Hilbert Problem 15 and Ritt-Wu Method (II)*

LI Banghe · WANG Dingkang

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Abstract This paper proves three statements of Schubert about cuspal cubic curves in a plane by using the concept of generic point of Van der Waerden and Weil and Ritt-Wu methods. They are relations of some special lines: 1) For a given point, all the curves containing this point are considered. For any such curve, there are five lines. Two of them are the tangent lines of the curve passing through the given point. The other three are the lines connecting the given point with the cusp, the inflexion point and the intersection point of the tangent line at the cusp and the inflexion line. 2) For a given point, the curves whose tangent line at the cusp passes through this point are considered. For any such curve, there are four lines. Three of them are the tangent lines passing through this point and the other is the line connect the given point and the inflexion point. 3) For a given point, the curves whose cusp, inflexion point and the given point are collinear are considered. For any such curve, there are five lines. Three of them are tangent lines passing through the given point. The other two are the lines connecting the given point with the cusp and the intersection point of the tangent line at the cusp and the inflexion line.

Keywords Cubic curves with cusp, Hilbert problem 15, Ritt-Wu method.

1 Introduction

This paper is a subsequent one to [1].

For any planar cubic curves with cusp in $\mathbb{C}P^3$, there are three special points, the first is the cusp point, the second is the inflexion point, the third is the intersection point of the inflexion tangent line and the tangent line at the cusp point. The triangle decided by these three points are called singular triangle.

LI Banghe
KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190*, China.*
Email: libh@amss.ac.cn.
WANG Dingkang
KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190*, China;*
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049*, China.*
Email: dwang@mmrc.iss.ac.cn.
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In [1], we let

$$
\begin{aligned}
f = &(a_3(x-a_1)+(y-a_2))^2+a_4(x-a_1)^3+a_5(x-a_1)^2(y-a_2)\\
&+a_6(x-a_1)(y-a_2)^2+a_7(y-a_2)^3
\end{aligned}
$$

with $a_i \in \mathbb{C}$, then $f=0$ defines a curve in $\mathbb{C}^2 \subset \mathbb{C}P^2$.

In this paper, we always assume that the point S has the projective coordinate $[0,1,0]$ in $\mathbb{C}P^2$. A line in $\mathbb{C}P^2$ is define by

$$ax+by+cz=0,$$

with a,b,c not all being zero. So it passes through $[0,1,0]$ if and only if $b=0$. If $a=0$, then $c \neq 0$. The line must be the infinity line $z=0$. Now, if it is not the infinity line, a must be nonzero. So the line can be represented by

$$x+cz=0.$$

Thus, in the affine space with $z=1$, it can be represented by x equals a constant in $\mathbb{C}$.

In [2], 129–130, there are four theorems about the relation of the position of the three special points: I, II, III, IV. The proof of Theorem I has been given in [1]. In this paper, we will give the proofs of Theorem II-IV by using the concept of generic point of Van der Waerden[3] and Weil[4] and Ritt-Wu methods.

2 Theorem II

Theorem II *For a given point S in this curve, we can draw five lines passing though S, the first is the tangent line at S and the second is another tangent line of the curve which passes through S. The other three lines can be drawn by connecting S with the three special points. These five lines all passing through point S form a group and they have the following relationship: If three lines are given, then all the five lines are decided uniquely.*

Proof In this theorem, S is a point in this curve. The homogeneous equation of the curve in $\mathbb{C}P^2$ is

$$
\begin{aligned}
f([x,y,z]) = &(a_3(x-a_1z)+(y-a_2z))^2z+a_4(x-a_1z)^3+a_5(x-a_1z)^2(y-a_2z)\\
&+a_6(x-a_1z)(y-a_2z)^2+a_7(y-a_2z)^3.
\end{aligned}
$$

It is easy to verify that $f([x,y,z])=0$ has the solution $[0,1,0]$ if and only if $a_7=0$. That is to say the curve passes S if and only if $a_7=0$.

As in [1], the line passing through the cusp and S is defined by $x=c_4$, the line passing through the inflexion point and S will be defined by $x=c_5$, and the line pass passing through S and the intersection point of the inflexion tangent line and the tangent line at the cusp will be defined by $x = c_6$. c_4, c_5 and c_6 are given by the following formula:

$$c_4 = a_1,$$
$$c_5 = \frac{C_{50}}{C_{51}},$$
$$c_6 = \frac{C_{60}}{C_{61}},$$

where the expression of $C_{50}, C_{51}, C_{60}, C_{61}$ can be found in [1] for generic a_7:

