.

In algebraic geometry, many questions are related to the local properties of varieties. Such as given a zero-dimensional ideal I in $k[x_1, \ldots, x_n]$, we want to know the multiplicity of an isolated singular point p in the variety $\mathbb{V}(I) \subset k^n$, or the Milnor and Tjurina numbers of the point. The local ring is useful for solving these questions.

As Gröbner bases in polynomial ring, there is a similar notation called standard bases in local ring. Through computing a standard bases G of the ideal I in local ring, the local properties of original point can be got. For other point p, we only need change the coordinates to translate the point p to the origin.

Given a collection $f_1, \ldots, f_s$ of polynomials which generate the ideal I, we would like to find a standard bases of I in a local ring with respect to some semigroup orders. There are two main algorithms to compute the standard bases of the ideal I in the local ring. One is based on the Lazard's homogeneous idea, and the other one is based on the Mora normal form algorithm. Let f^H be the homogenization of f in $k[t, x_1, \ldots, x_n]$. According to Lazard's idea, we only need to compute a Gröbner basis of $\langle f_1^H, \ldots, f_s^H \rangle$ with respect to some special global semigroup orders, then the dehomogenizations of elements of the Gröbner basis is a standard basis of I in the local ring. The other one is combining the Mora normal form algorithm [22] with Buchberger algorithm to compute the standard basis. The algorithm has been implemented in Singular and REDUCE, but not in Maple or Mathematica. The experience seems to indicate that standard bases computation with Mora's normal form algorithm is more efficient than computations using Lazard's method (quote from [6]).

Since the GVW algorithm is more efficient than the Buchberger algorithm for computing Gröbner bases, it is asked naturally whether the GVW algorithm can be used to compute the standard bases instead of Buchberger algorithm. The answer is yes. In this paper, we will combining the Mora normal form algorithm with GVW algorithm to compute the standard bases in the local ring. What's more, we have implemented the idea in the Maple.

The paper is organized as follows. Some basic notations about local ring, signature, and strong standard basis are introduced in the section 2. In section 3, we present the GVW algorithm in local ring. The correctness and finite termination of the algorithm are proved in this section. An example is given for illustrating our method in the section 4. We conclude this paper in the last section.

2 PRELIMINARIES

In this section, we first review some basic definitions about local ring. The details can refer to [6]. Then we give the definition of strong standard bases in local ring, which is similar to strong Gröbner bases [15] in polynomial ring. Finally, we propose the term orders that we should consider in this paper.

2.1 Local Ring

Let X be the n variables $x_1, \ldots, x_n$; $k[X]$ be the polynomial ring in variables X with coefficients in a field k; $\{X^\alpha : \alpha \in \mathbb{Z}^n_{\geq 0}\}$ be the set of monomials in $k[X]$.

Definition 2.1 (Semigroup Order). An order $>$ on $\mathbb{Z}^n_{\geq 0}$ or, equivalently, on $\{X^\alpha : \alpha \in \mathbb{Z}^n_{\geq 0}\}$, is said to be a *semigroup order* if it satisfies:

(1) $>$ is a total order on $\mathbb{Z}^n_{\geq 0}$;
(2) $>$ is compatible with multiplication of monomials.

As in Definition 2.1, being a total order means that for any $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, exactly one of the following is true:

$$X^\alpha > X^\beta, \ X^\alpha = X^\beta, \ \text{or} \ X^\alpha < X^\beta.$$

Compatibility with multiplication means that for any X^γ in $\{X^\alpha : \alpha \in \mathbb{Z}^n_{\geq 0}\}$, if $X^\alpha > X^\beta$, then $X^\alpha X^\gamma > X^\beta X^\gamma$.

For any $\alpha \neq (0, \ldots, 0)$, if $X^\alpha > 1$, the semigroup order is called *global order*; and if $X^\alpha < 1$, it is called *local order*. For example, the lexicographic order is a global order and the antigraded lexicographic order (abbreviated *alex*) is a local order. The definition of *alex* is as follows.

Definition 2.2. Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say $X^\alpha >_{alex} X^\beta$ if $|\alpha| = \sum_{i=1}^n \alpha_i < |\beta| = \sum_{i=1}^n \beta_i$, or if $|\alpha| = |\beta|$ and $X^\alpha >_{lex} X^\beta$.

Let f be a polynomial in $k[X]$, $>$ be a semigroup order on the monomials in $k[X]$, the leading power product, the leading coefficient of f is denoted by $\mathrm{lpp}(f)$, $\mathrm{lc}(f)$ respectively, and the leading term, $\mathrm{lt}(f) = \mathrm{lc}(f)\mathrm{lpp}(f)$. The localization of $k[X]$ with respect to $>$ is defined as follows.

Definition 2.3 (Localization of Ring). Let $>$ be a semigroup order on monomials in $k[X]$, and let $S = \{1 + g : g = 0 \text{ or } \mathrm{lt}(g) < 1\}$. The *localization* of $k[X]$ w.r.t. $>$ is the ring

$$\mathrm{Loc}_>(k[X]) = \{f/(1+g) : f, g \in k[X] \text{ and } 1 + g \in S\}.$$

Notes that, if $>$ is a global order, $\mathrm{Loc}_>(k[X]) = k[X]$. On the other hand, if $>$ is a local order, $\mathrm{Loc}_>(k[X]) = k[X]_{\langle x_1, \ldots, x_n \rangle}$. For briefly, we denote $\mathrm{Loc}_>(k[X])$ by R w.r.t. a local order $>$ in the following.

The semigroup order $>$ on the monomials in $k[X]$ can be naturally extended to R. For any $h = f/(1+g) \in R$, the leading power product, the leading coefficient, and the leading term of h are defined to be same as those of f, that is, $\mathrm{lpp}(h) = \mathrm{lpp}(f)$, $\mathrm{lc}(h) = \mathrm{lc}(f)$, and $\mathrm{lt}(h) = \mathrm{lt}(f)$.

For any $h_1, \ldots, h_m$ in R, an ideal $I \subset R$ generated by them is $I = \langle h_1, \ldots, h_m \rangle = \{\sum_{i=1}^m u_i h_i : \forall u_1, \ldots, u_m \in R\}$. The n-tuple $(u_1, \ldots, u_m)$ is called a *syzygy* of $\{h_1, \ldots, h_m\}$, if $\sum_{i=1}^m u_i h_i = 0$.

Definition 2.4 (Standard basis). Let $>$ be a semigroup order on the monomials in $k[X]$, and I be an ideal in R. A *standard basis* of I is a set $\{g_1, \ldots, g_s\}$ in I such that $\langle \mathrm{lt}(I) \rangle = \langle \mathrm{lt}(g_1), \ldots, \mathrm{lt}(g_s) \rangle$.

Since k is a field, the set $\{g_1, \ldots, g_s\}$ is a standard basis of I if and only if $\langle \mathrm{lpp}(I) \rangle = \langle \mathrm{lpp}(g_1), \ldots, \mathrm{lpp}(g_s) \rangle$. If $>$ is a global order, the standard basis is exactly the Gröbner basis. So the standard bases in R is more extensive than Gröbner bases.

In order to compute a standard basis of $I \subset R$ w.r.t. a local order, we need to develop an extension of the division algorithm in $k[X]$

which will yield information about ideals in R. Since we deal with orders that are not well-orderings, the difficult part is to give a division process that is guaranteed to terminate. We can evade this difficulty with a splendid idea of Mora, and obtain the Mora normal form algorithm in R.

Corollary 2.5 (Mora Normal Form Algorithm). *Let $\succ$ be a semigroup order on the monomials in $k[X]$, $g \in R$ and $g_1, \dots, g_s \in k[X]$ be nonzero. Then there is an algorithm for producing polynomials $a_1, \dots, a_s, h \in R$ such that $g = a_1 g_1 + \cdots + a_s g_s + h$, where $\mathrm{lpp}(a_i)\mathrm{lpp}(g_i) \preceq \mathrm{lpp}(g)$ for all i with $a_i \neq 0$, and either $h = 0$, or $\mathrm{lpp}(h) \preceq \mathrm{lpp}(g)$ and $\mathrm{lpp}(h)$ is not divisible by any of $\mathrm{lpp}(g_1), \dots, \mathrm{lpp}(g_s)$.*

Remark 1. Based on Mora's research, Greuel and Pfister in [19] obtained a normal form algorithm in R^m, when they studied the standard bases for modules. That is, for any given module order $\prec'$ in R^m, $\mathbf{u} \in R^m$ and $\mathbf{u}_1, \dots, \mathbf{u}_s \in (k[X])^m$ are nonzero, then there is an algorithm for producing polynomials $b_1, \dots, b_s \in R$ and $\mathbf{r} \in R^m$ such that $\mathbf{u} = b_1\mathbf{u}_1 + \cdots + b_s\mathbf{u}_s + \mathbf{r}$, where $\mathrm{lpp}(b_i)\mathrm{lpp}(\mathbf{u}_i) \preceq' \mathrm{lpp}(\mathbf{u})$ for all i with $b_i \neq 0$, and either $\mathbf{r} = \mathbf{0}$, or $\mathrm{lpp}(\mathbf{r})$ is not divisible by any $\mathrm{lpp}(\mathbf{u}_i)$, $i = 1, \dots, s$.

