Further Results on the Factorization and Equivalence for Multivariate Polynomial Matrices

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ABSTRACT

This paper is concerned with the factorization and equivalence problems of multivariate polynomial matrices. We present a new criterion for the existence of matrix factorizations for a class of multivariate polynomial matrices, and prove that these matrix factorizations are unique. Based on this new criterion and the constructive proof process, we give an algorithm to compute a matrix factorization of a multivariate polynomial matrix. After that, we put forward a sufficient and necessary condition for the equivalence of square polynomial matrices: a square polynomial matrix is equivalent to a diagonal triangle if it satisfies the condition. An illustrative example is given to show the effectiveness of the matrix equivalence theorem.

CCS CONCEPTS

• **Computing methodologies** → **Symbolic and algebraic algorithms**; **Algebraic algorithms**.

KEYWORDS

Polynomial matrices, Matrix factorization, Matrix equivalence, Minors, Gröbner basis

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1 INTRODUCTION

Multidimensional systems have wide applications in image, signal processing, and other areas (see, e.g., [1, 2]). A multidimensional system may be represented by a multivariate polynomial matrix, and we can obtain some important properties of the system by studying the matrix. Therefore, the factorization problem and the equivalence problem related to multivariate polynomial matrices have attracted much attention over the past decades.

Up to now, the factorization problem for univariate and bivariate polynomial matrices has been completely solved by [23, 41, 46], but the case of more than two variables is still open. In [60], Youla and Gnavi first introduced three important concepts according to different properties of polynomial matrices, namely zero prime matrix factorization, minor prime matrix factorization and factor prime matrix factorization. Based on the work of [60] on basic structures of multivariate polynomial matrices, the factorization problem for multivariate (more than two variables) polynomial matrices has made great progress.

When multivariate polynomial matrices satisfy several special properties, there are some results about the existence problem of zero prime matrix factorizations for the polynomial matrices (see, e.g., [8, 31, 33]). After that, Lin and Bose in [34] proposed the famous Lin-Bose conjecture: a multivariate polynomial matrix admits a zero prime matrix factorization if all its maximal reduced minors generate a unit ideal. This conjecture was proved by Liu et al. [39], Pommaret [48], Wang and Feng [58], respectively. Wang and Kwong in [59] gave a sufficient and necessary condition for a multivariate polynomial with full row (column) rank to have a minor prime matrix factorization. They extracted an algorithm from Pommaret's proof of the Lin-Bose conjecture, and examples showed the effectiveness of the algorithm. Guan et al. in [22] generalized the main results in [59] to the case of polynomial matrices without full row (column) rank. For the existence problem of factor prime matrix factorizations for multivariate polynomial matrices with full row (column) rank, Wang and Liu have achieved some important results (see, e.g., [40, 56]). Then Guan et al. in [21] gave an algorithm to decide whether a class of polynomial matrices has a factor prime matrix factorization. However, the existence problem of factor prime matrix factorizations for multivariate polynomial matrices remains a challenging open problem so far.

Comparing to the factorization problem of multivariate polynomial matrices which has been widely investigated during the past years, less attention has been paid to the equivalence problem of multivariate polynomial matrices. For any given multidimensional system, our goal is to simplify it into a simpler equivalent form.

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Since a univariate polynomial ring is a principal ideal domain, a univariate polynomial matrix is always equivalent to its Smith form. This implies that the equivalence problem has been solved [24, 51]. For any given bivariate polynomial matrix, conditions under which it is equivalent to its Smith form have been investigated in [18, 19, 26]. Note that the equivalence problem of two multivariate polynomial matrices is equivalent to the isomorphism problem for two finitely presented modules, Boudellioua and Quadrat [6] and Cluzeau and Quadrat [9–11] obtained some important results by using module theory and homological algebra. According to the previous works in [6], Boudellioua in [3, 5] designed some algorithms based on Maple to compute Smith forms for some classes of multivariate polynomial matrices. For the case of multivariate polynomial matrices with more than one variable, however, the equivalence problem is not yet fully solved due to the lack of a mature polynomial matrix theory (see, e.g., [25, 46, 49]).

From our personal viewpoint, new ideas need to be injected into these areas to obtain new theoretical results and effective algorithms. Therefore, it would be significant to provide some new criteria to study the factorization problem and the equivalence problem for some classes of multivariate polynomial matrices.

From the 1990s to the present, there is a class of multivariate polynomial matrices that has always attracted attention. That is,

$$\mathcal{M} = \{\mathbf{F} \in k[\mathbf{z}]^{l\times m} : (z_1 - f(\mathbf{z}_2)) \mid d_l(\mathbf{F}) \text{ with } f(\mathbf{z}_2) \in k[\mathbf{z}_2]\},$$

where $\mathbf{z}_2 = \{z_2, \ldots, z_n\}$ and $d_l(\mathbf{F})$ is the GCD of all the $l\times l$ minors of $\mathbf{F}$. Many people tried to solve the factorization problem and the equivalence problem of multivariate polynomial matrices in $\mathcal{M}$.

Let $\mathbf{F} \in \mathcal{M}$ and $h = z_1 - f(\mathbf{z}_2)$. Lin and coauthors presented some criteria on the existence problem of a matrix factorization for $\mathbf{F}$ w.r.t. h (see, e.g., [29, 30, 36, 37]). Moreover, Lin et al. in [37] proposed a constructive algorithm to factorize $\mathbf{F}$ w.r.t. h. When $d_l(\mathbf{F}) = h$, Wang [57] gave a new result for $\mathbf{F}$ to have a minor prime matrix factorization using methods from computer algebra. Based on the pioneering work of Lin et al., Liu et al. [38] and Lu et al. [44, 45] obtained some new criteria for factorizing $\mathbf{F}$ w.r.t. h. When $l = m$ and $\det(\mathbf{F}) = h$, Lin et al. [35] proved that $\mathbf{F}$ is equivalent to the diagonal triangle $\mathrm{diag}(1, \ldots, 1, h)$. After that, Li et al. [27] generalized the main results in [35] to the case of $\det(\mathbf{F}) = h^q$.

Through research, we find that there are still many multivariate polynomial matrices in $\mathcal{M}$ which do not satisfy previous results and can be factorized or are equivalent to some diagonal triangles. As a consequence, we continue to study the factorization problem and the equivalence problem of multivariate polynomial matrices in $\mathcal{M}$ in this paper.

The rest of the paper is organized as follows. After a brief introduction to matrix factorization and matrix equivalence in Section 2, we use two examples to propose two problems that we shall consider. We present in Section 3 a new criterion for factorizing $\mathbf{F}$ w.r.t. h, then we study the uniqueness of the matrix factorization and construct an algorithm to factorize $\mathbf{F}$. A sufficient and necessary condition for a square multivariate polynomial matrix being equivalent to a diagonal triangle is described in Section 4, and we use an example to illustrate the effectiveness of the new matrix equivalence theorem. The paper contains a summary of contributions and some remarks in Section 5.

2 PRELIMINARIES AND PROBLEMS

In this section we first recall some basic notions which will be used in the following sections. For those notions which are not formally introduced in the paper, the reader may consult the references [27, 37, 38, 45]. And then, we use two examples to put forward two problems that we are considering.

2.1 Basic Notions

We denote by k an algebraically closed field, $\mathbf{z}$ the n variables $z_1, z_2, \ldots, z_n$, $\mathbf{z}_2$ the $(n-1)$ variables $z_2, \ldots, z_n$, where $n \geq 3$. Let $k[\mathbf{z}]$ and $k[\mathbf{z}_2]$ be the ring of polynomials in variables $\mathbf{z}$ and $\mathbf{z}_2$ with coefficients in k, respectively. Let $k[\mathbf{z}]^{l\times m}$ be the set of $l \times m$ matrices with entries in $k[\mathbf{z}]$. Without loss of generality, we assume that $l \leq m$, and for convenience we use uppercase bold letters to denote polynomial matrices. In addition, "w.r.t." and "GCD" stand for "with respect to" and "greatest common divisor", respectively.

Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ and $f \in k[\mathbf{z}_2]$, then $\mathbf{F}(f, \mathbf{z}_2)$ denotes a polynomial matrix in $k[\mathbf{z}_2]^{l\times m}$ which is formed by transforming z_1 in $\mathbf{F}$ into f. Moreover, $\mathbf{F}^{\mathrm{T}}$ represents the transposed matrix of $\mathbf{F}$. Throughout the paper, we use $d_i(\mathbf{F})$ to denote the GCD of all the $i \times i$ minors of $\mathbf{F}$ with the convention that $d_0(\mathbf{F}) = 1$, where $i = 1, \ldots, l$. Assume that $f_1, \ldots, f_s \in k[\mathbf{z}]$, we use $\langle f_1, \ldots, f_s\rangle$ to denote the ideal generated by $f_1, \ldots, f_s$ in $k[\mathbf{z}]$. Let $g, h \in k[\mathbf{z}]$, then $g \mid h$ means that g is a divisor of h.

The following concepts are from multidimensional systems theory.

Definition 2.1 ([28, 54]). Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank. For any given integer i with $1 \leq i \leq l$, let $a_1, \ldots, a_\beta$ denote all the $i\times i$ minors of $\mathbf{F}$, where $\beta = \binom{l}{i} \cdot \binom{m}{i}$. Extracting $d_i(\mathbf{F})$ from $a_1, \ldots, a_\beta$ yields

$$a_j = d_i(\mathbf{F}) \cdot b_j,\ j = 1, \ldots, \beta,$$

where $b_1, \ldots, b_\beta$ are called all the $i \times i$ reduced minors of $\mathbf{F}$.

Definition 2.2 ([60]). Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank.

(1) If all the $l \times l$ minors of $\mathbf{F}$ generate $k[\mathbf{z}]$, then $\mathbf{F}$ is said to be a zero left prime (ZLP) matrix.
(2) If all the $l \times l$ minors of $\mathbf{F}$ are relatively prime, i.e., $d_l(\mathbf{F})$ is a nonzero constant in k, then $\mathbf{F}$ is said to be a minor left prime (MLP) matrix.
(3) If for any polynomial matrix factorization $\mathbf{F} = \mathbf{F}_1\mathbf{F}_2$ with $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times l}$, $\mathbf{F}_1$ is necessarily a unimodular matrix, i.e., $\det(\mathbf{F}_1)$ is a nonzero constant in k, then $\mathbf{F}$ is said to be a factor left prime (FLP) matrix.

Zero right prime (ZRP) matrices, minor right prime (MRP) matrices and factor right prime (FRP) matrices can be similarly defined for matrices $\mathbf{F} \in k[\mathbf{z}]^{m\times l}$ with $m \geq l$. We refer to [60] for more details about the relationship among ZLP matrices, MLP matrices and FLP matrices.

For any given ZLP matrix $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$, Quillen [50] and Suslin [55] proved that an $m \times m$ unimodular matrix can be constructed such that $\mathbf{F}$ is its first l rows, respectively. This result is called Quillen-Suslin theorem, and it solved the question raised by Serre in [52].

Lemma 2.3 ([50, 55]). *If* $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ *is a ZLP matrix, then a unimodular matrix* $\mathbf{U} \in k[\mathbf{z}]^{m\times m}$ *can be constructed such that* $\mathbf{F}$ *is its first* l *rows.*

There are many algorithms for the Quillen-Suslin theorem, we refer to [43, 47, 61] for more details. In [16], Fabiańska and Quadrat first designed a Maple package, which is called QUILLENSUSLIN [17], to implement the Quillen-Suslin theorem.

Let W be a $k[\mathbf{z}]$-module generated by $\vec{u}_1, \ldots, \vec{u}_l \in k[\mathbf{z}]^{1\times m}$. The set of all $(b_1, \ldots, b_l) \in k[\mathbf{z}]^{1\times l}$ such that $b_1\vec{u}_1 + \cdots + b_l\vec{u}_l = \vec{0}$ is a $k[\mathbf{z}]$-module of $k[\mathbf{z}]^{1\times l}$, is called the (first) syzygy module of W, and denoted by $\mathrm{Syz}(W)$. Lin in [32] proposed several interesting structural properties of syzygy modules. Let $\mathbf{F} = \left[\vec{u}_1^{\mathrm{T}}, \ldots, \vec{u}_l^{\mathrm{T}}\right]^{\mathrm{T}}$. The rank of W is defined as the rank of $\mathbf{F}$ that is denoted by $\mathrm{rank}(\mathbf{F})$. Guan et al. in [21] proved that the rank of W does not depend on the choice of generators of W.

Lemma 2.4. *With above notations. If* $\mathrm{rank}(W) = r$ *with* $1 \le r \le l$*, then the rank of* $\mathrm{Syz}(W)$ *is* $l - r$*.*

Proof. Let $k(\mathbf{z})$ be the fraction field of $k[\mathbf{z}]$, and $\mathrm{Syz}^*(W) = \{\vec{v} \in k(\mathbf{z})^{1\times l} : \vec{v}\cdot\mathbf{F} = \vec{0}\}$. Then, $\mathrm{Syz}^*(W)$ is a $k(\mathbf{z})$-vector space of dimension $l - r$. For any given $l - r + 1$ different vectors $\vec{v}_1, \ldots, \vec{v}_{l-r+1} \in k[\mathbf{z}]^{1\times l}$ in $\mathrm{Syz}(W)$, it is obvious that $\vec{v}_i \in \mathrm{Syz}^*(W)$ for each i, and they are $k(\mathbf{z})$-linearly dependent. This implies that $\vec{v}_1, \ldots, \vec{v}_{l-r+1}$ are $k[\mathbf{z}]$-linearly dependent. Thus $\mathrm{rank}(\mathrm{Syz}(W)) \le l - r$.

Assume that $\vec{p}_1, \ldots, \vec{p}_{l-r} \in k(\mathbf{z})^{1\times l}$ are $l-r$ vectors in $\mathrm{Syz}^*(W)$, and they are $k(\mathbf{z})$-linearly independent. For each j, we have

$$p_{j1}\vec{u}_1 + \cdots + p_{jl}\vec{u}_l = \vec{0}, \tag{1}$$

where $\vec{p}_j = (p_{j1}, \ldots, p_{jl})$. Multiplying both sides of Equation (1) by the least common multiple of the denominators of $p_{j1}, \ldots, p_{jl}$, we obtain $\bar{p}_j = (\bar{p}_{j1}, \ldots, \bar{p}_{jl}) \in k[\mathbf{z}]$ such that $\bar{p}_{j1}\vec{u}_1 + \cdots + \bar{p}_{jl}\vec{u}_l = \vec{0}$. Then, $\bar{p}_j \in \mathrm{Syz}(W)$, where $j = 1, \ldots, l - r$. Moreover, $\bar{p}_1, \ldots, \bar{p}_{l-r}$ are $k[\mathbf{z}]$-linearly independent. Thus, $\mathrm{rank}(\mathrm{Syz}(W)) \ge l - r$.

As a consequence, the rank of $\mathrm{Syz}(W)$ is $l - r$ and the proof is completed. □

Remark 1. *Assume that* $\mathrm{Syz}(W)$ *is generated by* $\vec{v}_1, \ldots, \vec{v}_t \in k[\mathbf{z}]^{1\times l}$*, and* $\mathbf{H} = \left[\vec{v}_1^{\mathrm{T}}, \ldots, \vec{v}_t^{\mathrm{T}}\right]^{\mathrm{T}}$*. It follows from* $\mathrm{rank}(\mathbf{H}) = l - r$ *that* $t \ge l - r$*. That is, the number of vectors in any given generators of* $\mathrm{Syz}(W)$ *is greater than or equal to* $l - r$*.*

Definition 2.5 ([7]). Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$. For each $1 \le i \le l$, the ideal generated by all the $i\times i$ minors of $\mathbf{F}$ is called the i-th determinantal ideal of $\mathbf{F}$, and denoted by $I_i(\mathbf{F})$. For convenience, let $I_0(\mathbf{F}) = k[\mathbf{z}]$.