$$\begin{aligned}C_{50} = {} & 27a_1a_3^3a_4a_7^2 - 9a_1a_3^3a_5a_6a_7 + 2a_1a_3^3a_6^3 - 27a_3^5a_7^2 - 27a_1a_3^2a_4a_6a_7 + 18a_1a_3^2a_5^2a_7 \\ & - 3a_1a_3^2a_5a_6^2 + 45a_3^4a_6a_7 - 27a_1a_3a_4a_5a_7 + 18a_1a_3a_4a_6^2 - 3a_1a_3a_5^2a_6 - 36a_3^3a_5a_7 - 18a_3^3a_6^2 \\ & + 27a_1a_4^2a_7 - 9a_1a_4a_5a_6 + 2a_1a_5^3 + 27a_3^2a_4a_7 + 27a_3^2a_5a_6 - 18a_3a_4a_6 - 9a_3a_5^2 + 9a_4a_5, \\ C_{51} = {} & 27a_3^3a_4a_7^2 - 9a_3^3a_5a_6a_7 + 2a_3^3a_6^3 - 27a_3^2a_4a_6a_7 + 18a_3^2a_5^2a_7 - 3a_3^2a_5a_6^2 - 27a_3a_4a_5a_7 \\ & + 18a_3a_4a_6^2 - 3a_3a_5^2a_6 + 27a_4^2a_7 - 9a_4a_5a_6 + 2a_5^3, \\ C_{60} = {} & 3a_1a_3^2a_5a_7 - a_1a_3^2a_6^2 - 9a_1a_3a_4a_7 + a_1a_3a_5a_6 + 3a_3^3a_7 + 3a_1a_4a_6 - a_1a_5^2 - 3a_3^2a_6 + 3a_3a_5 - 3a_4, \\ C_{61} = {} & 3a_3^2a_5a_7 - a_3^2a_6^2 - 9a_3a_4a_7 + a_3a_5a_6 + 3a_4a_6 - a_5^2.\end{aligned}$$

Here, let $a_7 = 0$ and we get:

$$\begin{aligned}C_{50} &= (2a_3a_6 - a_5)(a_1a_3^2a_6^2 - a_1a_3a_5a_6 + 9a_1a_4a_6 - 2a_1a_5^2 - 9a_3^2a_6 + 9a_3a_5 - 9a_4), \\ C_{51} &= (2a_3a_6 - a_5)(a_3^2a_6^2 - a_3a_5a_6 + 9a_4a_6 - 2a_5^2), \\ C_{60} &= -(a_1a_3^2a_6^2 - a_1a_3a_5a_6 - 3a_1a_4a_6 + a_1a_5^2 + 3a_3^2a_6 - 3a_3a_5 + 3a_4), \\ C_{61} &= -(a_3^2a_6^2 - a_3a_5a_6 - 3a_4a_6 + a_5^2).\end{aligned}$$

The common factor between C_{50} and C_{51} can be deleted without changing the value of c_5, so in this paper we let

$$\begin{aligned}C_{50} &= a_1a_3^2a_6^2 - a_1a_3a_5a_6 + 9a_1a_4a_6 - 2a_1a_5^2 - 9a_3^2a_6 + 9a_3a_5 - 9a_4, \\ C_{51} &= a_3^2a_6^2 - a_3a_5a_6 + 9a_4a_6 - 2a_5^2,\end{aligned}$$

and this will not cause any confusion.

There are two tangent lines passing through S. One is tangent at S, the other is not. From the above, the two lines should be $x = c_1$ and $x = c_2$. In order to compute the two tangent lines, we need to compute the resultant of f and $f_1 = \frac{\partial f}{\partial y}$ w.r.t. variable y.

$$\begin{aligned}resultant(f, f_1, y) = {} & (x - a_1)^3(a_1a_6 - a_6x - 1) \\ & (4a_1a_4a_6 - a_1a_5^2 - 4a_3^2a_6 - 4a_4a_6x + a_5^2x + 4a_3a_5 - 4a_4).\end{aligned}$$

Then, c_1 is the solution of $a_1a_6 - a_6x - 1 = 0$ and c_2 is the solution of $4a_1a_4a_6 - a_1a_5^2 - 4a_3^2a_6 - 4a_4a_6x + a_5^2x + 4a_3a_5 - 4a_4 = 0$.

Now we get all the expressions of c_i for $i = 1, 2, 4, 5, 6$. All of them are expressed as rational functions on the variables a_1, a_3, a_4, a_5, a_6. In other words, $(a_1, a_3, a_4, a_5, a_6) \rightarrow$

$(c_1, c_2, c_4, c_5, c_6)$ gives a rational map. This theorem says that the image of this map is a variety of dimension three in $\mathbb{C}^5$, and for any three coordinates, the other two are rational functions of them.

We use "wsolve"[5] and get

$$wsolve(PS, [a_6, a_5, a_1, c_5, c_6], \{C_{61}, C_{51}, a_6, a_5^2 - 4a_4a_6\}) = [[A_1, A_2, A_3, A_4, A_5]],$$

where $PS = [a_1a_6 - a_6c_1 - 1,\ 4a_1a_4a_6 - a_1a_5^2 - 4a_3^2a_6 - 4a_4a_6c_2 + a_5^2c_2 + 4a_3a_5 - 4a_4,\ c_4 - a_1,\ C_{51}c_5 - C_{50}, C_{61}c_6 - C_{60}]$ and

$$\begin{aligned}
A_1 &= a_6c_1 - a_6c_4 + 1,\\
A_2 &= a_5^2c_1c_2 - a_5^2c_1c_4 - a_5^2c_2c_4 + a_5^2c_4^2 + 4a_3a_5c_1 - 4a_3a_5c_4 + 4a_3^2 - 4a_4c_1 + 4a_4c_2,\\
A_3 &= -c_4 + a_1,\\
A_4 &= 9c_1c_2 - c_1c_4 - 8c_1c_5 - 8c_2c_4 - c_2c_5 + 9c_4c_5,\\
A_5 &= 3c_1c_2 + c_1c_4 - 4c_1c_6 - 4c_2c_4 + c_2c_6 + 3c_4c_6.
\end{aligned}$$