2.2 Strong Standard Basis

By analogy with the notation of strong Gröbner bases [15] in $(k[X])^m \times k[X]$, we will define the strong standard bases in $R^m \times R$ w.r.t a local order $\succ$. Cox et.al. [6] proved that every ideal $I \subset k[X]_{\langle x_1, \dots, x_n \rangle}$ has a generating set consisting of polynomials in $k[X]$. By the above fact, restricting to ideals generated by polynomials in this paper entails loss of generality when we are studying ideals in $R = k[X]_{\langle x_1, \dots, x_n \rangle}$ for a local order $\succ$.

In this paper, elements in R^m are denoted by the bold letters $\mathbf{f}, \mathbf{u}$ etc., while elements in R are denoted by the letters v, r etc. Let $\mathbf{f} = (f_1, \dots, f_m)$ in $(k[X])^m$, we can define a subset in $R^m \times R$:

$$M = \{(\mathbf{u}, v) \in R^m \times R : \mathbf{u} \cdot \mathbf{f} = v, \mathbf{u} \in R^m\},$$

For any $(\mathbf{u}_1, v_1), (\mathbf{u}_2, v_2) \in M$, and $r \in R$, since $r\mathbf{u}_1 \cdot \mathbf{f} + \mathbf{u}_2 \cdot \mathbf{f} = (r\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{f} = rv_1 + v_2$, so $(r\mathbf{u}_1 + \mathbf{u}_2, rv_1 + v_2) \in M$, and M is a R-submodule in $R^m \times R$. It is obvious that M is generated by $(\mathbf{e}_1, f_1), \dots, (\mathbf{e}_m, f_m)$, where $\mathbf{e}_i$ is the i-th unit vector of R^m, i.e., $(\mathbf{e}_i)_j = \delta_{ij}$, δ_{ij} is the Kronecker delta. We say $X^\alpha \mathbf{e}_i$ *divides* $X^\beta \mathbf{e}_j$ if X^α divides X^β and $i = j$.

Fix any local order $\prec_1$ in R, and any module order $\prec_2$ in R^m. For any element $v \in R$, the leading power product, the leading coefficient of v w.r.t. $\prec_1$ is denoted by $\mathrm{lpp}_{\prec_1}(v)$, $\mathrm{lc}_{\prec_1}(v)$ respectively. Similarly, any element $\mathbf{u} \in R^m$, the leading power product, the leading coefficient of $\mathbf{u}$ w.r.t. $\prec_2$ is denoted by $\mathrm{lpp}_{\prec_2}(\mathbf{u})$, $\mathrm{lc}_{\prec_2}(\mathbf{u})$ respectively. For convenient, we denote them by $\mathrm{lpp}(v)$, $\mathrm{lc}(v)$, $\mathrm{lpp}(\mathbf{u})$, $\mathrm{lc}(\mathbf{u})$ with no confusion. For any $p = (\mathbf{u}, v)$ in M, the $\mathrm{lpp}(\mathbf{u})$ is called the *signature* of p.

We say $\prec_2$ is *compatible* with $\prec_1$, if it satisfies that: $X^\alpha \prec_1 X^\beta$ if and only if $X^\alpha \mathbf{e}_i \prec_2 X^\beta \mathbf{e}_i$ for all $i = 1 \dots m$.

Definition 2.6 (Top-reducible). Let $p_1 = (\mathbf{u}_1, v_1)$, $p_2 = (\mathbf{u}_2, v_2)$ be two elements in M. We say p_1 is *top-reducible* by p_2, if it satisfies:

(1) when $v_2 = 0$, $\mathrm{lpp}(\mathbf{u}_2)$ divides $\mathrm{lpp}(\mathbf{u}_1)$; and
(2) when $v_1 v_2 \neq 0$, $\mathrm{lpp}(v_2)$ divides $\mathrm{lpp}(v_1)$ and $t\mathrm{lpp}(\mathbf{u}_2) \preceq_2 \mathrm{lpp}(\mathbf{u}_1)$, where $t = \mathrm{lpp}(v_1)/\mathrm{lpp}(v_2)$.

When $v_1 v_2 \neq 0$, the corresponding one-step top-reduction is

$$\mathrm{OneRed}(p_1, p_2) = p_1 - ctp_2 = (\mathbf{u}_1 - ct\mathbf{u}_2, v_1 - ctv_2),$$

where $c = \mathrm{lc}(v_1)/\mathrm{lc}(v_2)$. Such a top-reduction is called *regular* if $\mathrm{lpp}(\mathbf{u}_1 - ct\mathbf{u}_2) = \mathrm{lpp}(\mathbf{u}_1)$, and *super* otherwise. When v_1 is zero, the corresponding top-reduction is always called *super*. Let G be any set of pairs in $R^m \times R$, we call a pair $(\mathbf{u}, v)$ *eventually super top-reducible* by G if there is a sequence of regular top-reductions by pairs in G that reduce $(\mathbf{u}, v)$ to a pair $(\hat{\mathbf{u}}, \hat{v})$ that is no longer regular top-reducible by G but is super top-reducible by at least one pair in G.

Definition 2.7 (Strong standard bases). Let $G = \{(\mathbf{u}_1, v_1), \dots, (\mathbf{u}_s, v_s)\}$ be a finite subset of M, where $\mathbf{u}_1, \dots, \mathbf{u}_s \in (k[X])^m$ and $v_1, \dots, v_s \in k[X]$. Then G is called a *strong standard basis* for M, if for any nonzero $(\mathbf{u}, v)$ in M, $(\mathbf{u}, v)$ is top-reducible by some element in G.

In Gao et. al. [15], the authors have proved that if $G = \{(\mathbf{u}_1, v_1), \dots, (\mathbf{u}_s, v_s)\}$ is a strong standard basis for M, then $\{v_i : 1 \leqslant i \leqslant s\}$ is a Gröbner basis for $I = \langle f_1, \dots, f_m \rangle$ in $k[X]$ w.r.t. a global order. In their proof, they can select a minimal $\mathrm{lpp}(\mathbf{u})$ such that $\mathbf{u} \cdot \mathbf{f} = v$, since the monomials order in $k[X]$ satisfies the well-ordering relation. However, local orders are not well-orderings, and we can not get a minimal $\mathrm{lpp}(\mathbf{u})$. Therefore, we need a new method to solve this problem.

Proposition 2.8. *Let $\prec_1$ be an arbitrary local order in R and $\prec_2$ be a module order in R^m. Suppose that $G = \{(\mathbf{u}_1, v_1), \dots, (\mathbf{u}_s, v_s)\}$ is a strong standard basis for M, then*

(1) *$\mathrm{G}_0 = \{\mathbf{u}_i : v_i = 0, 1 \leqslant i \leqslant s\}$ is a standard basis for the syzygy module of $\{f_1, \dots, f_m\}$, and*
(2) *$G_1 = \{v_i : 1 \leqslant i \leqslant s\}$ is a standard basis for ideal $I = \langle f_1, \dots, f_m \rangle$ in R.*

Proof. Since the proof of (1) is same as the proposition 2.2 in Gao et. al. [15], we only prove the second assertion.