Definition 2.6 ([15]). Let W be a finitely generated $k[\mathbf{z}]$-module, and $k[\mathbf{z}]^{1\times l} \xrightarrow{\phi} k[\mathbf{z}]^{1\times m} \to W \to 0$ be a presentation of W, where ϕ acts on the right on row vectors, i.e., $\phi(\vec{u}) = \vec{u}\cdot\mathbf{F}$ for $\vec{u} \in k[\mathbf{z}]^{1\times l}$ with $\mathbf{F}$ being a presentation matrix corresponding to the linear mapping ϕ. Then the ideal $Fitt_j(W) = I_{m-j}(\mathbf{F})$ is called the j-th Fitting ideal of W. Here, we make the convention that $Fitt_j(W) = k[\mathbf{z}]$ for $j \ge m$, and that $Fitt_j(W) = 0$ for $j < \max\{m - l, 0\}$.

We remark that $Fitt_j(W)$ only depend on W (see, e.g., [15, 20]). In addition, the chain

$$0 = Fitt_{-1}(W) \subseteq Fitt_0(W) \subseteq \ldots \subseteq Fitt_m(W) = k[\mathbf{z}]$$

of Fitting ideals is increasing. We can use SINGULAR procedures to compute Fitting ideals of modules [13, 14]. Cox et al. in [12] showed that one obtains the presentation matrix $\mathbf{F}$ for W by arranging the generators of $\mathrm{Syz}(W)$ as rows. We denote the submodule of $k[\mathbf{z}]^{1\times m}$ generated by all the row vectors of $\mathbf{F}$ by $\mathrm{Im}(\mathbf{F})$, then $\mathrm{Im}(\mathbf{F}) = \mathrm{Syz}(W)$.

2.2 Matrix Factorization Problem

A matrix factorization of a multivariate polynomial matrix is formulated as follows.

Definition 2.7. Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ and $h_0 \mid d_l(\mathbf{F})$. $\mathbf{F}$ is said to admit a matrix factorization w.r.t. h_0 if $\mathbf{F}$ can be factorized as

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 \tag{2}$$

such that $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$, $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$, and $\det(\mathbf{G}_1) = h_0$. In particular, Equation (2) is said to be a ZLP (MLP, FLP) matrix factorization if $\mathbf{F}_1$ is a ZLP (MLP, FLP) matrix.

Throughout the paper, let $h = z_1 - f(\mathbf{z}_2)$ with $f(\mathbf{z}_2) \in k[\mathbf{z}_2]$. Combining Definition 2.7 and the type of polynomial matrices we mentioned in Section 1, this paper will address the following specific matrix factorization problem.

Problem 1. *Let* $\mathbf{F} \in \mathcal{M}$*. Under what condition does* $\mathbf{F}$ *have a matrix factorization w.r.t.* h*.*

So far, some results have been made on Problem 1, and the latest progress on this problem was obtained by Lu et al. [45].

Lemma 2.8 ([45]). *Let* $\mathbf{F} \in \mathcal{M}$*. If* $h \nmid d_{l-1}(\mathbf{F})$ *and the ideal generated by* h *and all the* $(l-1)\times(l-1)$ *reduced minors of* $\mathbf{F}$ *is* $k[\mathbf{z}]$*, then* $\mathbf{F}$ *admits a matrix factorization w.r.t.* h*.*

Although Lemma 2.8 gives a criterion to determine whether $\mathbf{F}$ has a matrix factorization w.r.t. h, we found that there exist some polynomial matrices in $\mathcal{M}$ which do not satisfy the conditions of Lemma 2.8, but still admit matrix factorizations w.r.t. h. Now, we use an example to illustrate this situation.

Example 2.9. Let

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}[1,1] & z_1^3 - z_2^3 - z_1^2z_3 + z_2z_3^2 & z_1z_2 - z_2z_3 & z_2^2 \\ -z_1z_2 + z_3^2 & -z_2^2 + z_1z_3 & 0 & z_2 \end{bmatrix}$$

be a polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{2\times 4}$, where $\mathbf{F}[1,1] = -2z_1z_2^2 + z_1^2z_3 + z_2^2z_3 - z_1z_3^2 + z_2z_3^2$ and $\mathbb{C}$ is the complex field.

It is easy to compute that $d_2(\mathbf{F}) = z_2(z_1 - z_3)$ and $d_1(\mathbf{F}) = 1$. Let $h = z_1 - z_3$, then $h \mid d_2(\mathbf{F})$ implies that $\mathbf{F} \in \mathcal{M}$. Obviously, $h \nmid d_1(\mathbf{F})$. Since $d_1(\mathbf{F}) = 1$, the entries in $\mathbf{F}$ are all the 1×1 reduced minors of $\mathbf{F}$. Let $\prec_{\mathbf{z}}$ be the degree reverse lexicographic order, then the reduced Gröbner basis G of the ideal generated by h and all the 1×1 reduced minors of $\mathbf{F}$ w.r.t. $\prec_{\mathbf{z}}$ is $\{z_1 - z_3, z_2, z_3^2\}$. It follows from $G \neq \{1\}$ that Lemma 2.8 cannot be applied. However, $\mathbf{F}$ has a matrix factorization w.r.t. h, i.e., there exist polynomial matrices $\mathbf{G}_1 \in \mathbb{C}[z_1, z_2, z_3]^{2\times 2}$ and $\mathbf{F}_1 \in \mathbb{C}[z_1, z_2, z_3]^{2\times 4}$ such that

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 = \begin{bmatrix} h & z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1z_3 - z_2^2 & z_1^2 - z_2z_3 & z_2 & 0 \\ -z_1z_2 + z_3^2 & -z_2^2 + z_1z_3 & 0 & z_2 \end{bmatrix},$$

where $\det(\mathbf{G}_1) = h$.

From the above example we see that Problem 1 is far from being resolved. So, in the next section we make a detailed analysis on this problem.

2.3 Matrix Equivalence Problem

Now we introduce the concept of the equivalence of two multivariate polynomial matrices.

Definition 2.10. Two polynomial matrices $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$ and $\mathbf{F}_2 \in k[\mathbf{z}]^{l\times m}$ are said to be equivalent if there exist two unimodular matrices $\mathbf{U} \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{V} \in k[\mathbf{z}]^{m\times m}$ such that

$$\mathbf{F}_1 = \mathbf{U}\mathbf{F}_2\mathbf{V}. \tag{3}$$

In fact, a univariate polynomial matrix is equivalent to its Smith form. However, this result is not valid for the case of more than one variable, and there are many counter-examples (see, e.g., [4, 26]). Hence, people began to consider under what conditions multivariate polynomial matrices in $k[\mathbf{z}]$ with $n \geq 2$ are equivalent to simpler forms. In [27], Li et al. investigated the equivalence problem of a class of multivariate polynomial matrices and obtained the following result.

Lemma 2.11 ([27]). *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times l}$ with $\det(\mathbf{F}) = h^q$, where $h = z_1 - f(\mathbf{z}_2)$ and q is a positive integer. Then $\mathbf{F}$ is equivalent to $\mathrm{diag}(1,\ldots,1,h^q)$ if and only if h^q and all the $(l-1)\times(l-1)$ minors of $\mathbf{F}$ generate $k[\mathbf{z}]$.*

For a given matrix that does not satisfy the condition of Lemma 2.11, we use the following example to illustrate that it can be equivalent to another diagonal triangle.