Since $A_4 = 0, A_5 = 0$ are linear equations for c_5 and c_6, it is seen that c_5, c_6 are decided uniquely if c_1, c_2, c_4 are algebraically independent. And hence the lines $x = c_5$ and $x = c_6$ are uniquely determined by the other three lines $x = c_1$, $x = c_2$ and $x = c_4$, i.e., the last three generically uniquely determine the previous two. Here, we treat c_1, c_2, c_4, a_3, a_4 as algebraically independent variables. Thus, A_1, A_2, A_3 indicate that the inverse image of the above rational map of a point $(c_1, c_2, c_4, c_5, c_6)$ is a Zariski closed set of dimension two over the field $\mathbb{C}(c_1, c_2, c_4)$.

To see that any three of c_1, c_2, c_4, c_5, c_6 are given the other two are rational functions of the three, we need to consider $C_5^3 = C_5^2 = 10$ cases. One of them is given as above, and the remaining nine cases are treated as follows (the variables indicated are regarded as functions of the unindicated)

1) Case $\{c_1, c_2\}$. We get

$$wsolve([A_5, A_4], [c_1, c_2], \{-4c_1 + c_2 + 3c_4, -8c_1 - c_2 + 9c_4\}) = [[B_1^{(1)}, B_2^{(1)}]],$$

where

$$\begin{aligned}
B_1^{(1)} &= c_1c_4 + 2c_1c_5 - 3c_1c_6 - 3c_4c_5 + 2c_4c_6 + c_5c_6,\\
B_2^{(1)} &= 4c_2c_4 - c_2c_5 - 3c_2c_6 - 3c_4c_5 - c_4c_6 + 4c_5c_6.
\end{aligned}$$

2) Case $\{c_1, c_4\}$. We have

$$wsolve([A_5, A_4], [c_1, c_4], \{-4c_1 + c_2 + 3c_4, -8c_1 - c_2 + 9c_4\}) = [[B_1^{(2)}, B_2^{(2)}]],$$

where

$$\begin{aligned}
B_1^{(2)} &= 3c_1c_2 - 2c_1c_5 - c_1c_6 - c_2c_5 - 2c_2c_6 + 3c_5c_6,\\
B_2^{(2)} &= 4c_2c_4 - c_2c_5 - 3c_2c_6 - 3c_4c_5 - c_4c_6 + 4c_5c_6.
\end{aligned}$$

3) Case $\{c_1, c_5\}$. We obtain

$$wsolve([A_5, A_4], [c_1, c_5], \{-4c_1 + c_2 + 3c_4, -8c_1 - c_2 + 9c_4\}) = [[B_1^{(3)}, B_2^{(3)}]],$$

where

$$B_1^{(3)} = 3c_1c_2 + c_1c_4 - 4c_1c_6 - 4c_2c_4 + c_2c_6 + 3c_4c_6,$$
$$B_2^{(3)} = 4c_2c_4 - c_2c_5 - 3c_2c_6 - 3c_4c_5 - c_4c_6 + 4c_5c_6.$$

4) Case $\{c_1, c_6\}$. We have

$$wsolve([A_5, A_4], [c_1, c_6], \{-4c_1 + c_2 + 3c_4, -8c_1 - c_2 + 9c_4\}) = [[B_1^{(4)}, B_2^{(4)}]],$$

where

$$B_1^{(4)} = 9c_1c_2 - c_1c_4 - 8c_1c_5 - 8c_2c_4 - c_2c_5 + 9c_4c_5,$$
$$B_2^{(4)} = 4c_2c_4 - c_2c_5 - 3c_2c_6 - 3c_4c_5 - c_4c_6 + 4c_5c_6.$$

5) Case $\{c_2, c_4\}$. Using "wsolve", we get

$$wsolve([A_5, A_4], [c_2, c_4], \{-4c_1 + c_2 + 3c_4, -8c_1 - c_2 + 9c_4\}) = [[B_1^{(5)}, B_2^{(5)}]],$$

where

$$B_1^{(5)} = 3c_1c_2 - 2c_1c_5 - c_1c_6 - c_2c_5 - 2c_2c_6 + 3c_5c_6,$$
$$B_2^{(5)} = c_1c_4 + 2c_1c_5 - 3c_1c_6 - 3c_4c_5 + 2c_4c_6 + c_5c_6.$$

6) Case $\{c_2, c_5\}$. Using "wsolve", we obtain

$$wsolve([A_5, A_4], [c_2, c_5], \{-4c_1 + c_2 + 3c_4, -8c_1 - c_2 + 9c_4\}) = [[B_1^{(6)}, B_2^{(6)}]],$$

where

$$B_1^{(6)} = 3c_1c_2 + c_1c_4 - 4c_1c_6 - 4c_2c_4 + c_2c_6 + 3c_4c_6,$$
$$B_2^{(6)} = c_1c_4 + 2c_1c_5 - 3c_1c_6 - 3c_4c_5 + 2c_4c_6 + c_5c_6.$$