Without loss of generality, let $\mathbf{G}_0 = \{\mathbf{u}_i : v_i = 0, 1 \leqslant i \leqslant k\}$ and $G_1 = \{v_i : k+1 \leqslant i \leqslant s\}$, where $1 \leq k < s$. We select $v \in I$ such that $v \neq 0$. Then there exists $\mathbf{u} \in R^m$ so that $\mathbf{u} \cdot \mathbf{f} = v$. Remark 1 implies that there exist $a_1, \dots, a_k \in R$ and $\mathbf{h} \in R^m$ such that $\mathbf{u} = a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k + \mathbf{h}$, where $\mathrm{lpp}(a_i)\mathrm{lpp}(\mathbf{u}_i) \preceq_2 \mathrm{lpp}(\mathbf{u})$ for all i with $a_i \neq 0$, and either $\mathbf{h} = \mathbf{0}$, or $\mathbf{h} \notin \langle \mathbf{G}_0 \rangle$. It follows from $v \neq 0$ that $\mathbf{h} \neq 0$. Hence, $(\mathbf{u}, v) \in M$ can be top-reducible to $(\mathbf{h}, v)$ by $\{(\mathbf{u}_i, 0) : \mathbf{u}_i \in \mathbf{G}_0\}$. Since $(\mathbf{h}, v) \in M$ and $\mathbf{h} \notin \langle \mathbf{G}_0 \rangle$, it can be top-reducible by some $(\mathbf{u}_i, v_i) \in G$ with $v_i \in G_1$. So $v_i \neq 0$ and $\mathrm{lpp}(v_i)$ divides $\mathrm{lpp}(v)$. Hence G_1 is a standard basis for I. □

2.3 Term Orders

In the following, we consider a local order $\prec_1$ in R and a module order $\prec_2$ in R^m. For any $\prec_1$, there are many ways that we can extend $\prec_1$ to $\prec_2$. For example, we get $\prec_2$ as follows.

(1) Position Over Term (POT). We say that $X^\beta \mathbf{e}_j \prec_2 X^\alpha \mathbf{e}_i$ if $j > i$, or if $j = i$ and $X^\beta \prec_1 X^\alpha$.
(2) Term Over Position (TOP). We say that $X^\beta \mathbf{e}_j \prec_2 X^\alpha \mathbf{e}_i$ if $X^\beta \prec_1 X^\alpha$, or if $X^\beta = X^\alpha$ and $j > i$.
(3) $\mathbf{f}$-weighted anti-degree followed by TOP. We say that $X^\beta \mathbf{e}_j \prec_2 X^\alpha \mathbf{e}_i$ if $\mathrm{tdeg}(X^\beta f_j) > \mathrm{tdeg}(X^\alpha f_i)$, or if $\mathrm{tdeg}(X^\beta f_j) =$

$\mathrm{tdeg}(X^{\alpha} f_i)$ and $X^{\beta}\mathbf{e}_j \prec_{TOP} X^{\alpha}\mathbf{e}_i$, where tdeg is for total degree.

(4) $\mathbf{f}$-weighted $\prec_1$ followed by POT. We say that $X^{\beta}\mathbf{e}_j \prec_2 X^{\alpha}\mathbf{e}_i$ if $\mathrm{lpp}(X^{\beta} f_j) \prec_1 \mathrm{lpp}(X^{\alpha} f_i)$, or if $\mathrm{lpp}(X^{\beta} f_j) = \mathrm{lpp}\,(X^{\alpha} f_i)$ and $X^{\beta}\mathbf{e}_j \prec_{POT} X^{\alpha}\mathbf{e}_i$.

For any $(\mathbf{u}_0, v_0) \in M$, we consider the set

$$L(\mathrm{lpp}(v_0)) = \{\mathrm{lpp}(\mathbf{u}) : (\mathbf{u}, v) \in M \text{ and } \mathrm{lpp}(v) = \mathrm{lpp}(v_0)\}.$$

Note that $L(\mathrm{lpp}(v_0))$ is a nonempty set. But, $L(\mathrm{lpp}(v_0))$ may not have a minimal element. For example, let $\prec_1$ be an anti-graded lex order with $x_2 \prec_1 x_1$ on R, and $\prec_2$ be a POT order with $\mathbf{e}_2 = (0, 1) \prec_2 \mathbf{e}_1 = (1, 0)$ on R^2, where $R = k[x_1, x_2]_{\langle x_1, x_2\rangle}$. Consider M generated by $(\mathbf{e}_1, x_1)$ and $(\mathbf{e}_2, x_2)$. Let $p_0 = (\mathbf{u}_0, v_0) = ((x_1, x_1 + 1), x_1^2 + x_1x_2 + x_2)$, then $p_0 \in M$ and $\mathrm{lpp}(\mathbf{u}_0) = x_1\mathbf{e}_1, \mathrm{lpp}(v_0) = x_2$. We can construct $p_i = (\mathbf{u}_i, v_i) = ((x_1^{1+i}, x_1 + 1), x_1^{2+i} + x_1x_2 + x_2)$, where $i \in \mathbb{Z}_{\geq 1}$. Then $p_i \in M$, $\mathrm{lpp}(v_i) = \mathrm{lpp}(v_0)$ and $L(\mathrm{lpp}(v_0)) \supseteq \{x_1^i \mathbf{e}_1 : i \in \mathbb{Z}_{\geq 1}\}$. Obviously, $L(\mathrm{lpp}(v_0))$ has not a minimal element. Moreover, if G is a subset of M and p_0 is not top-reducible by any pair in G, then p_i is also not top-reducible by any pair in G.

Nevertheless, if $\prec_1$ is an anti-graded order in R and $\prec_2$ is a TOP order in R^m, then we can prove that $L(\mathrm{lpp}(v_0))$ has a minimal element.

Lemma 2.9. *Let $\prec_1$ be an anti-graded order in R, and $\prec_2$ be a TOP order in R^m, where $\prec_2$ is compatible with $\prec_1$. Then for any $(\boldsymbol{u}_0, v_0) \in M$, $L(\mathrm{lpp}(v_0))$ has a minimal element.*

Proof. Without loss of generality, we suppose $\mathrm{lpp}(f_1)$ is maximal in $\{\mathrm{lpp}(f_1), \ldots, \mathrm{lpp}(f_m)\}$. For any $(\mathbf{u}, v) \in M$ which satisfies $\mathrm{lpp}(v) = \mathrm{lpp}(v_0)$, we have $u_1 f_1 + \cdots + u_m f_m = v$, where $\mathbf{u} = (u_1, \ldots, u_m)$. Let $\mathrm{lpp}(\mathbf{u}) = \mathrm{lpp}(u_i)\mathbf{e}_i$ for some i, where $1 \leq i \leq m$. Since $\prec_2$ is a TOP order in R^m and is compatible with $\prec_1$, $\mathrm{lpp}(u_i) = \max\{\mathrm{lpp}(u_1), \ldots, \mathrm{lpp}(u_m)\}$. It follows from $v = \sum_{j=1}^{m} u_j f_j$ that there exists some j such that $\mathrm{lpp}(v) = \mathrm{lpp}(v_0) \preceq_1 \mathrm{lpp}(u_j)\mathrm{lpp}(f_j)$, where $1 \leq j \leq m$. Then we have $\mathrm{lpp}(v_0) \preceq_1 \mathrm{lpp}(u_j)\mathrm{lpp}(f_1) \preceq_1 \mathrm{lpp}(u_i)\mathrm{lpp}(f_1)$. Since $\preceq_1$ is an anti-graded order, there are a finite number of $\mathrm{lpp}(u_i)$ for which the inequality $\mathrm{lpp}(v_0) \preceq_1 \mathrm{lpp}(u_i)\mathrm{lpp}(f_1)$ holds. Therefore, $L(\mathrm{lpp}(v_0))$ is a finite set, and has a minimal element. □

Remark 2. If $\prec_2$ is not a TOP order in R^m, then $\mathrm{lpp}(u_i) = \max\{\mathrm{lpp}(u_1), \ldots, \mathrm{lpp}(u_m)\}$ may not hold. Moreover, if $\prec_1$ is not an anti-graded order in R, there may be an infinite number of $\mathrm{lpp}(u_i)$ for which the inequality $\mathrm{lpp}(v_0) \preceq_1 \mathrm{lpp}(u_i)\mathrm{lpp}(f_1)$ holds. In either case, $L(\mathrm{lpp}(v_0))$ may not have a minimal element.

3 THE GVW ALGORITHM IN LOCAL RING

In order to compute the strong standard basis for M, we need to define a concept of J-pair which is similar to S-polynomial in Buchberger's algorithm. Suppose $p_1 = (\mathbf{u}_1, v_1), p_2 = (\mathbf{u}_2, v_2)$ are two pairs in M with $v_1v_2 \neq 0$. Let $t = \mathrm{lcm}(\mathrm{lpp}(v_1), \mathrm{lpp}(v_2))$, $t_1 = t/\mathrm{lpp}(v_1)$, $t_2 = t/\mathrm{lpp}(v_2)$, $c = \mathrm{lc}(v_1)/\mathrm{lc}(v_2)$, and $T = \max\{t_1\mathrm{lpp}(\mathbf{u}_1), t_2\mathrm{lpp}(\mathbf{u}_2)\}$. Without loss of generality, we assume $T = t_1\mathrm{lpp}(\mathbf{u}_1)$. If

$$\mathrm{lpp}(t_1\mathbf{u}_1 - ct_2\mathbf{u}_2) = T,$$

then t_1p_1 is called the *J-pair* of p_1 and p_2, and T is called the *J-signature* of the J-pair. It is obvious that the J-pair t_1p_1 is regular top-reducible by p_2.