Example 2.12. Let $\mathbf{F} \in \mathbb{C}[z_1,z_2,z_3]^{3\times 3}$ with $\mathbb{C}$ being the complex field, where

$$\begin{cases} \mathbf{F}[1,1] = z_1z_2 - z_2^2 + z_2z_3 + z_2 - z_3 - 1, \\ \mathbf{F}[1,2] = z_1z_2z_3 - z_2^2z_3 + z_1z_2 - z_2^2 + z_2z_3 - z_3, \\ \mathbf{F}[1,3] = z_1z_2z_3 - z_2^2z_3, \\ \mathbf{F}[2,1] = z_1z_2 - z_2^2 + z_1 - z_2 + z_3 + 1, \\ \mathbf{F}[2,2] = (z_1 - z_2)(z_2z_3 + 2z_2 + z_3 + 1) + z_3, \\ \mathbf{F}[2,3] = z_1z_2z_3 - z_2^2z_3 + z_1z_2 - z_2^2 + z_1z_3 - z_2z_3, \\ \mathbf{F}[3,1] = z_1 - z_2, \\ \mathbf{F}[3,2] = z_1z_3 - z_2z_3 + 2z_1 - 2z_2, \\ \mathbf{F}[3,3] = z_1z_3 - z_2z_3 + z_1 - z_2. \end{cases}$$

It is easy to compute that $\det(\mathbf{F}) = (z_1 - z_2)^2$. Let $h = z_1 - z_2$ and $\prec_{\mathbf{z}}$ be the degree reverse lexicographic order, then the reduced Gröbner basis G of the ideal generated by h^2 and all the 2×2 minors of $\mathbf{F}$ w.r.t. $\prec_{\mathbf{z}}$ is $\{z_1 - z_2\}$. It follows from $G \neq \{1\}$ that Lemma 2.11 cannot be applied. However, $\mathbf{F}$ is equivalent to $\mathrm{diag}(1,h,h)$, i.e., there exist two unimodular polynomial matrices $\mathbf{U} \in \mathbb{C}[z_1,z_2,z_3]^{3\times 3}$ and $\mathbf{V} \in \mathbb{C}[z_1,z_2,z_3]^{3\times 3}$ such that $\mathbf{F} = \mathbf{U}\cdot\mathrm{diag}(1,h,h)\cdot\mathbf{V} =$

$$\begin{bmatrix} z_2 - 1 & z_2 & 0 \\ 1 & z_2+1 & z_2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{bmatrix}\begin{bmatrix} z_3+1 & z_3 & 0 \\ 1 & z_3+1 & z_3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Based on the phenomenon of Example 2.12, we consider the following matrix equivalence problem in this paper.

Problem 2. *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times l}$ with $\det(\mathbf{F}) = h^r$, where $h = z_1 - f(\mathbf{z}_2)$ and $1 \leq r \leq l$. What is the sufficient and necessary condition for the equivalence of $\mathbf{F}$ and $\mathrm{diag}(\underbrace{1,\ldots,1}_{l-r},\underbrace{h,\ldots,h}_{r})$?*

3 FACTORIZATION FOR POLYNOMIAL MATRICES

In this section, we first propose a new criterion to judge whether $\mathbf{F} \in \mathcal{M}$ has a matrix factorization w.r.t. h, and then we study the uniqueness of this matrix factorization. Based on the constructive algorithm proposed by Lin et al. [37] and the new criterion, we finally present a polynomial matrix factorization algorithm and use a non-trivial example to demonstrate the detailed process of the algorithm.

3.1 Matrix Factorization Theorem

We first introduce an important result, which is an answer to the generalized Serre problem proposed by Lin and Bose [31, 34].

Lemma 3.1 ([58]). *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with $\mathrm{rank}(\mathbf{F}) = r$, and all the $r\times r$ reduced minors of $\mathbf{F}$ generate $k[\mathbf{z}]$. Then there exist $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times r}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{r\times m}$ such that $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ with $\mathbf{F}_1$ being a ZLP matrix.*

Remark 2. *Since $\mathrm{rank}(\mathbf{F}) \leq \min\{\mathrm{rank}(\mathbf{G}_1), \mathrm{rank}(\mathbf{F}_1)\}$, we have $\mathrm{rank}(\mathbf{G}_1) = r$ in Lemma 3.1. This implies that $\mathbf{G}_1$ is a polynomial matrix with full column rank.*

Lemma 3.2 ([37]). *Let $p \in k[\mathbf{z}]$ and $f(\mathbf{z}_2) \in k[\mathbf{z}_2]$. If $p(f,\mathbf{z}_2)$ is a zero polynomial in $k[\mathbf{z}_2]$, then $(z_1 - f(\mathbf{z}_2))$ is a divisor of p.*

Now, we propose a new criterion to solve Problem 1.

Theorem 3.3. *Let $\mathbf{F} \in \mathcal{M}$ and $W = \mathrm{Im}(\mathbf{F}(f,\mathbf{z}_2))$. If $Fitt_{l-2}(W) = 0$ and $Fitt_{l-1}(W) = \langle d\rangle$ with $d \in k[\mathbf{z}_2]\setminus\{0\}$, then $\mathbf{F}$ admits a matrix factorization w.r.t. h.*

Proof. Let $k[\mathbf{z}_2]^{1\times s} \xrightarrow{\phi} k[\mathbf{z}_2]^{1\times l} \to W \to 0$ be a presentation of W, and $\mathbf{H} \in k[\mathbf{z}_2]^{s\times l}$ be a matrix corresponding to the linear mapping ϕ. Then $\mathrm{Syz}(W) = \mathrm{Im}(\mathbf{H})$.

It follows from $Fitt_{l-2}(W) = 0$ that all the 2×2 minors of $\mathbf{H}$ are zero polynomials. Then, $\mathrm{rank}(\mathbf{H}) \leq 1$. Moreover, $Fitt_{l-1}(W) = \langle d\rangle$ with $d \in k[\mathbf{z}_2]\setminus\{0\}$ implies that $\mathrm{rank}(\mathbf{H}) \geq 1$. As a consequence, we have $\mathrm{rank}(\mathbf{H}) = 1$.

Let $a_1,\ldots,a_\beta \in k[\mathbf{z}_2]$ and $b_1,\ldots,b_\beta \in k[\mathbf{z}_2]$ be all the 1×1 minors and reduced minors of $\mathbf{H}$, respectively. Then, $a_i = d_1(\mathbf{H})\cdot b_i$ for $i = 1,\ldots,\beta$. Since $\langle a_1,\ldots,a_\beta\rangle = \langle d\rangle$, it is obvious that $d \mid d_1(\mathbf{H})$. Moreover, we have $d = \sum_{i=1}^{\beta} c_ia_i$ for some $c_i \in k[\mathbf{z}_2]$. Thus $d = d_1(\mathbf{H})\cdot(\sum_{i=1}^{\beta} c_ib_i)$. This implies that $d_1(\mathbf{H}) \mid d$. Hence $d = \delta\cdot d_1(\mathbf{H})$, where δ is a nonzero constant. Therefore, $\langle b_1,\ldots,b_\beta\rangle = k[\mathbf{z}_2]$.

According to Lemma 3.1, there exist $\mathbf{G} \in k[\mathbf{z}_2]^{s\times 1}$ and $\mathbf{H}_1 \in k[\mathbf{z}_2]^{1\times l}$ such that $\mathbf{H} = \mathbf{G}\mathbf{H}_1$ with $\mathbf{H}_1$ being a ZLP matrix. It follows from $\mathrm{Syz}(W) = \mathrm{Im}(\mathbf{H})$ that $\mathbf{G}\mathbf{H}_1\mathbf{F}(f,\mathbf{z}_2) = \mathbf{0}_{s\times m}$. Since $\mathbf{G}$ is a matrix with full column rank, we have $\mathbf{H}_1\mathbf{F}(f,\mathbf{z}_2) = \mathbf{0}_{1\times m}$.

Using the Quillen-Suslin theorem, we can construct a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{l\times l}$ such that $\mathbf{H}_1$ is its first row. Let $\mathbf{F}_0 = \mathbf{U}\mathbf{F}$, then the first row of $\mathbf{F}_0(f,\mathbf{z}_2) = \mathbf{U}\mathbf{F}(f,\mathbf{z}_2)$ is zero vector. By

Lemma 3.2, h is a common divisor of the polynomials in the first row of $\mathbf{F}_0$, thus

$$\mathbf{F}_0 = \mathbf{U}\mathbf{F} = \mathbf{D}\mathbf{F}_1 = \mathrm{diag}(h, \underbrace{1, \ldots, 1}_{l-1}) \cdot \begin{bmatrix} \bar{f}_{11} & \bar{f}_{12} & \cdots & \bar{f}_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{f}_{l1} & \bar{f}_{l2} & \cdots & \bar{f}_{lm} \end{bmatrix}.$$

Consequently, we can now derive the matrix factorization of $\mathbf{F}$ w.r.t. h:

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1,$$

where $\mathbf{G}_1 = \mathbf{U}^{-1}\mathbf{D} \in k[\mathbf{z}]^{l\times l}$, $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$ and $\det(\mathbf{G}_1) = h$. □

According to the proof of Theorem 3.3, it is easy to get a more general result below.