7) Case $\{c_2, c_6\}$. We also find out that

$$wsolve([A_5, A_4], [c_2, c_6], \{-4c_1 + c_2 + 3c_4, -8c_1 - c_2 + 9c_4\}) = [[B_1^{(7)}, B_2^{(7)}]],$$

where

$$B_1^{(7)} = 9c_1c_2 - c_1c_4 - 8c_1c_5 - 8c_2c_4 - c_2c_5 + 9c_4c_5,$$
$$B_2^{(7)} = c_1c_4 + 2c_1c_5 - 3c_1c_6 - 3c_4c_5 + 2c_4c_6 + c_5c_6.$$

8) Case $\{c_4, c_5\}$. "wsolve" will lead to

$$wsolve([A_5, A_4], [c_4, c_5], \{-4c_1 + c_2 + 3c_4, -8c_1 - c_2 + 9c_4\}) = [[B_1^{(8)}, B_2^{(8)}]],$$

where

$$B_1^{(8)} = 3c_1c_2 + c_1c_4 - 4c_1c_6 - 4c_2c_4 + c_2c_6 + 3c_4c_6,$$
$$B_2^{(8)} = 3c_1c_2 - 2c_1c_5 - c_1c_6 - c_2c_5 - 2c_2c_6 + 3c_5c_6.$$

9) Case $\{c_4, c_6\}$. Using "wsolve", we get

$$wsolve([A_5, A_4], [c_4, c_6], \{-4c_1 + c_2 + 3c_4, -8c_1 - c_2 + 9c_4\}) = [[B_1^{(9)}, B_2^{(9)}]],$$

where

$$B_1^{(9)} = 9c_1c_2 - c_1c_4 - 8c_1c_5 - 8c_2c_4 - c_2c_5 + 9c_4c_5,$$
$$B_2^{(9)} = 3c_1c_2 - 2c_1c_5 - c_1c_6 - c_2c_5 - 2c_2c_6 + 3c_5c_6.$$

From the above we see that the theorem is correct. ∎

Figure 1 is an example for the curve described by this theorem, where $a_1 = 0, a_2 = 0, a_3 = \sqrt{2}, a_4 = 1, a_5 = 0, a_6 = -1, a_7 = 0$ and the curve $f = 0$ is $x^3 - xy^2 + 2\sqrt{2}xy + 2x^2 + y^2 = 0$. The real part of the curve is shown by the curve with label 1. Here, $S = [0, 1, 0]$ is an infinity point which the straight lines parallelled to the y-axis pass through. The origin is the cusp point. We can see that the straight line with label 2 is a tangent line of the curve which passes through S. The straight line with label 3 is the tangent line at S. The straight line with label 4 is the inflexion tangent line and the straight line with label 5 is the tangent line at the cusp.

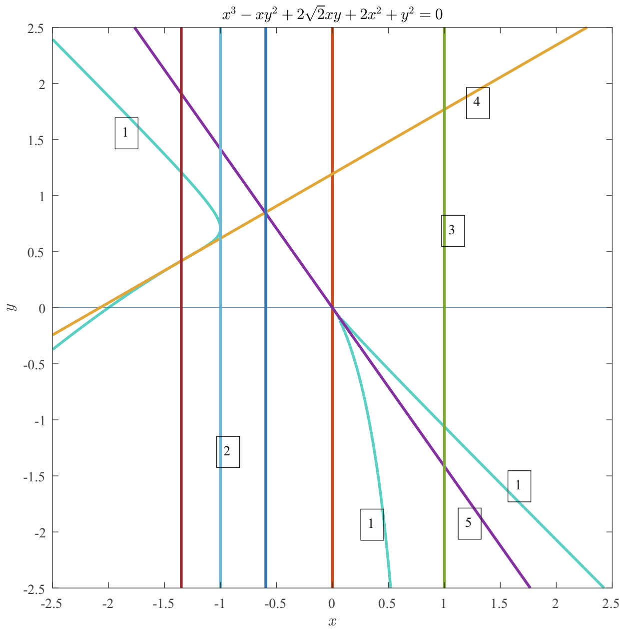

Figure 1

3 Theorem III

Theorem III *For a planar cubic curve with a cusp and a point S on the tangent line at the cusp which is not on the inflexion tangent line, two other tangent lines of the curve can be drawn from S. Another line can be drawn by connecting S and the inflexion point. The above four lines all pass through the point S and form a group. The position of the four lines are relevant: If three lines are given, then the group is determined completely.*

Proof In this case, since S is on the tangent line at the cusp (a_1, a_2), the curve $f = 0$ can be given by

$$f = (x-a_1)^2 + a_3(x-a_1)^3 + a_4(x-a_1)^2(y-a_2) + a_5(x-a_1)(y-a_2)^2 + a_6(y-a_2)^3.$$

Here $x = a_1$ is the tangent line at the cusp. We let $B_1 = c_1 - a_1$.