We say that a pair $(\mathbf{u}, v) \in M$ is *covered* by $G \subset M$, if there is a pair $(\mathbf{u}_i, v_i) \in G$ such that $\mathrm{lpp}(\mathbf{u}_i)$ divides $\mathrm{lpp}(\mathbf{u})$ and $t\mathrm{lpp}(v_i) \prec_1 \mathrm{lpp}(v)$, where $t = \mathrm{lpp}(\mathbf{u})/\mathrm{lpp}(\mathbf{u}_i)$.

3.1 The Algorithm

The following theorem is the theoretical foundation of the GVW algorithm in local ring.

Theorem 3.1 (Cover Theorem). *Suppose the TOP order $\prec_2$ in R^m is compatible with the anti-graded order $\prec_1$ in R. Let G be a finite subset of M such that, for any term $T \in R^m$, there is a pair $(\boldsymbol{u}, v) \in G$ and a monomial t such that $T = t\mathrm{lpp}(\boldsymbol{u})$, where for every pair $(\boldsymbol{u}, v) \in G$, $\boldsymbol{u} \in (k[X])^m$ and $v \in k[X]$. Then the following are equivalent:*

(*a*) *G is a strong standard basis for M;*
(*b*) *every J-pair of G is eventually super top-reducible by G;*
(*c*) *every J-pair of G is covered by G.*

Proof. We only prove $(c) \Rightarrow (a)$, other proofs are same as the Theorem 2.4 in Gao et. al. [15]. We prove by contradiction.

Let $W = \{(\mathbf{u}, v) \in M : (\mathbf{u}, v)$ is not top-reducible by any pair in $G\}$. Since $\prec_1$ is a local order in R, we can construct a subset $W_1 \subset W$ such that $W_1 = \{(\mathbf{u}, v) \in W : \mathrm{lpp}(v)$ is maximal$\}$. Then, for any element $(\mathbf{u}, v) \in W_1$, the leading power product of v is equal and maximal in W. Since $\prec_2$ in R^m is compatible with $\prec_1$, according to Lemma 2.9 we can also construct a subset $W_2 \subset W_1$ such that $W_2 = \{(\mathbf{u}, v) \in W_1 : \mathrm{lpp}(\mathbf{u})$ is minimal$\}$. Therefore, we can pick a pair $p_0 = (\mathbf{u}_0, v_0) \in W_2$ such that $\mathrm{lpp}(v_0)$ is maximal in W and $\mathrm{lpp}(\mathbf{u}_0)$ is minimal in W_1. Next, we select a pair $p_1 = (\mathbf{u}_1, v_1)$ from G such that

(*i*) $\mathrm{lpp}(\mathbf{u}_0) = t\mathrm{lpp}(\mathbf{u}_1)$ for some monomial t, and
(*ii*) $t\mathrm{lpp}(v_1)$ is minimal among all $p_1 \in G$ satisfying (*i*).

Then $t(\mathbf{u}_1, v_1)$ is not regular top-reducible by G (this proof can be found in Theorem 2.4, [15]). Consider

$$(\mathbf{u}_*, v_*) := (\mathbf{u}_0, v_0) - ct(\mathbf{u}_1, v_1), \tag{1}$$

where $c = \mathrm{lc}(\mathbf{u}_0)/\mathrm{lc}(\mathbf{u}_1)$ so that $\mathrm{lpp}(\mathbf{u}_*) \prec_2 \mathrm{lpp}(\mathbf{u}_0)$. Note that $\mathrm{lpp}(v_0) \neq t\mathrm{lpp}(v_1)$, since otherwise $(\mathbf{u}_0, v_0)$ would be top-reducible by $p_1 \in G$ contradicting the choice of $(\mathbf{u}_0, v_0)$. Then, we consider the following two cases:

- If $\mathrm{lpp}(v_0) \prec_1 t\mathrm{lpp}(v_1)$, then $\mathrm{lpp}(v_*) = t\mathrm{lpp}(v_1)$. Since every element $(\mathbf{u}, v) \in W$ satisfies that $\mathrm{lpp}(v) \preceq_1 \mathrm{lpp}(v_0)$, we have that $(\mathbf{u}_*, v_*) \notin W$ and it is top-reducible by G. Without loss of generality, we assume that $(\mathbf{u}_*, v_*)$ is top-reducible by $p_2 = (\mathbf{u}_2, v_2) \in G$ with $v_2 \neq 0$. Hence, $\mathrm{lpp}(v_2) \mid t\mathrm{lpp}(v_1)$ and $t_2\mathrm{lpp}(\mathbf{u}_2) \preceq_2 \mathrm{lpp}(\mathbf{u}_*) \prec_2 t\mathrm{lpp}(\mathbf{u}_1)$, $t_2 = t\mathrm{lpp}(v_1)/\mathrm{lpp}(v_2)$. It follows that $t(\mathbf{u}_1, v_1)$ is regular top-reducible by $p_2 \in G$. Since $t(\mathbf{u}_1, v_1)$ is not regular top-reducible by any pair in G, this case impossible.
- If $t\mathrm{lpp}(v_1) \prec_1 \mathrm{lpp}(v_0)$, then $\mathrm{lpp}(v_*) = \mathrm{lpp}(v_0)$. We assert that $(\mathbf{u}_*, v_*) \notin W$. If otherwise, $\mathrm{lpp}(v_*) = \mathrm{lpp}(v_0)$ implies that $(\mathbf{u}_*, v_*) \in W_1$. It follows that $\mathrm{lpp}(\mathbf{u}_*) \succeq_2 \mathrm{lpp}(\mathbf{u}_0)$, which leads to a contradiction. So $(\mathbf{u}_*, v_*) \notin W$ is top-reducible by G. Without loss of generality, we assume that $(\mathbf{u}_*, v_*)$ is top-reducible by $p_3 = (\mathbf{u}_3, v_3) \in G$ with $v_3 \neq 0$. We have $\mathrm{lpp}(v_3) \mid \mathrm{lpp}(v_0)$ and $t_3\mathrm{lpp}(\mathbf{u}_3) \preceq_2 \mathrm{lpp}(\mathbf{u}_*) \prec_2 \mathrm{lpp}(\mathbf{u}_0)$, where $t_3 = \mathrm{lpp}(v_0)/\mathrm{lpp}(v_3)$. Therefore, $(\mathbf{u}_0, v_0)$ is regular

top-reducible by $p_3 \in G$, contradicting the fact that $(\mathbf{u}_0, v_0)$ is not top-reducible by any pair in G.

Therefore such a pair $(\mathbf{u}_0, v_0)$ does not exist in M, so every pair in M is top-reducible by G. This proves $(c) \Rightarrow (a)$. □

Remark 3. If $L(\text{lpp}(v_0))$ has not a minimal element, then $(\mathbf{u}_*, v_*)$ may not be top-reducible by any pair in G under the case of $t\text{lpp}(v_1) \prec_1 \text{lpp}(v_0)$. If $(\mathbf{u}_*, v_*) \in W$, we need to select another pair $p_4 = (\mathbf{u}_4, v_4)$ from G and repeat the process of equation (1). Since $\prec_2$ is a local order, the process of equation (1) may not terminate, and the above theorem can not be justified.

It follows from Theorem 3.1 that any J-pair that is covered by G can be discarded without performing any reductions. As a consequence, there are three criteria used to discard superfluous J-pairs.

Corollary 3.2 (Syzygy Criterion). *For any J-pair $(\boldsymbol{u}, v)$ of G, it can be discarded if* $\text{lpp}(\boldsymbol{u})$ *is divided by* $\text{lpp}(\boldsymbol{w})$ *for some* $(\boldsymbol{w}, 0)$ *in* M.