Theorem 3.4. *Let* $\mathbf{F} \in \mathcal{M}$ *and* $W = \mathrm{Im}(\mathbf{F}(f, \mathbf{z}_2))$. *If* $Fitt_{r-1}(W) = 0$ *and* $Fitt_r(W) = \langle d \rangle$ *with* $d \in k[\mathbf{z}_2] \setminus \{0\}$ *and* $0 \le r \le l-1$, *then* $\mathbf{F}$ *admits a matrix factorization w.r.t.* h^{l-r}.

Remark 3. *In Theorem 3.4, it follows from* $Fitt_{r-1}(W) = 0$ *and* $Fitt_r(W) = \langle d \rangle$ *that* $\mathrm{rank}(\mathbf{H}) = l - r$, *where* $\mathrm{Syz}(W) = \mathrm{Im}(\mathbf{H})$. *Based on Lemma 2.4, we have* $\mathrm{rank}(\mathbf{F}(f, \mathbf{z}_2)) = \mathrm{rank}(W) = r$. $\mathbf{F} \in \mathcal{M}$ *implies that* $h = z_1 - f(\mathbf{z}_2)$ *is a divisor of* $d_l(\mathbf{F})$, *and it is easy to show that* $\mathrm{rank}(\mathbf{F}(f, \mathbf{z}_2)) \le l - 1$. *Thus, we have* $r \le l - 1$. *When* $r = 0$, $\mathrm{rank}(\mathbf{F}(f, \mathbf{z}_2)) = 0$ *implies that* $h \mid d_1(\mathbf{F})$. *Then, we can extract* h *from each row of* $\mathbf{F}$ *and obtain a matrix factorization of* $\mathbf{F}$ *w.r.t.* h^l.

Let $k[\bar{\mathbf{z}}_j] = k[z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n]$, where $1 \le j \le n$. We construct a new set of polynomial matrices: $\mathcal{M}_j = \{\mathbf{F} \in k[\mathbf{z}]^{l\times m} : h_j \mid d_l(\mathbf{F})\}$, where $h_j = z_j - f(\bar{\mathbf{z}}_j)$ with $f(\bar{\mathbf{z}}_j)$ being a polynomial in $k[\bar{\mathbf{z}}_j]$. Then, we can get the following corollary.

Corollary 3.5. *Let* $\mathbf{F} \in \mathcal{M}_j$ *and* $W = \mathrm{Im}(\mathbf{F}(z_1, \ldots, z_{j-1}, f, z_{j+1}, \ldots, z_n))$. *If* $Fitt_{r-1}(W) = 0$ *and* $Fitt_r(W) = \langle d \rangle$ *with* $d \in k[\bar{\mathbf{z}}_j] \setminus \{0\}$ *and* $0 \le r \le l - 1$, *then* $\mathbf{F}$ *admits a matrix factorization w.r.t.* h_j^{l-r}.

3.2 Uniqueness of Polynomial Matrix Factorizations

In [42], Liu and Wang studied the uniqueness problem of polynomial matrix factorizations. They pointed out that for a non-regular factor h_0 of $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$, under the condition that there exists a matrix factorization $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ with $\det(\mathbf{G}_1) = h_0$, $\mathrm{Im}(\mathbf{F}_1)$ is not uniquely determined. In other words, when $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 = \mathbf{G}_2\mathbf{F}_2$ with $\det(\mathbf{G}_1) = \det(\mathbf{G}_2) = h_0$, $\mathrm{Im}(\mathbf{F}_1)$ and $\mathrm{Im}(\mathbf{F}_2)$ might not be the same.

Let $\mathbf{F} \in \mathcal{M}$ satisfy the conditions of Theorem 3.4. According to the proof of Theorem 3.3, we can select different generators of $\mathrm{Syz}(W)$ and obtain different presentation matrices of W. Then, we can construct different unimodular matrices and get different matrix factorizations of $\mathbf{F}$ w.r.t. h^{l-r}. Hence, in the following we study the uniqueness of matrix factorizations of $\mathbf{F}$ w.r.t. h^{l-r}.

Theorem 3.6. *Let* $\mathbf{F} \in \mathcal{M}$ *satisfy* $\mathbf{F} = \mathbf{U}_1^{-1}\mathbf{D}\mathbf{F}_1 = \mathbf{U}_2^{-1}\mathbf{D}\mathbf{F}_2$, *where* $\mathbf{U}_1, \mathbf{U}_2$ *are two unimodular matrices in* $k[\mathbf{z}_2]^{l\times l}$, *and* $\mathbf{D} = \mathrm{diag}(\underbrace{h, \ldots, h}_{l-r}, \underbrace{1, \ldots, 1}_{r})$. *Then,* $\mathrm{Im}(\mathbf{F}_1) = \mathrm{Im}(\mathbf{F}_2)$.

Proof. Let $\mathbf{F}_1 = \left[\vec{u}_1^{\mathrm{T}}, \ldots, \vec{u}_l^{\mathrm{T}}\right]^{\mathrm{T}}$ and $\mathbf{F}_2 = \left[\vec{v}_1^{\mathrm{T}}, \ldots, \vec{v}_l^{\mathrm{T}}\right]^{\mathrm{T}}$, where $\vec{u}_1, \ldots, \vec{u}_l, \vec{v}_1, \ldots, \vec{v}_l \in k[\mathbf{z}]^{1\times m}$. So, $\mathrm{Im}(\mathbf{F}_1) = \langle \vec{u}_1, \ldots, \vec{u}_l \rangle$ and $\mathrm{Im}(\mathbf{F}_2) = \langle \vec{v}_1, \ldots, \vec{v}_l \rangle$.

Let $\mathbf{F}_{01} = \mathbf{U}_1\mathbf{F}$ and $\mathbf{F}_{02} = \mathbf{U}_2\mathbf{F}$. Then $\mathbf{F}_{01} = \mathbf{D}\mathbf{F}_1$ and $\mathbf{F}_{02} = \mathbf{D}\mathbf{F}_2$. It follows that $\mathbf{F}_{01} = \left[h\vec{u}_1^{\mathrm{T}}, \ldots, h\vec{u}_{l-r}^{\mathrm{T}}, \vec{u}_{l-r+1}^{\mathrm{T}}, \ldots, \vec{u}_l^{\mathrm{T}}\right]^{\mathrm{T}}$ and $\mathbf{F}_{02} = \left[h\vec{v}_1^{\mathrm{T}}, \ldots, h\vec{v}_{l-r}^{\mathrm{T}}, \vec{v}_{l-r+1}^{\mathrm{T}}, \ldots, \vec{v}_l^{\mathrm{T}}\right]^{\mathrm{T}}$. Since $\mathbf{U}_1$ and $\mathbf{U}_2$ are two unimodular matrices in $k[\mathbf{z}_2]^{l\times l}$, we have $\mathbf{F}_{01} = \mathbf{U}_1\mathbf{U}_2^{-1}\mathbf{F}_{02}$. This implies that there exist polynomials $a_{i1}, \ldots, a_{il} \in k[\mathbf{z}_2]$ such that

$$h\vec{u}_i = h \cdot (\sum_{j=1}^{l-r} a_{ij}\vec{v}_j) + \sum_{j=l-r+1}^{l} a_{ij}\vec{v}_j,$$

where $i = 1, \ldots, l - r$. Then, for each i setting z_1 of the above equation to $f(\mathbf{z}_2)$, we have

$$a_{i(l-r+1)}\vec{v}_{l-r+1}(f, \mathbf{z}_2) + \cdots + a_{il}\vec{v}_l(f, \mathbf{z}_2) = \vec{0}.$$

As $\mathrm{rank}(\mathbf{F}(f, \mathbf{z}_2)) = r$ and $\mathrm{rank}(\mathbf{F}_{02}(f, \mathbf{z}_2)) = \mathrm{rank}(\mathbf{F}(f, \mathbf{z}_2))$, we have that $\vec{v}_{l-r+1}(f, \mathbf{z}_2), \ldots, \vec{v}_l(f, \mathbf{z}_2)$ are $k[\mathbf{z}_2]$-linearly independent. This implies that $a_{i(l-r+1)} = \cdots = a_{il} = 0$. Hence,

$$\vec{u}_i = a_{i1}\vec{v}_1 + \cdots + a_{i(l-r)}\vec{v}_{l-r},$$

where $i = 1, \ldots, l - r$. Obviously, $\vec{u}_j$ is a $k[\mathbf{z}]$-linear combination of $\vec{v}_1, \ldots, \vec{v}_l$, where $j = l - r + 1, \ldots, l$. As a consequence, $\langle \vec{u}_1, \ldots, \vec{u}_l \rangle \subset \langle \vec{v}_1, \ldots, \vec{v}_l \rangle$. We can use the same method to prove that $\langle \vec{v}_1, \ldots, \vec{v}_l \rangle \subset \langle \vec{u}_1, \ldots, \vec{u}_l \rangle$.