Now, we use the same method in [1] to get the x-coordinate of the inflexion point c_5. Consider those $f = 0$ which have inflexion tangent lines in the form $y = ax + b$, $a, b \in \mathbb{C}$. Let

$$\begin{aligned} g(x) &\triangleq f(x, ax+b), \\ &= b_3x^3 + b_2x^2 + b_1x + b_0, \end{aligned}$$

where

$$\begin{aligned} b_3 =& a^3a_6 + a^2a_5 + aa_4 + a_3, \\ b_2 =& 1 - 3a_3a_1 - 2a_4a_1a + a_4(-a_2+b) - a_5a_1a^2 + 2a_5(-a_2+b)a + 3a_6(-a_2+b)a^2, \\ b_1 =& -2a_1 + 3a_3a_1^2 + a_4a_1^2a - 2a_4a_1(-a_2+b) - 2a_5a_1(-a_2+b)a + a_5(-a_2+b)^2 + 3a_6(-a_2+b)^2a, \\ b_0 =& a_1^2 - a_3a_1^3 + a_4a_1^2(-a_2+b) - a_5a_1(-a_2+b)^2 + a_6(-a_2+b)^3. \end{aligned}$$

Then $f = 0$ having such inflexion tangent lines with tangent points in $\mathbb{C}^2$ implies $b_3(a, b, a_1, \cdots, a_6) \neq 0$ and $g(x) = b_3(x-v)^3 = b_3(x^3 - 3x^2v + 3xv^2 - v^3)$ for some $v \in \mathbb{C}$. The latter is equivalent to

$$\begin{aligned} -3b_3v &= b_2, \\ 3b_3v^2 &= b_1, \\ -b_3v^3 &= b_0, \end{aligned}$$

and equivalent further to $g_1 \triangleq b_2^2 - 3b_1b_3 = 0$, $g_2 \triangleq b_2^3 - 27b_0b_3^2 = 0$. Thus, use "wsolve", we get

$$wsolve([g_1, g_2], [b, a], \{b_3\}) = [[C_1b + C_0, A_1a + A_0]],$$

where

$$\begin{aligned} A_0 :=& 27a_3a_6^2 - a_5^3, \\ A_1 :=& 9a_6(3a_4a_6 - a_5^2), \\ C_0 :=& 27a_1a_3a_6^2 - a_1a_5^3 + 27a_2a_4a_6^2 - 9a_2a_5^2a_6 - 27a_6^2, \\ C_1 :=& -9a_6(3a_4a_6 - a_5^2). \end{aligned}$$

Then, we substitute the values of a and b in $g(x)$, and find out the c_5 satisfies $B_2 = 0$, where

$$B_2 = (27a_3a_6^2 - 9a_4a_5a_6 + 2a_5^3)c_5 - (27a_1a_3a_6^2 - 9a_1a_4a_5a_6 + 2a_1a_5^3 - 27a_6^2).$$

To give the tangent lines passing through S, we need to compute the resultant of f and $f_1 = \frac{\partial f}{\partial y}$ with respective to the variable y:

$$resultant(f, f_1, y) = a_6(x - a_1)^4 R,$$

where $R = r_2x^2 + r_1x + r_0$, and

$$\begin{aligned} r_2 &= 27a_3^2a_6^2 - 18a_3a_4a_5a_6 + 4a_3a_5^3 + 4a_4^3a_6 - a_4^2a_5^2, \\ r_1 &= -54a_1a_3^2a_6^2 + 36a_1a_3a_4a_5a_6 - 8a_1a_3a_5^3 - 8a_1a_4^3a_6 + 2a_1a_4^2a_5^2 + 54a_3a_6^2, \\ &\quad - 18a_4a_5a_6 + 4a_5^3, \\ r_0 &= 27a_1^2a_3^2a_6^2 - 18a_1^2a_3a_4a_5a_6 + 4a_1^2a_3a_5^3 + 4a_1^2a_4^3a_6 - a_1^2a_4^2a_5^2 - 54a_1a_3a_6^2, \\ &\quad + 18a_1a_4a_5a_6 - 4a_1a_5^3 + 27a_6^2. \end{aligned}$$

Because $x = a_1$ is one of the three tangent lines passing through S, other two tangent lines must be given by the solutions of the equation $R = 0$. Let $x = c_2$ and $x = c_3$ be the other two tangent lines, then R can be rewritten as $R = r_2(x - c_2)(x - c_3)$. Comparing the two representation of R, we get $B_3 = 0, B_4 = 0$ where