Corollary 3.3 (Signature Criterion). *Among all J-pairs with a same signature, only one (with the polynomial part minimal) needs to be stored.*

Corollary 3.4 (Rewrite Criterion). *For any J-pair $(\boldsymbol{u}, v)$ of G, it can be discarded if $(\boldsymbol{u}, v)$ is covered by G.*

Before presenting the GVW algorithm in local ring, we need to make some explanations. Since storing and updating vectors $\mathbf{u} \in R^m$ are expensive, we will store $\text{lpp}(\mathbf{u})$ instead of $\mathbf{u}$ in our computation, which does not effect the correctness and termination of the algorithm. That is, for any given set $G' = \{(\mathbf{u}_1, v_1), \ldots, (\mathbf{u}_s, v_s)\} \subset M$, we will use the set $G = \{(\text{lpp}(\mathbf{u}_1), v_1), \ldots, (\text{lpp}(\mathbf{u}_s), v_s)\}$ instead of G' to compute a standard basis of $\langle f_1, \ldots, f_m\rangle \subset R$. Assume that the J-pair of $(\mathbf{u}_i, v_i)$ and $(\mathbf{u}_j, v_j)$ is $(\mathbf{u}, v)$, then the J-pair (T, v) of $(\text{lpp}(\mathbf{u}_i), v_i)$ and $(\text{lpp}(\mathbf{u}_j), v_j)$ is defined as $(\text{lpp}(\mathbf{u}), v)$, where $1 \le i \ne j \le s$. For simplicity, we use $\overline{(T, v)}^G$ to denote the remainder obtained by using G to regular top-reduce (T, v) repeatedly until it is not regular top-reducible, we will prove that this process is terminated within a finite number of steps in Section 3.2.

According to the Theorem 3.1, and the Corollary 3.2, 3.3, 3.4, the GVW algorithm in local ring is presented below.

$^\diamond$: The trivial principle syzygies are used to delete the redundant J-pairs.

$^\clubsuit$: Only storing the J-pairs whose signatures are not divided by $\{(T, 0) \mid T \in H\}$ and only storing one J-pair for each distinct signature with v-part minimal. (syzygy criterion and signature criterion)

$^\spadesuit$: The principle syzygy is stored only when $\text{lpp}(v_j T_0 - v_0 T_j) = \max\{\text{lpp}(v_j T_0), \text{lpp}(v_0 T_j)\}$.

The correctness of the algorithm follows directly from the theorem 3.1. The algorithm can terminate if the regular top-reduction can terminate in the local ring.

Algorithm 1: GVW algorithm in local ring

Input : $F = \{f_1, \ldots, f_m\} \subset k[X]$, an anti-graded order $\prec_1$ in R, and a TOP order $\prec_2$ in R^m, where $\prec_2$ is compatible with $\prec_1$.

Output: two sets V and H, where V is the set of a standard basis for $\langle f_1, \ldots, f_m\rangle \subset R$, and H is the set consisting of the leading power products of a standard basis for the syzygy of F.

```
begin
  Initial:
    G := {(e_1, f_1), ..., (e_m, f_m)};
    H := {lpp(f_i e_j − f_j e_i) | 1 ⩽ i < j ⩽ m}^◇;
    JP := {J-pairs of G}^♣;
  while JP ≠ ∅ do
    choose (T, v) ∈ JP, and JP := JP \ {(T, v)};
    if (T, v) is covered by G then
      next;
    else
      (T_0, v_0) := \overline{(T, v)}^G;
      if v_0 = 0 then
        H := H ∪ {T_0};
        JP := JP \ {(T′, v′) ∈ JP satisfies T_0 divides T′};
      else
        H := H ∪ {lpp(v_0 T_j − v_j T_0) | (T_j, v_j) ∈ G}^♠;
        JP := JP ∪ {J-pairs between (T_0, v_0) and G}^♣;
        G := G ∪ {(T_0, v_0)};
      end if
    end if
  end while
  return V := {v | (T, v) ∈ G} and H.
end
```

Example 3.5. Let $p_1 = (\mathbf{u}_1, v_1) = (\mathbf{e}_1, x_1)$, $p_2 = (\mathbf{u}_2, v_2) = (\mathbf{e}_2, x_1 - x_1^2)$, $\prec_1$ is the anti-graded lexicographic order, $\prec_2$ is a TOP order and compatible with $\prec_1$, where $\mathbf{e}_2 \prec_2 \mathbf{e}_1$.

Since $\text{lpp}(v_2) = \text{lpp}(v_1) = x_1$ and $\text{lpp}(\mathbf{u}_2) = \mathbf{e}_2 \prec_2 \text{lpp}(\mathbf{u}_1) = \mathbf{e}_1$, p_1 is regular top-reducible by p_2. Then we have

$$p_3 = (\mathbf{u}_3, v_3) = \text{OneRed}(p_1, p_2) = (\mathbf{e}_1 - \mathbf{e}_2, x_1^2).$$

Similarly, $x_1\text{lpp}(v_2) = \text{lpp}(v_3)$ and $x_1\text{lpp}(\mathbf{u}_2) = x_1\mathbf{e}_2 \prec_2 \text{lpp}(\mathbf{u}_3) = \mathbf{e}_1$ imply that p_3 is still regular top-reducible by p_2:

$$p_4 = \text{OneRed}(p_3, p_2) = (\mathbf{e}_1 - (1 + x_1)\mathbf{e}_2, x_1^3).$$

Continue the regular top-reduction steps, we have:

$p_5 = \text{OneRed}(p_4, p_2) = (\mathbf{e}_1 - (1 + x_1 + x_1^2)\mathbf{e}_2, x_1^4)$;

$p_6 = \text{OneRed}(p_5, p_2) = (\mathbf{e}_1 - (1 + x_1 + x_1^2 + x_1^3)\mathbf{e}_2, x_1^5)$;

$p_7 = \text{OneRed}(p_6, p_2) = (\mathbf{e}_1 - (1 + x_1 + x_1^2 + x_1^3 + x_1^4)\mathbf{e}_2, x_1^6)$;

$\cdots$

The above example shows that the top-reduction steps may not terminate in the local ring, if we use the usual division algorithm in the polynomial ring [7]. Thanks to the splendid idea of Mora, the termination problem can be solved by the Mora Normal Form Algorithm [6]. The notation écart will be used in the algorithm. Let

3.2 Regular Top-Reduction in Local Ring

Since the local order is not a well-ordering, a sequence of successive one-step regular top-reductions may not terminate.

$f \in k[X]$, the écart of f is

$$ecart(f) = \deg(f) - \deg(\mathrm{lpp}(f)),$$

where $\deg(f)$ is the total degree of f. For an element $p = (\mathbf{u}, f)$ in $(k[X])^m \times k[X]$, we define the écart of p is equal to $ecart(f)$.

Theorem 3.6. *Assume $\prec_1, \prec_2$ be the semigroup orders on the monomials in the ring $k[X]$ and $(k[X])^m$ respectively, where $\prec_1$ is a local order and $\prec_2$ is compatible with $\prec_1$. Let $p = (\boldsymbol{u}, f)$ be a J-pair of $G = \{p_1 = (\boldsymbol{u}_1, f_1), \dots, p_s = (\boldsymbol{u}_s, f_s)\} \subset (k[X])^m \times k[X]$ and p is not covered by G. Then there is an algorithm for producing polynomials $h, a_1, \dots, a_s$ in $k[X]$ and $r = (\mathbf{w}, v)$ in $(k[X])^m \times k[X]$ such that*

$$hp = a_1 p_1 + \cdots + a_s p_s + r, \tag{2}$$

where $\mathrm{lpp}(h) = 1$ (so h is a unit in R), $\mathrm{lpp}(a_i f_i) \preceq_1 \mathrm{lpp}(f)$, $\mathrm{lpp}(a_i \boldsymbol{u}_i) \preceq_2 \mathrm{lpp}(\boldsymbol{u})$ for all i with $a_i \neq 0$, $\mathrm{lpp}(\mathbf{w}) = \mathrm{lpp}(\boldsymbol{u})$, and either $v = 0$ or $\mathrm{lpp}(v)$ is not divisible by any $\mathrm{lpp}(f_i)$. The r is called the remainder of p regular top-reduced by G.

Proof. Since $p = (\mathbf{u}, f)$ is a J-pair of G, there exists a pair $p_i = (\mathbf{u}_i, f_i) \in G$ such that p is regular top-reducible by p_i. Let $r_0 := (\mathbf{w}_0, v_0) = p - c_i^{(0)} t_i^{(0)} p_i$, where $c_i^{(0)} = \mathrm{lc}(f)/\mathrm{lc}(f_i)$ and $t_i^{(0)} = \mathrm{lpp}(f)/\mathrm{lpp}(f_i)$, then $\mathrm{lpp}(v_0) \prec_1 \mathrm{lpp}(f)$ and $\mathrm{lpp}(\mathbf{w}_0) = \mathrm{lpp}(\mathbf{u})$. If r_0 can be expressed as $hr_0 = a_1 p_1 + \cdots + a_s p_s + r$, then the equation (2) holds for $(\mathbf{u}, f)$. We give a constructive proof by the following algorithm, which is similar to the algorithm in page 173 of [6].