Therefore, we have $\mathrm{Im}(\mathbf{F}_1) = \mathrm{Im}(\mathbf{F}_2)$. □

Based on Theorem 3.4 and Theorem 3.6, we can now derive the conclusion: if $\mathbf{F} \in \mathcal{M}$ satisfies the conditions of Theorem 3.4, then we have $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ with $\det(\mathbf{G}_1) = h^{l-r}$ and $\mathrm{Im}(\mathbf{F}_1)$ uniquely determined, where $\mathbf{G}_1 = \mathbf{U}^{-1}\mathbf{D}$ with $\mathbf{U} \in k[\mathbf{z}_2]^{l\times l}$ a unimodular matrix and $\mathbf{D} = \mathrm{diag}(h, \ldots, h, 1, \ldots, 1)$.

3.3 Algorithm

Combining the algorithm proposed in [37] and the matrix factorization conditions of Theorem 3.4, we get the following algorithm for factoring matrices in $\mathcal{M}$.

Before proceeding further, let us remark on Algorithm 1.

- Step 2 implies that $\mathrm{rank}(W) = r$.
- In Step 7, $\mathbf{H}$ is a presentation matrix of W. By Lemma 2.4, we have $\mathrm{rank}(\mathbf{H}) = l - r$. Thus, $Fitt_{r-1}(W) = I_{l-r+1}(\mathbf{H}) = 0$.
- In Step 9, $\#(\mathcal{G})$ stands for the number of generators in $\mathcal{G}$, $\#(\mathcal{G}) = 1$ implies that $Fitt_r(W)$ is a principal ideal in $k[\mathbf{z}_2]$.
- From Step 10 to Step 12, we refer to [44, 45] for more details.
- In Step 15, we need to find another new criterion to judge whether $\mathbf{F}$ has a matrix factorization w.r.t. h^{l-r}.

Now, we use an example to illustrate the calculation process of Algorithm 1.

Example 3.7. Let

$$\mathbf{F} = \begin{bmatrix} z_1^2 - z_1z_2 & z_2z_3 + z_3^2 + z_2 + z_3 & -z_2z_3 - z_2 \\ z_1z_2 - z_2^2 & -z_1z_3 + z_2z_3 & z_1^3 - z_1^2z_2 + z_1z_2 - z_2^2 \\ 0 & z_2 + z_3 & -z_2 \end{bmatrix}$$

be a multivariate polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{3\times 3}$, where $z_1 > z_2 > z_3$ and $\mathbb{C}$ is the complex field.

Algorithm 1: polynomial matrix factorization algorithm

Input : $\mathbf{F} \in \mathcal{M}$, $h = z_1 - f(\mathbf{z}_2)$ and a monomial order $\prec_{\mathbf{z}_2}$ in $k[\mathbf{z}_2]$.

Output : a matrix factorization of $\mathbf{F}$ w.r.t. h^{l-r}, where r is the rank of $\mathbf{F}(f, \mathbf{z}_2)$.

begin
- compute the rank r of $\mathbf{F}(f, \mathbf{z}_2)$;
- **if** $r = 0$ **then**
  - extract h from each row of $\mathbf{F}$ and obtain $\mathbf{F}_1$, i.e., $\mathbf{F} = \mathrm{diag}(h, \ldots, h) \cdot \mathbf{F}_1$;
  - **return** $\mathrm{diag}(h, \ldots, h)$ and $\mathbf{F}_1$.
- compute a Gröbner basis $\{\vec{h}_1, \ldots, \vec{h}_s\}$ of the syzygy module of $W = \mathrm{Im}(\mathbf{F}(f, \mathbf{z}_2))$;
- let $\mathbf{H}$ be a matrix in $k[\mathbf{z}_2]^{s\times l}$ composed of $\vec{h}_1, \ldots, \vec{h}_s$;
- compute a reduced Gröbner basis $\mathcal{G}$ of the $(l-r)$-th determinantal ideal of $\mathbf{H}$ w.r.t. $\prec_{\mathbf{z}_2}$;
- **if** $\#(\mathcal{G}) = 1$ **then**
  - compute a ZLP matrix factorization of $\mathbf{H}$ and obtain a ZLP matrix $\mathbf{H}_1 \in k[\mathbf{z}_2]^{(l-r)\times l}$;
  - construct a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{l\times l}$ such that $\mathbf{H}_1$ is its first $l-r$ rows;
  - extract h from the first $l-r$ rows of $\mathbf{UF}$ and obtain $\mathbf{F}_1$, i.e., $\mathbf{UF} = \mathrm{diag}(h, \ldots, h, 1, \ldots, 1) \cdot \mathbf{F}_1$;
  - **return** $\mathbf{U}^{-1} \cdot \mathrm{diag}(h, \ldots, h, 1, \ldots, 1)$ and $\mathbf{F}_1$.
- **else**
  - **return** unable to judge.

It is easy to compute that $d_3(\mathbf{F}) = -z_1(z_1 - z_2)^2(z_1^2 z_2 + z_1^2 z_3 + z_2^2)$, $d_2(\mathbf{F}) = z_1 - z_2$ and $d_1(\mathbf{F}) = 1$. Let $h = z_1 - z_2$ and $\prec_{z_2,z_3}$ be the degree reverse lexicographic order. Then, the input of Algorithm 1 are $\mathbf{F}$, $h = z_1 - z_2$ and $\prec_{z_2,z_3}$.

Note that

$$\mathbf{F}(z_2, z_2, z_3) = \begin{bmatrix} 0 & (z_2+z_3)(z_3+1) & -z_2(z_3+1) \\ 0 & 0 & 0 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix},$$

the rank of $\mathbf{F}(z_2, z_2, z_3)$ is $r = 1$. Let $W = \mathrm{Im}(\mathbf{F}(z_2, z_2, z_3))$. Then, we use *Singular* command "syz" to compute a Gröbner basis of the syzygy module of W, and obtain

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & -z_3-1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy to check that the reduced Gröbner basis of all the 2×2 minors of $\mathbf{H}$ w.r.t. $\prec_{z_2,z_3}$ is $\mathcal{G} = \{1\}$. Then, $Fitt_1(W) = I_2(\mathbf{H}) = \langle 1\rangle$ and $\mathbf{H}$ is a ZLP matrix. This implies that $\mathbf{H}_1 = \mathbf{H}$. $\mathbf{H}_1$ can be easily extended as the first 2 rows of a unimodular matrix

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & -z_3-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can extract h from the first 2 rows of $\mathbf{UF}$, and get

$$\mathbf{UF} = \mathbf{DF}_1 = \begin{bmatrix} z_1-z_2 & 0 & 0 \\ 0 & z_1-z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 & 0 & 0 \\ z_2 & -z_3 & z_1^2+z_2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix}.$$

Then, we obtain a matrix factorization of $\mathbf{F}$ w.r.t. h^2: $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 = (\mathbf{U}^{-1}\mathbf{D})\mathbf{F}_1 =$

$$\begin{bmatrix} z_1-z_2 & 0 & z_3+1 \\ 0 & z_1-z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 & 0 & 0 \\ z_2 & -z_3 & z_1^2+z_2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix},$$

where $\det(\mathbf{G}_1) = \det(\mathbf{U}^{-1}\mathbf{D}) = h^2$.