$$\begin{aligned} B_3 &= -54a_1a_3^2a_6^2 + 36a_1a_3a_4a_5a_6 - 8a_1a_3a_5^3 - 8a_1a_4^3a_6 + 2a_1a_4^2a_5^2 + 27a_3^2a_6^2c_2 \\ &\quad + 27a_3^2a_6^2c_3 - 18a_3a_4a_5a_6c_2 - 18a_3a_4a_5a_6c_3 + 4a_3a_5^3c_2 + 4a_3a_5^3c_3 + 4a_4^3a_6c_2 \\ &\quad + 4a_4^3a_6c_3 - a_4^2a_5^2c_2 - a_4^2a_5^2c_3 + 54a_3a_6^2 - 18a_4a_5a_6 + 4a_5^3, \\ B_4 &= 27a_1^2a_3^2a_6^2 - 18a_1^2a_3a_4a_5a_6 + 4a_1^2a_3a_5^3 + 4a_1^2a_4^3a_6 - a_1^2a_4^2a_5^2 - 27a_3^2a_6^2c_2c_3 \\ &\quad + 18a_3a_4a_5a_6c_2c_3 - 4a_3a_5^3c_2c_3 - 4a_4^3a_6c_2c_3 + a_4^2a_5^2c_2c_3 - 54a_1a_3a_6^2 \\ &\quad + 18a_1a_4a_5a_6 - 4a_1a_5^3 + 27a_6^2. \end{aligned}$$

The four lines passing through S are $x = c_1$, $x = c_2$, $x = c_3$ and $x = c_5$, and c_1, c_2, c_3, c_5 satisfy the equations $B_1 = 0, B_2 = 0, B_3 = 0$ and $B_4 = 0$. If the leading coefficient of $resultant(f, f_1, y)$ is zero, at least one of the tangent lines will be the infinity line, so in the generic case, $a_6r_2 \neq 0$.

Now we will find the relationship among the variables c_1, c_2, c_3, c_5 under the conditions $B_1 = 0, B_2 = 0, B_3 = 0, B_4 = 0$ and $a_6r_2 \neq 0$. We get

$$wsolve([B_1, B_2, B_3, B_4], [a_1, a_3, a_4, c_5], \{a_6r_2, -27a_3a_6^2 + 9a_4a_5a_6 - 2a_5^3\}) = [[C_1, C_2, C_3, C_4]],$$

where $C_1 = a_1 - c_1$,

$$\begin{aligned} C_2 &= (54a_6^2c_1^2 - 54a_6^2c_1c_2 - 54a_6^2c_1c_3 + 54a_6^2c_2c_3)a_3 - 18a_4a_5a_6c_1^2 + 18a_4a_5a_6c_2c_1 \\ &\quad + 18a_4a_5a_6c_3c_1 - 18a_4a_5a_6c_2c_3 + 4a_5^3c_1^2 - 4a_5^3c_2c_1 - 4a_5^3c_3c_1 + 4a_5^3c_2c_3 \\ &\quad - 54c_1a_6^2 + 27a_6^2c_2 + 27a_6^2c_3, \\ C_3 &= c_{33}a_4^3 + c_{32}a_4^2 + c_{31}a_4 + c_{30}, \\ C_4 &= c_1c_2 + c_1c_3 - 2c_1c_5 - 2c_2c_3 + c_2c_5 + c_3c_5, \end{aligned}$$

and

$$\begin{aligned}
c_{33} &= 432a_6^3(c_1-c_3)^2(c_1-c_2)^2,\\
c_{32} &= -432a_5^2a_6^2(c_1-c_3)^2(c_1-c_2)^2,\\
c_{31} &= 144a_5^4a_6(c_1-c_3)^2(c_1-c_2)^2,\\
c_{30} &= -(4a_5^3c_1^2-4a_5^3c_1c_2-4a_5^3c_1c_3+4a_5^3c_2c_3+27a_6^2c_2-27a_6^2c_3)\\
&\quad(4a_5^3c_1^2-4a_5^3c_1c_2-4a_5^3c_1c_3+4a_5^3c_2c_3-27a_6^2c_2+27a_6^2c_3).
\end{aligned}$$

From $C_4=0$, we know that c_5 is uniquely determined by the values of c_1,c_2,c_3 since C_4 is degree one on c_5.

To see that any three of c_1,c_2,c_3,c_5 are given the other one is a rational function of the three, we need to consider $C_4^3=C_4^1=4$ cases. One of them is given as above, and the remaining three cases are treated as follows (the variable indicated is regarded as functions of the unindicated)

1) Case $\{c_1\}$. We get

$$wsolve([C_4],[c_1],\{c_2+c_3-2c_1\})=[[C^{(1)}]],$$

where

$$C^{(1)}=c_1c_2+c_1c_3-2c_1c_5-2c_2c_3+c_2c_5+c_3c_5.$$

2) Case $\{c_2\}$. We have

$$wsolve([C_4],[c_2],\{c_2+c_3-2c_1\})=[[C^{(2)}]],$$

where

$$C^{(2)}=c_1c_2+c_1c_3-2c_1c_5-2c_2c_3+c_2c_5+c_3c_5.$$

3) Case $\{c_3\}$. We obtain

$$wsolve([C_4],[c_3],\{c_2+c_3-2c_1\})=[[C^{(3)}]],$$

where

$$C^{(3)}=c_1c_2+c_1c_3-2c_1c_5-2c_2c_3+c_2c_5+c_3c_5.$$

From the above we see that the theorem is correct. ∎

Figure 2 is an example for this kind of curve, where $a_1=0, a_2=0, a_3=-\frac{1}{2}, a_4=0, a_5=\frac{3}{2}\sqrt[3]{3}, a_6=1$ and the curve $f=0$ is $x^2-\frac{1}{2}x^3+\frac{3}{2}\sqrt[3]{3}xy^2+y^3=0$. The real part of the curve is shown by the curve with label 1. The origin is the cusp point. The y-axis is the tangent line at the cusp. The straight line with label 2 and the the one with label 3 are tangent lines of the curve passing through S. The straight line with label 4 is the inflexion tangent line. The straight line with label 5 is the straight line connecting S and the inflexion point.