Input: $r_0 = (\mathbf{w}_0, v_0), p_1 = (\mathbf{u}_1, f_1), \dots, p_s = (\mathbf{u}_s, f_s)$;
Output: r as statement of theorem 3.6.
Initial: $r := (\mathbf{w}, v)$; $\mathbf{w} := \mathbf{w}_0$; $v := v_0$; $L := \{p_1, \dots, p_s\}$;
    $M := \{g \in L : r$ is regular top-reducible by $g\}$.
WHILE ($v \neq 0$ AND $M \neq \emptyset$) THEN
  SELECT $g \in M$ with ecart(g) minimal;
  IF ecart(g) > ecart(r) THEN
    $L := L \cup \{r\}$;
  END IF;
  $r :=$ OneRed(r, g);
  IF $v \neq 0$ THEN
    $M := \{g \in L : r$ is regular top-reducible by $g\}$;
  END IF;
END DO.

To prove the correctness, we will prove by induction on $j \geq 0$ that we have identities of the form

$$h_j r_0 = a_1^{(j)} p_1 + \cdots + a_s^{(j)} p_s + r_j, \tag{3}$$

where $\mathrm{lpp}(h_j) = 1$, $\mathrm{lpp}(a_i^{(j)} f_i) \preceq_1 \mathrm{lpp}(v_0)$, $\mathrm{lpp}(a_i^{(j)} \mathbf{u}_i) \preceq_2 \mathrm{lpp}(\mathbf{w}_0)$, and $\mathrm{lpp}(\mathbf{w}_j) = \mathrm{lpp}(\mathbf{w}_0)$. Setting $h_0 = 1$ and $a_i^{(0)} = 0$ for all i shows that everything works for $j = 0$. Now suppose that in the first $l - 1$ steps, the equation (3) is satisfied, where $l \geq 1$. Then we need to prove that $r_l = (\mathbf{w}_l, v_l)$ produced by the l-th pass through the loop satisfies the above conditions.

If $v_{l-1} \neq 0$ and $M_{l-1} \neq \emptyset$, in the step l, there is $g_l = (\mathbf{s}_l, b_l) \in M_{l-1}$ such that r_{l-1} is regular top-reducible by g_l. Then

$$r_l = \mathrm{OneRed}(r_{l-1}, g_l) = r_{l-1} - c_l t_l g_l, \tag{4}$$

where $t_l = \mathrm{lpp}(v_{l-1})/\mathrm{lpp}(b_l)$, $c_l = \mathrm{lc}(v_{l-1})/\mathrm{lc}(b_l)$ and $\mathrm{lpp}(\mathbf{w}_l) = \mathrm{lpp}(\mathbf{w}_{l-1})$. For g_l, there are two cases:

($\star$) $g_l = p_i \in \{p_1, \dots, p_s\}$;
($\star\star$) $g_l = r_n \in \{r_0, r_1, \dots, r_{l-2}\}$.

In case ($\star$), substituting $r_{l-1} = c_l t_l p_i + r_l$ to the right-side of equation (3) for $j = l - 1$, we have

$$h_{l-1} r_0 = a_1^{(l-1)} p_1 + \cdots + a_s^{(l-1)} p_s + c_l t_l p_i + r_l.$$

Setting $h_l := h_{l-1}$, $a_i^{(l)} := a_i^{(l-1)} + c_l t_l$ and $a_k^{(l)} := a_k^{(l-1)}$ for $k \in \{1, \dots, s\} \setminus \{i\}$, the equation (3) holds for $j = l$.

In case ($\star\star$), $g_l = r_n = h_n r_0 - \sum_{i=1}^s a_i^{(n)} p_i$ and $\mathrm{lpp}(b_l) \succ_1 \mathrm{lpp}(v_{l-1})$, where $n \in \{0, \dots, l-2\}$. Substituting g_l to the right-hand side of (4), we have $r_{l-1} = c_l t_l (h_n r_0 - \sum_{i=1}^s a_i^{(n)} p_i) + r_l$. Substituting r_{l-1} to the right-hand side of (3) for $j = l - 1$, we have

$$h_{l-1} r_0 = a_1^{(l-1)} p_1 + \cdots + a_s^{(l-1)} p_s + c_l t_l (h_n r_0 - \sum_{i=1}^{s} a_i^{(n)} p_i) + r_l,$$

i.e., $(h_{l-1} - c_l t_l h_n) r_0 = \sum_{i=1}^s (a_i^{(l-1)} - c_l t_l a_i^{(n)}) p_i + r_l$. Setting $h_l := h_{l-1} - c_l t_l h_n$ and $a_i^{(l)} := a_i^{(l-1)} - c_l t_l a_i^{(n)}$. $\mathrm{lpp}(b_l) \succ_1 \mathrm{lpp}(v_{l-1})$ implies that $\mathrm{lpp}(c_l t_l) = \mathrm{lpp}(v_{l-1})/\mathrm{lpp}(b_l) \neq 1$. Since $\prec_1$ is a local order, $1 = \mathrm{lpp}(h_{l-1}) \succ_1 \mathrm{lpp}(c_l t_l h_n)$, where $\mathrm{lpp}(h_n) = 1$. Therefore, $\mathrm{lpp}(h_l) = \mathrm{lpp}(h_{l-1} - c_l t_l h_n) = 1$, and the equation (3) also holds for $j = l$.

If the algorithm terminates after N steps, then $h := h_N$, $a_i := a_i^{(N)}$ and $r := r_N$ satisfy the conditions in Theorem 3.6, so the algorithm is correct.

To prove the termination, the order $\prec_1$ extends to a semigroup order $\prec'$ on monomials in $t, x_1, \dots, x_n$ in the following way. Define $t^a X^\alpha \prec' t^b X^\beta$, if either $a + |\alpha| < b + |\beta|$, or $a + |\alpha| = b + |\beta|$ and $X^\alpha \prec_1 X^\beta$. The order $\prec'$ is a global order. Let f^H denote the homogenization of f with respect to a new variable t. For any $f \in k[X]$, we have $\mathrm{lpp}_{\prec'}(f^H) = t^{ecart(f)} \mathrm{lpp}_{\prec_1}(f)$. For any $r = (\mathbf{w}, v) \in (k[X])^m \times k[X]$, the homogenization of r is defined by $r^H = (t^{ecart(v)} \mathbf{w}, v^H)$. And for any pairs $r_1 = (\mathbf{w}_1, v_1), r_2 = (\mathbf{w}_2, v_2)$, we say that r_1 *divides* r_2 if $\mathrm{lpp}(\mathbf{w}_1) \mid \mathrm{lpp}(\mathbf{w}_2)$ and $\mathrm{lpp}(v_1) \mid \mathrm{lpp}(v_2)$.

Let $\mathrm{IniHom(L)} = \{(t^{ecart(v)} \mathrm{lpp}(\mathbf{w}), \mathrm{lpp}_{\prec'}(v^H)) : (\mathbf{w}, v) \in L\}$, we claim that if $r_{l-1} = (\mathbf{w}_{l-1}, v_{l-1})$ is added to the set L_l in the step l, the $(t^{ecart(v_{l-1})} \mathrm{lpp}(\mathbf{w}_{l-1}), \mathrm{lpp}_{\prec'}(v_{l-1}^H))$ is not divisible by any element in $\mathrm{IniHom}(L_{l-1})$. We prove it by contradiction. Assume that $(t^{ecart(v_{l-1})} \mathrm{lpp}(\mathbf{w}_{l-1}), \mathrm{lpp}_{\prec'}(v_{l-1}^H))$ is divisible by some element in $\mathrm{IniHom}(L_{l-1})$, then there exists $g = (\mathbf{s}, b) \in L_{l-1}$ such that

$$\begin{cases} t^{ecart(b)} \mathrm{lpp}(\mathbf{s}) \mid t^{ecart(v_{l-1})} \mathrm{lpp}(\mathbf{w}_{l-1}), \\ \mathrm{lpp}_{\prec'}(b^H) \mid \mathrm{lpp}_{\prec'}(v_{l-1}^H). \end{cases}$$

Therefore, $\mathrm{lpp}(\mathbf{s}) \mid \mathrm{lpp}(\mathbf{w}_{l-1})$, $\mathrm{lpp}(b) \mid \mathrm{lpp}(v_{l-1})$ and $ecart(b) \leqslant ecart(v_{l-1})$. Let $\mathrm{lpp}(v_{l-1}) = X^\alpha \mathrm{lpp}(b)$ and $\mathrm{lpp}(\mathbf{w}_{l-1}) = X^\beta \mathrm{lpp}(\mathbf{s})$. For g, there are two cases:

- $g \in L_{l-1} \backslash G \subset \{r_0, r_1, \dots, r_{l-2}\}$;
- $g \in G$.