At this moment, $d_3(\mathbf{F}_1) = -z_1(z_1^2 z_2 + z_1^2 z_3 + z_2^2)$. We reuse Algorithm 1 to judge whether $\mathbf{F}_1$ has a matrix factorization w.r.t. z_1. Similarly, we obtain

$$\mathbf{F}_1 = \mathbf{G}_2\mathbf{F}_2 = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ z_2 & -z_3 & z_1^2+z_2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix},$$

where $\det(\mathbf{G}_2) = z_1$.

Therefore, we obtain a matrix factorization of $\mathbf{F}$ w.r.t. $z_1(z_1 - z_2)^2$, i.e., $\mathbf{F} = \mathbf{GF}_2 = (\mathbf{G}_1\mathbf{G}_2)\mathbf{F}_2 =$

$$\begin{bmatrix} z_1(z_1-z_2) & 0 & z_3+1 \\ 0 & z_1-z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ z_2 & -z_3 & z_1^2+z_2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix},$$

where $\det(\mathbf{G}) = z_1(z_1 - z_2)^2$.

Remark 4. *In Example 3.7, we can first judge whether* $\mathbf{F}$ *has a matrix factorization w.r.t.* z_1. *Note that*

$$\mathbf{F}(0, z_2, z_3) = \begin{bmatrix} 0 & (z_2+z_3)(z_3+1) & -z_2(z_3+1) \\ -z_2^2 & z_2 z_3 & -z_2^2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix},$$

the rank of $\mathbf{F}(0, z_2, z_3)$ *is* $r = 2$. *We compute a Gröbner basis of the syzygy module of* $\mathrm{Im}(\mathbf{F}(0, z_2, z_3))$ *and get* $\mathbf{H} = \begin{bmatrix} -1 & 0 & z_3+1 \end{bmatrix}$. *Since the reduced Gröbner basis of all the* 1×1 *minors of* $\mathbf{H}$ *w.r.t.* $\prec_{z_2,z_3}$ *is* $\mathcal{G} = \{1\}$. *Then,* $Fitt_2(W) = I_1(\mathbf{H}) = \langle 1\rangle$. *This implies that* $\mathbf{F}$ *has a matrix factorization w.r.t.* z_1. *According to the above calculations, we have the following conclusions:* $\mathbf{F}$ *has matrix factorizations w.r.t.* z_1, $z_1 - z_2$, $z_1(z_1 - z_2)$, $(z_1 - z_2)^2$ *and* $z_1(z_1 - z_2)^2$, *respectively.*

4 EQUIVALENCE FOR POLYNOMIAL MATRICES

In this section, we first put forward a sufficient and necessary condition to solve Problem 2, and then we use an example to illustrate the effectiveness of the new matrix equivalence theorem.

4.1 Matrix Equivalence Theorem

We first introduce a lemma, which is a generalization of Binet-Cauchy formula in [53].

Lemma 4.1 ([53]). *Let* $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$, *where* $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$ *and* $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$. *Then an* $i \times i$ $(1 \le i \le l)$ *minor of* $\mathbf{F}$ *is*

$$\det\left(\mathbf{F}\begin{pmatrix} r_1\cdots r_i \\ j_1\cdots j_i \end{pmatrix}\right) = \sum_{1\le s_1<\cdots<s_i\le l} \det\left(\mathbf{G}_1\begin{pmatrix} r_1\cdots r_i \\ s_1\cdots s_i \end{pmatrix}\right)\cdot\det\left(\mathbf{F}_1\begin{pmatrix} s_1\cdots s_i \\ j_1\cdots j_i \end{pmatrix}\right).$$

In Lemma 4.1, $\mathbf{F}\begin{pmatrix} r_1\cdots r_i \\ j_1\cdots j_i \end{pmatrix}$ denotes an $i \times i$ sub-matrix consisting of the $r_1, \ldots, r_i$ rows and $j_1, \ldots, j_i$ columns of $\mathbf{F}$. Based on this lemma, we can obtain the following two results.

Lemma 4.2. *Let* $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ *be of full row rank with* $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$, *where* $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$ *and* $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$. *Then* $d_i(\mathbf{F}_1) \mid d_i(\mathbf{F})$ *and* $d_i(\mathbf{G}_1) \mid d_i(\mathbf{F})$ *for each* $i \in \{1,\ldots,l\}$.

Proof. We only prove $d_i(\mathbf{F}_1) \mid d_i(\mathbf{F})$, since the proof of $d_i(\mathbf{G}_1) \mid d_i(\mathbf{F})$ follows in a similar manner. For any given $i \in \{1,\ldots,l\}$, let $a_{i,1},\ldots,a_{i,t_i}$ and $\bar{a}_{i,1},\ldots,\bar{a}_{i,t_i}$ be all the $i\times i$ minors of $\mathbf{F}$ and $\mathbf{F}_1$ respectively, where $t_i = \binom{l}{i}\binom{m}{i}$. For each $a_{i,j}$, it is a $k[\mathbf{z}]$-linear combination of $\bar{a}_{i,1},\ldots,\bar{a}_{i,t_i}$ by using Lemma 4.1, where $j = 1,\ldots,t_i$. Since $d_i(\mathbf{F}_1) = \mathrm{GCD}(\bar{a}_{i,1},\ldots,\bar{a}_{i,t_i})$, for each j we have $d_i(\mathbf{F}_1) \mid a_{i,j}$. Then, $d_i(\mathbf{F}_1) \mid d_i(\mathbf{F})$. □

Lemma 4.3. *Let* $\mathbf{F}_1, \mathbf{F}_2 \in k[\mathbf{z}]^{l\times m}$ *be of full row rank. If* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *are equivalent, then* $d_i(\mathbf{F}_1) = d_i(\mathbf{F}_2)$ *for each* $i \in \{1,\ldots,l\}$.

Proof. Since $\mathbf{F}_1$ and $\mathbf{F}_2$ are equivalent, then there exist two unimodular matrices $\mathbf{U} \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{V} \in k[\mathbf{z}]^{m\times m}$ such that $\mathbf{F}_1 = \mathbf{U}\mathbf{F}_2\mathbf{V}$. For each $i \in \{1,\ldots,l\}$, it follows from Lemma 4.2 that $d_i(\mathbf{F}_2) \mid d_i(\mathbf{U}\mathbf{F}_2) \mid d_i(\mathbf{F}_1)$. Furthermore, we have $\mathbf{F}_2 = \mathbf{U}^{-1}\mathbf{F}_1\mathbf{V}^{-1}$ since $\mathbf{U}$ and $\mathbf{V}$ are two unimodular matrices. Similarly, we obtain $d_i(\mathbf{F}_1) \mid d_i(\mathbf{U}^{-1}\mathbf{F}_1) \mid d_i(\mathbf{F}_2)$. Therefore, $d_i(\mathbf{F}_1) = d_i(\mathbf{F}_2)$. □

Before presenting the matrix equivalence theorem, we introduce a lemma which plays an important role in our proof.

Lemma 4.4 ([44]). *Let* $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ *with* $\mathrm{rank}(\mathbf{F}) = r$. *If all the* $r\times r$ *minors of* $\mathbf{F}$ *generate* $k[\mathbf{z}]$, *then there exists a ZLP matrix* $\mathbf{H} \in k[\mathbf{z}]^{(l-r)\times l}$ *such that* $\mathbf{H}\mathbf{F} = \mathbf{0}_{(l-r)\times m}$.

Combining Lemma 4.4 and the Quillen-Suslin theorem, we can now solve Problem 2.

Theorem 4.5. *Let* $\mathbf{F} \in k[\mathbf{z}]^{l\times l}$ *with* $\det(\mathbf{F}) = h^r$, *where* $h = z_1 - f(\mathbf{z}_2)$ *and* $1 \le r \le l$. *Then* $\mathbf{F}$ *and* $\mathrm{diag}(1,\ldots,1,h,\ldots,h)$ *are equivalent if and only if* $h \mid d_{l-r+1}(\mathbf{F})$ *and the ideal generated by* h *and all the* $(l-r)\times(l-r)$ *minors of* $\mathbf{F}$ *is* $k[\mathbf{z}]$.