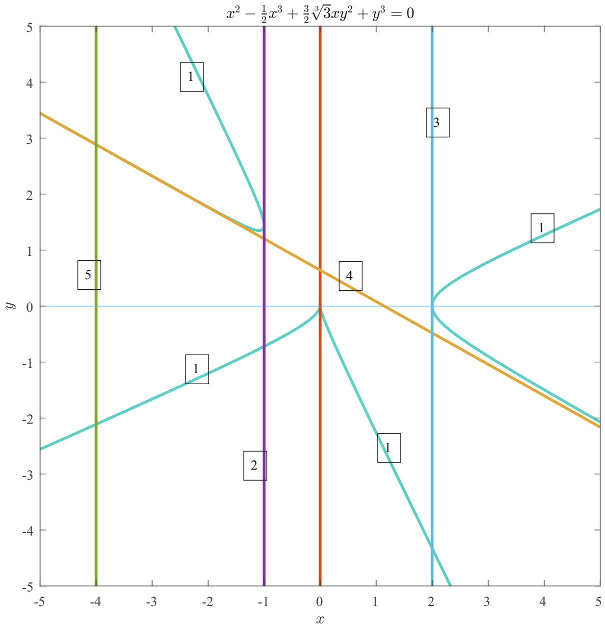

Figure 2

4 Theorem IV

Theorem IV *For a planar cubic curve with a cusp and a point S on the line connecting the cusp and the inflexion point, three tangent lines of the curve can be drawn from S, another line can be draw by connecting S and the intersection point of the inflexion tangent line and the tangent line at the cusp point. The above five lines all pass through the point S and form a group. The position of the five lines are relevant.*

1) *If the three tangent lines are given, then there are two possibilities.*

2) *If two tangent lines and the line connecting S and the intersection point of the inflexion tangent line and the tangent line at the cusp point are given, there are two possibilities.*

3) *If two tangent lines and the line connecting the cusp and the inflexion point are given, then there are two possibilities.*

4) *If one tangent line, the line connecting S and the intersection point of the inflexion tangent line and the tangent line at the cusp point and the line connecting the cusp and the inflexion point are given, then there is only one possibilities.*

Proof Here, we also use the

$$\begin{aligned} f = &(a_3(x-a_1)+(y-a_2))^2 + a_4(x-a_1)^3 + a_5(x-a_1)^2(y-a_2) \\ &+a_6(x-a_1)(y-a_2)^2 + a_7(y-a_2)^3 = 0 \end{aligned}$$

to represent this curve as in [1].

In [1], Theorem I says that the image of the map $(a_1, a_3, a_4, a_5, a_6, a_7) \to (c_1, c_2, \cdots, c_6)$ is a variety of four dimension. For this theorem, point S is on the line connecting the cusp and the inflexion point implies that $c_5 - a_1 = 0$. c_5 is a rational function of $a_1, a_3, \cdots, a_7$, i.e.,

$$c_5 = \frac{C_{50}}{C_{51}}.$$

Thus, the parameters of the curve $f = 0$ must lie in the hyper-surface given by the equation

$$C_{50} - a_1 C_{51} = 0.$$

The image is included in the hyperplane $c_4 - c_5 = 0$. We use c_1, c_2, c_3, c_4, c_6 as the parameters of the hyperplane. Using Maple,

$$factor(C_{50} - a_1 C_{51}) = D_1 D_2,$$

where

$$D_1 = a_3^3 a_7 - a_3^2 a_6 + a_3 a_5 - a_4,$$
$$D_2 = 3a_3^2 a_7 - 2a_3 a_6 + a_5.$$

Thus, the hyper-surface $C_{50} - a_1 C_{51} = 0$ can be decomposed into two irreducible hyper-surfaces, which are represented by $D_1 = 0$ and $D_2 = 0$, respectively. What we get from Maple only tell us that $C_{50} - a_1 C_{51} = D_1 D_2$ in the field $\mathbb{Q}$. However, as there is only one linear item of a_5 in D_1 and there is only one linear item of a_4 in D_2, we can conclude that both D_1 and D_2 are indecomposable in the field $\mathbb{C}$.

Lemma *The parameters of the curve $f = 0$ satisfy the equation*

$$D_1 = 0,$$

if and only this curve is degenerate into a conic plus a line tangent to it.