If $g \in L_{l-1} \backslash G$, then $\mathrm{lpp}(\mathbf{s}) = \mathrm{lpp}(\mathbf{w}_{l-1}) = \mathrm{lpp}(\mathbf{w}_0)$ and $\mathrm{lpp}(v_{l-1}) \neq \mathrm{lpp}(b)$. Then $X^\alpha \prec_1 X^\beta$ since $\prec_1$ is a local order, $X^\alpha \neq 1$ and $X^\beta = 1$. We have:

$$\frac{\mathrm{lpp}(v_{l-1})}{\mathrm{lpp}(b)} \mathrm{lpp}(\mathbf{s}) = X^\alpha \mathrm{lpp}(\mathbf{s}) \prec_2 X^\beta \mathrm{lpp}(\mathbf{s}) = \mathrm{lpp}(\mathbf{w}_{l-1}),$$

and $\text{lpp}(b) \mid \text{lpp}(v_{l-1})$, so r_{l-1} is regular top-reducible by g and $g \in M_{l-1}$. But $ecart(g) \leqslant ecart(r_{l-1})$, this contradicts that r_{l-1} is added to L_l only when $ecart(r_{l-1}) < ecart(g')$ for any $g' \in M_{l-1}$.

If $g \in G$ and $X^\alpha \prec_1 X^\beta$, it is contradictory by the same analysis as above. If $g \in G$ and $X^\alpha \succeq_1 X^\beta$, then

$$X^\beta \text{lpp}(b) \preceq_1 X^\alpha \text{lpp}(b) = \text{lpp}(v_{l-1}) \preceq_1 \text{lpp}(v_0) \prec_1 \text{lpp}(f),$$

and $\text{lpp}(\mathbf{s})|\text{lpp}(\mathbf{w}_{l-1}) = \text{lpp}(\mathbf{w}_0) = \text{lpp}(\mathbf{u})$. This contradicts that $p = (\mathbf{u}, f)$ is not cover by G.

Above all, $(t^{ecart(v_{l-1})}\text{lpp}(\mathbf{w}_{l-1}), \text{lpp}_{>'}(v^H_{l-1}))$ is not divisible by $\text{IniHom}(L_{l-1})$. Therefore, we have a sequence $r_{j_1}, r_{j_2}, \ldots, r_{j_i}, \ldots \in L$, which corresponds to a sequence

$$\begin{aligned}&(t^{ecart(v_{j_1})}\text{lpp}(\mathbf{w}_{j_1}), \text{lpp}(v^H_{j_1})), (t^{ecart(v_{j_2})}\text{lpp}(\mathbf{w}_{j_2}),\\ &\text{lpp}(v^H_{j_2})), \ldots, (t^{ecart(v_{j_i})}\text{lpp}(\mathbf{w}_{j_i}), \text{lpp}(v^H_{j_i})), \ldots\end{aligned} \tag{5}$$

with no pair divisible by any previous one.

We introduce new variables $\vec{y}_i = (y_{i_0}, y_{i_1}, \ldots, y_{i_n})$. Each pair $(t^a X^\alpha \mathbf{e}_i, t^a X^\beta)$ corresponds to a term $\vec{y}_i^{(a,\alpha)} t^a X^\beta$ in the variables x_i, t, y_{i_j} (this idea is similar to that on Page 4 of the paper [10]), where $i = 1, \ldots, n, j = 0, \ldots, n$. Then the pairs in (5) give us a list of monomials in $x_i, t, y_{i_j}, i = 1, \ldots, n, j = 0, \ldots, n$ with no one divisible by any previous one. Since every polynomial ring over a field is Noetherian, the list of monomials must be finite. So there is some N such that $L_N = L_{N+1} = L_{N+2} = \cdots$. Then the algorithm continues with a fixed set of L. For $m \geqslant N$, since any regular top-reduction of r_m by L_m corresponds to a regular top-reduction of r^H_m by $L^H_m = L^H_N$, the reduction must terminate after finite steps. □

Example 3.7 (Continue Example 3.5). The J-pair of p_1 and p_2 is $p = (\mathbf{u}, v) = (\mathbf{e}_1, x_1)$, which is not covered by p_1 and p_2. We start the division algorithm with $r_0 := p - p_2 = (\mathbf{e}_1 - \mathbf{e}_2, x_1^2)$, and $L_0 = \{p_1, p_2\}$. Since r_0 is regular top-reducible by p_1 and p_2, $M_0 = \{p_1, p_2\}$. In the step 1, p_1 is chosen to reduce r_0 since $ecart(p_1) = 0 < 1 = ecart(p_2)$. $r_1 := OneRed(r_0, p_1) = r_0 - x_1 p_1 = ((1 - x_1)\mathbf{e}_1 - \mathbf{e}_2, 0)$. The division algorithm terminates, and $p = x_1 p_1 + p_2 + ((1 - x_1)\mathbf{e}_1 - \mathbf{e}_2, 0)$.

4 AN ILLUSTRATIVE EXAMPLE

The following is an example to illustrate our algorithm in local ring.

Example 4.1. Let $R = \text{Loc}_{\prec_1}(\mathbb{C}[x_1, x_2, x_3])$, and $I = \langle f_1, f_2, f_3 \rangle = \langle x_1^2 - 5x_2x_3 - 2x_2^2x_3, 2x_1x_2 + 2x_2^3 - x_3^3, -x_1x_2 + x_2x_3^2 \rangle \subset R$, where $\prec_1$ is the anti-graded revlex order with $x_1 \succ_1 x_2 \succ_1 x_3$. Suppose $\prec_2$ is a TOP order in R^3 and compatible with $\prec_1$, where $\mathbf{e}_1 \succ_2 \mathbf{e}_2 \succ_2 \mathbf{e}_3$. Computing a standard basis for I and the leading power products of a standard basis for the syzygy module of $\{f_1, f_2, f_3\}$.