Proof. For convenience, let $\mathbf{D} = \mathrm{diag}(1,\ldots,1,h,\ldots,h)$ and $\bar{\mathbf{F}} = \mathbf{F}(f,\mathbf{z}_2)$. Let $a_1,\ldots,a_\beta$ be all the $(l-r)\times(l-r)$ minors of $\mathbf{F}$. It is obvious that $a_1(f,\mathbf{z}_2),\ldots,a_\beta(f,\mathbf{z}_2)$ are all the $(l-r)\times(l-r)$ minors of $\bar{\mathbf{F}}$.

Sufficiency. It follows from $h \mid d_{l-r+1}(\mathbf{F})$ that $\mathrm{rank}(\bar{\mathbf{F}}) \le l-r$. Assume that there exists a point $(\varepsilon_2,\ldots,\varepsilon_n) \in k^{1\times(n-1)}$ such that

$$a_i(f(\varepsilon_2,\ldots,\varepsilon_n),\varepsilon_2,\ldots,\varepsilon_n) = 0,\ i = 1,\ldots,\beta. \tag{4}$$

Let $\varepsilon_1 = f(\varepsilon_2,\ldots,\varepsilon_n)$, then Equation (4) implies that $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n) \in k^{1\times n}$ is a common zero of the polynomial system $\{h = 0, a_1 = 0,\ldots,a_\beta = 0\}$. This contradicts the fact that h and all the $(l-r)\times(l-r)$ minors of $\mathbf{F}$ generate $k[\mathbf{z}]$. Then, all the $(l-r)\times(l-r)$ minors of $\bar{\mathbf{F}}$ generate $k[\mathbf{z}_2]$. According to Lemma 4.4, there exists a ZLP matrix $\mathbf{H} \in k[\mathbf{z}_2]^{r\times l}$ such that $\mathbf{H}\bar{\mathbf{F}} = \mathbf{0}_{r\times l}$. Based on the Quillen-Suslin theorem, we can construct a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{l\times l}$ such that $\mathbf{H}$ is its last r rows. Then, there is a polynomial matrix $\mathbf{V} \in k[\mathbf{z}]^{l\times l}$ such that $\mathbf{U}\mathbf{F} = \mathbf{D}\mathbf{V}$. Since $\det(\mathbf{F}) = h^r$ and $\mathbf{U}$ is a unimodular matrix, we have $\mathbf{F} = \mathbf{U}^{-1}\mathbf{D}\mathbf{V}$ and $\mathbf{V}$ is a unimodular matrix. Therefore, $\mathbf{F}$ and $\mathbf{D}$ are equivalent.

Necessity. If $\mathbf{F}$ and $\mathbf{D}$ are equivalent, then there exist two unimodular matrices $\mathbf{U} \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{V} \in k[\mathbf{z}]^{l\times l}$ such that $\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}$. It follows from Lemma 4.3 that $d_{l-r+1}(\mathbf{F}) = d_{l-r+1}(\mathbf{D}) = h$. If $\langle h, a_1,\ldots,a_\beta\rangle \ne k[\mathbf{z}]$, then there exists a point $\vec{\varepsilon} \in k^{1\times n}$ such that $h(\vec{\varepsilon}) = 0$ and $\mathrm{rank}(\mathbf{F}(\vec{\varepsilon})) < l-r$. Obviously, $\mathrm{rank}(\mathbf{D}(\vec{\varepsilon})) = l-r$ and $\mathrm{rank}(\mathbf{U}^{-1}(\vec{\varepsilon})) = \mathrm{rank}(\mathbf{V}^{-1}(\vec{\varepsilon})) = l$. Since $\mathbf{D} = \mathbf{U}^{-1}\mathbf{F}\mathbf{V}^{-1}$, we have

$$\mathrm{rank}(\mathbf{D}(\vec{\varepsilon})) \le \min\{\mathrm{rank}(\mathbf{U}^{-1}(\vec{\varepsilon})), \mathrm{rank}(\mathbf{F}(\vec{\varepsilon})), \mathrm{rank}(\mathbf{V}^{-1}(\vec{\varepsilon}))\},$$

which leads to a contradiction. Therefore, $\langle h, a_1,\ldots,a_\beta\rangle = k[\mathbf{z}]$ and the proof is completed. □

Remark 5. *When* $r = l$ *in Theorem 4.5, we just need to check whether* h *is a divisor of* $d_1(\mathbf{F})$.

4.2 Example

Now, we use Example 2.12 to illustrate a constructive method which follows Lin et al. in [35] and explain how to obtain the two unimodular matrices associated with equivalent matrices in Theorem 4.5.

Example 4.6. Let $\mathbf{F}$ be the same polynomial matrix as in Example 2.12. It is easy to compute that $\det(\mathbf{F}) = (z_1 - z_2)^2$ and $d_2(\mathbf{F}) = z_1 - z_2$. Let $h = z_1 - z_2$, it is obvious that $h \mid d_2(\mathbf{F})$. The reduced Gröbner basis of the ideal generated by h and all the 1×1 minors of $\mathbf{F}$ w.r.t. $\prec_{\mathbf{z}}$ is $\{1\}$. Then, $\mathbf{F}$ is equivalent to $\mathrm{diag}(1,h,h)$.

Note that

$$\mathbf{F}(z_2,z_2,z_3) = \begin{bmatrix} (z_3+1)(z_2-1) & z_3(z_2-1) & 0 \\ z_3+1 & z_3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the rank of $\mathbf{F}(z_2,z_2,z_3)$ is $r = 1$. According to the calculation process of Example 3.7, we can get a ZLP matrix

$$\mathbf{H} = \begin{bmatrix} 1 & -z_2+1 & z_2^2 - z_2 \\ -1 & z_2 - 1 & -z_2^2 + z_2 + 1 \end{bmatrix}$$

such that $\mathbf{H}\cdot\mathbf{F}(z_2,z_2,z_3) = \mathbf{0}_{2\times 3}$. Then, a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{3\times 3}$ can be constructed such that $\mathbf{H}$ is its the last 2 rows, where

$$\mathbf{U} = \begin{bmatrix} -1 & z_2 & -z_2^2 \\ 1 & -z_2+1 & z_2^2 - z_2 \\ -1 & z_2 - 1 & -z_2^2 + z_2 + 1 \end{bmatrix}.$$

Now we can extract h from the last 2 rows of $\mathbf{U}\mathbf{F}$, and get $\mathbf{F} = \mathbf{U}^{-1}\cdot\mathrm{diag}(1,h,h)\cdot\mathbf{V} =$

$$\begin{bmatrix} z_2-1 & z_2 & 0 \\ 1 & z_2+1 & z_2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{bmatrix}\begin{bmatrix} z_3+1 & z_3 & 0 \\ 1 & z_3+1 & z_3 \\ 0 & 1 & 1 \end{bmatrix}.$$

5 CONCLUDING REMARKS

In this paper, we point out two directions of research in which multivariate polynomial matrices have been explored. The first is concerned with the factorization problem of multivariate polynomial matrices in $\mathcal{M}$, and the second direction is devoted to the investigation of the equivalence problem of square matrices in $\mathcal{M}$.

The main contributions of this paper include: (1) a new criterion (Theorem 3.4) and an algorithm (Algorithm 1) are given to factorize $\mathbf{F} \in \mathcal{M}$ w.r.t. h^{l-r}, as a consequence, the application range of the constructive algorithm in [37] has been greatly extended; (2) Theorem 3.6 shows that the output of Algorithm 1 is unique if $\mathbf{F}$ satisfies the new criterion; (3) a sufficient and necessary condition (Theorem 4.5) is proposed to judge whether a square polynomial matrix $\mathbf{F}$ with $\det(\mathbf{F}) = h^r$ is equivalent to $\mathrm{diag}(1,\ldots,1,h,\ldots,h)$; (4) a generalization about the type of polynomial matrices has been presented (Corollary 3.5) and the implementation of two main theorems (Theorem 3.4 and Theorem 4.5) has been illustrated by two non-trivial examples.

If $\#(\mathcal{G}) \neq 1$, then Algorithm 1 returns "unable to judge". At this moment, how to establish a necessary and sufficient condition for $\mathbf{F} \in \mathcal{M}$ admitting a matrix factorization w.r.t. h^{l-r} is the question that remains for further investigation.

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