Initial:
$G_0 := \{p_1, p_2, p_3\} = \{(\mathbf{e}_1, f_1), (\mathbf{e}_2, f_2), (\mathbf{e}_3, f_3)\}$;
$H_0 := \{x_1^2\mathbf{e}_2, x_1^2\mathbf{e}_3, x_1x_2\mathbf{e}_2\}$ is the set of the leading power products of principle syzygies $\{\mathbf{e}_1 f_2 - \mathbf{e}_2 f_1, \mathbf{e}_1 f_3 - \mathbf{e}_3 f_1, \mathbf{e}_2 f_3 - \mathbf{e}_3 f_2\}$;
$JP_0 := \{(T_1, v_1), (T_2, v_2), (T_3, v_3)\} = \{(x_1\mathbf{e}_3, x_1 f_3), (x_1\mathbf{e}_2, x_1 f_2), (\mathbf{e}_2, f_2)\}$ is the J-pairs set of G_0.
First cycle:
We select the J-pair (T_1, v_1) from JP_0 and use G_0 to reduce it. By computing, (T_1, v_1) is not covered by G_0. So (T_1, v_1) can be regular top-reducible by G_0 to $p_4 = (T_1, \tilde{v}_1) = (x_1\mathbf{e}_3, -5x_2^2x_3 + x_1x_2x_3^2 - 2x_2^3x_3)$. Since $\tilde{v}_1 \neq 0$, computing the principle syzygies of p_4 with G_0, and adding the leading power product of these syzygies to H_0 (delete any redundant ones), we obtain $H_1 := H_0$. Computing the J-pairs of p_4 with elements in G_0 and getting $JP_1 := \{(T_2, v_2), (T_3, v_3)\}$. Moreover, $G_1 := G_0 \cup \{p_4\}$.
Second cycle:
We select (T_2, v_2) from JP_1 and $JP_2 := \{(T_3, v_3)\}$. (T_2, v_2) can be regular top-reducible by G_1 to $p_5 = (T_2, \tilde{v}_2) = (x_1\mathbf{e}_2, -x_1x_3^3 + 2x_2^3x_3^2 + 2x_2x_3^4)$. According to syzygy criterion and signature criterion, we obtain $H_2 := H_0$, $JP_2 := \{(T_3, v_3)\}$ and $G_2 := G_1 \cup \{p_5\}$.
Third cycle:
We select (T_3, v_3) from JP_2 and $JP_3 := \emptyset$. (T_3, v_3) can be regular top-reducible by G_2 to $p_6 = (T_3, \tilde{v}_3) = (\mathbf{e}_2, 2x_2^3 + 2x_2x_3^2 - x_3^3)$. According to syzygy criterion and signature criterion, we obtain $H_3 := H_0 \cup \{x_2^2x_3\mathbf{e}_2\}$, $JP_3 := \{(T_4, v_4), (T_5, v_5)\}$ and $G_3 := G_2 \cup \{p_6\}$, where $(T_4, v_4) = (x_3\mathbf{e}_2, x_3\tilde{v}_3)$ and $(T_5, v_5) = (x_1\mathbf{e}_2, x_1\tilde{v}_3)$.
Fourth cycle:
We select (T_4, v_4) from JP_3 and $JP_4 := \{(T_5, v_5)\}$. (T_4, v_4) can be regular top-reducible by G_3 to $p_7 = (T_4, \tilde{v}_4) = (x_3\mathbf{e}_2, 2x_2x_3^3 - x_3^4 + \frac{2}{5}x_1x_2^2x_3^2 - \frac{4}{5}x_2^4x_3)$. According to syzygy criterion and signature criterion, we obtain $H_4 := H_3$, $JP_4 := \{(T_6, v_6), (T_7, v_7), (T_5, v_5)\}$ and $G_4 := G_3 \cup \{p_7\}$, where $(T_6, v_6) = (x_2x_3\mathbf{e}_2, x_2\tilde{v}_4)$ and $(T_7, v_7) = (x_1x_3\mathbf{e}_2, x_1\tilde{v}_4)$.
Fifth cycle:
We select (T_6, v_6) from JP_4 and $JP_5 := \{(T_7, v_7), (T_5, v_5)\}$. (T_6, v_6) can be regular top-reducible by G_4 to $p_8 = (T_6, \tilde{v}_6) = (x_2x_3\mathbf{e}_2, (-\frac{1}{2}x_3^5 - \frac{4}{5}x_2^5x_3 + \frac{2}{5}x_1x_2^3x_3^2 - \frac{2}{5}x_2^4x_3^2 + \frac{1}{5}x_1x_2^2x_3^2 - \frac{4}{5}x_2^3x_3^3 + \frac{2}{5}x_1x_2x_3^4)$. According to syzygy criterion and signature criterion, we obtain $H_5 := H_4$, $JP_5 := \{(T_7, v_7), (T_5, v_5)\}$ and $G_5 := G_4 \cup \{p_8\}$.
Sixth cycle:
We select (T_7, v_7) from JP_5 and $JP_6 := \{(T_5, v_5)\}$. By computing, (T_7, v_7) is covered by G_5. According to rewrite criterion, we get $H_6 := H_5$, $JP_6 := \{(T_5, v_5)\}$ and $G_6 := G_5$.
Seventh cycle:
We select (T_5, v_5) from JP_6 and $JP_7 := \emptyset$. By computing, (T_5, v_5) is covered by G_6. According to rewrite criterion, we get $H_7 := H_7$, $JP_7 := \emptyset$ and $G_7 := G_6$.
Output:
Since JP_7 is empty, the algorithm terminates. Therefore, the standard basis of I in R is $\{f_1, f_2, f_3, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4, \tilde{v}_6\}$, and the leading power products of the standard basis for the syzygy module is $\{x_1^2\mathbf{e}_2, x_1^2\mathbf{e}_3, x_1x_2\mathbf{e}_2, x_2^2x_3\mathbf{e}_2\}$.

It is apparent from the above example that we discard 23 J-pairs by using three criteria, and only do 5 regular top-reductions. In order to illustrate that the three criteria can improve the computational efficiency, we compare our algorithm with a classical Gröbner basis algorithm (non signature-based) [19] that uses standard criteria to discard useless S-polynomials. We randomly generate 10 ideals in $R = \text{Loc}_{\prec_1}(\mathbb{C}[x_1, x_2, x_3, x_4])$, and they are as follows.

- $I_1 = \langle -x_1^3 + x_3^3, -x_1x_3x_4 + x_2^2x_3, x_1^2x_4 - x_3x_4^2, x_2^2x_4 + x_2x_4 \rangle$;
- $I_2 = \langle x_1x_4^2, x_1^3 - x_3^2x_4 - x_4^3, x_1^2x_4 - x_2^2, -x_2^2x_3 - x_1x_2 \rangle$;
- $I_3 = \langle -x_1x_2^2, x_1^2x_3x_4 + x_1x_3, x_2^4 - x_1x_3 + x_4, x_3^4 + x_1x_2^2 + x_1 \rangle$;
- $I_4 = \langle x_1x_3^2 - x_1^2x_4, x_3^4 + x_3^3x_4, -x_3, -x_1^2x_2^2 - x_1x_4^2 - x_2x_4^2 \rangle$;
- $I_5 = \langle x_2x_3x_4 - 3x_2x_4^2 - 4x_2x_3, -4x_1^2x_3x_4 - 4x_2^3, -5x_4^2, -4x_1^2x_4^2 + 2x_3^3x_4 - 3x_1x_2x_4 \rangle$.

- $I_6 = \langle 8x_1^2x_3x_4 - 7x_1x_3^3 + 8x_3x_4, x_1x_2^2 - 6x_1x_3^2 - 7x_4, -5x_2^4 + 2x_1x_2x_4\rangle$;
- $I_7 = \langle 5x_1^2x_2^2 - 3x_1x_2x_3^2 + x_1^2x_2, 3x_2x_3^3 + 2x_3^2x_4^2 + x_3x_4^3, 5x_1^4 - x_1x_2x_3^2 - 7x_2x_4^2\rangle$.
- $I_8 = \langle -6x_4^3 - x_1x_3 + 7x_4^2, -x_2^4 + 4x_1x_2x_3 + 4x_3x_4, 2x_1x_3^2x_4 - 3x_1x_3x_4^2 + 7x_2^2x_4^2\rangle$.
- $I_9 = \langle -x_2^3x_3 + 6x_2^2x_4^2 - 4x_1x_2x_4, 2x_1^3x_4 - 4x_2x_3x_4 + 2x_3, 6x_1x_3^2 + 4x_1x_2^2x_4 + 3x_1x_4^2\rangle$.
- $I_{10} = \langle 3x_1x_4^2 + 7x_2^3 + 4x_2x_3^2, 5x_1x_3 - 10x_1x_4 - 5x_3^2, -8x_1x_2 + 3x_3^2 + 4x_2^2x_4\rangle$.

For all these examples, the term order in R and R^m ($3 \leq m \leq 4$) is anti-graded revlex order and TOP order, respectively. We implement the two algorithms on the computer algebra system *Maple*, and the codes and examples are available on the web: http://www.mmrc.iss.ac.cn/~dwang/software.html.

Table 1: examples

ideal	signature-based method			classical method		
	J-pairs	discard	ratio	S-polys	discard	ratio
I_1	21	14	67%	28	6	21%
I_2	21	14	67%	21	9	43%
I_3	15	12	80%	15	8	53%
I_4	20	16	80%	21	10	48%
I_5	15	9	60%	15	4	27%
I_6	20	16	80%	21	6	29%
I_7	14	11	79%	15	4	27%
I_8	35	29	83%	28	9	32%
I_9	10	7	70%	15	6	40%
I_{10}	21	17	81%	66	28	43%

The second column and fifth column in Table 1 represents the total number of J-pairs and S-polynomials (abbreviated S-polys) generated during the calculation, respectively. The third column (sixth column) represents the useless J-pairs (useless S-polys) that are discarded. The fourth column (last column) shows the percentage of the number of discarded J-pairs (S-polys) to the number of the total J-pairs (S-polys). Experimental data in Table 1 suggests that the proposed algorithm is superior in practice in comparison with the classical Gröbner basis algorithm.

5 CONCLUDING REMARKS

The paper proposed an efficient algorithm to compute the standard bases in local ring. In the process of extending the GVW algorithm from polynomial ring to local ring, we solved two key problems. First, an infinite set has not a minimal element in local ring. Under the situation that $\prec_1$ is an anti-graded order in $k[X]$ and $\prec_2$ is a TOP order in $(k[X])^m$, we proved that the signature set $L(\text{lpp}(v_0))$ w.r.t. v_0 has a minimal element. Then we generalized the cover theorem to local ring to discard the useless J-pairs. Second, since the general division algorithm may not terminate in local ring, Mora normal form algorithm is used to do regular top-reduction, and the proposed algorithm terminates in finite steps.

Although we only consider the case that $\prec_2$ is a TOP order in $(k[X])^m$, if $\prec_2$ is an $\mathbf{f}$-weighted anti-degree followed by TOP or an $\mathbf{f}$-weighted $\prec_1$ followed by POT, Lemma 2.9 and Theorem 3.1 are also established. Moreover, an alternative method to compute the standard bases is using the Lazard's homogeneous idea. In the future work, we will consider the case of $\prec_1$ is not an anti-graded order in $k[X]$. We hope that the results of this paper will motivate new progress in this research topic.

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