The lemma is proved as follows. The generic case of a conic plus a line tangent to it at (a_1, a_2) can be represented by

$$(d_0(x - a_1) + y - a_2)(d_0(x - a_1) + y - a_2 + d_1(x - a_1)^2 + d_2(x - a_1)(y - a_2) + d_3(y - a_2)^2) = 0.$$

If the curve $f = 0$ is in this case, comparing the coefficients of the two equations, we get

$$a_4 = d_0 d_1,$$
$$a_5 = d_1 + d_2 d_0,$$
$$a_6 = d_0 d_3 + d_2,$$
$$a_7 = d_3.$$

By Maple,

$$simplify(subs([a_4 = d_0 d_1, a_5 = d_1 + d_2 d_0, a_6 = d_0 d_3 + d_2, a_7 = d_3], D_1)) = 0.$$

This shows that the parameters in the degenerate case is on the surface $D_1 = 0$. Now we consider the map from $\mathbb{C}^6$ to the hyper-surface $D_1 = 0$, $(a_1, a_2, d_0, d_1, d_2, d_3) \mapsto (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ defined by

$$
\begin{aligned}
a_1 &= a_1, \\
a_2 &= a_2, \\
a_3 &= d_0, \\
a_4 &= d_0 d_1, \\
a_5 &= d_1 + d_2 d_0, \\
a_6 &= d_0 d_3 + d_2, \\
a_7 &= d_3.
\end{aligned}
$$

Using Maple,

$$
solve([a_7 - d_3, -d_0 d_3 + a_6 - d_2, -d_0 d_2 + a_5 - d_1, -d_0 d_1 + a_4, a_3 - d_0], [a_4, d_0, d_1, d_2, d_3]),
$$

we get the inverse map from $D_1 = 0$ to $\mathbb{C}^6$.

$$
\begin{aligned}
a_1 &= a_1, \\
a_2 &= a_2, \\
d_0 &= a_3, \\
d_1 &= a_3^2 a_7 - a_3 a_6 + a_5, \\
d_2 &= -a_3 a_7 + a_6, \\
d_3 &= a_7.
\end{aligned}
$$

This indicates that the map $(a_1, a_2, d_0, d_1, d_2, d_3) \mapsto (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ defined before is an isomorphism between $\mathbb{C}^6$ and the hyper-surface $D_1 = 0$. This shows that all the points on the hyper-surface represents a degenerate curve. The proof for the lemma is complete. ∎

The isomorphism between $\mathbb{C}^6$ and the hyper-surface $D_1 = 0$ is consistent with the Jacobian conjecture as the case of polynomials with degree two which has been already proved to be true.

Now we return to the proof of the theorem. As in [1], to find the x-coordinate of the tangent lines of $f = 0$ passing through S, c_1, c_2, c_3, we need to calculate

$$
resultant(f, f_1, y) = a_7 (x - a_1)^3 f_2(a_1, a_3, \cdots, a_7, x),
$$

where $f_2 = F_3 x^3 + F_2 x^2 + F_1 x + F_0$. Please see paper [1] for the details of F_3, F_2, F_1, F_0. Then, c_1, c_2, c_3 be the three solutions of $f_2 = 0$. We have $f_2 = F_3(x - c_1)(x - c_2)(x - c_3)$. Comparing the two representation of f_2, we have the following equations for c_1, c_2, c_3. $F_3 p_1 - F_2 = 0, F_3 p_2 - F_1 =$

$0, F_3p_3 - F_0 = 0$ where

$$p_1 = -(c_1 + c_2 + c_3),$$
$$p_2 = c_1c_2 + c_1c_3 + c_2c_3,$$
$$p_3 = -c_1c_2c_3.$$

As in [1], $c_4 = a_1$, $c_6 = \frac{C_{60}}{C_{61}}$. We have

$$wsolve([C_{61}c_6 - C_{60}, D_2, c_4 - a_1, F_3p_1 - F_2, F_3p_2 - F_1, F_3p_3 - F_0],$$
$$[a_7, a_6, a_5, a_1, c_6, c_4], \{C_{61}, C_{51}, F_3, a_7\}) = [\alpha_7, \alpha_6, \alpha_5, \alpha_1, \gamma_6, \gamma_4],$$

where

$$\gamma_6 = 3c_4c_6 + c_4p_1 + c_6p_1 + p_2,$$
$$\gamma_4 = c_4^2p_1^2 - 3c_4^2p_2 + c_4p_1p_2 - 9c_4p_3 - 3p_1p_3 + p_2^2.$$

Here, we regard a_3, a_4, p_1, p_2, p_3 as independent variables. Given p_1, p_2, p_3 is equivalent to given c_1, c_2, c_3. Note that from $\gamma_4 = 0$ and $\gamma_6 = 0$, for fixed p_1, p_2, p_3 we can only have two possible c_4, c_6. This means that there are two possibilities for the group of five lines passing through S if three tangent lines are given. This proves Case 1).

For Case 2), we use

$$wsolve([\gamma_4, \gamma_6], [c_4, c_3], (c_1 + c_2 + c_3)^2 - 3(c_1c_2 + c_1c_3 + c_2c_3), -c_1 - c_2 - c_3 + 3c_4) = [D_4^{(2)}, D_3^{(2)}],$$

where

$$D_4^{(2)} = c_1c_2 + c_1c_3 - c_1c_4 - c_1c_6 + c_2c_3 - c_2c_4 - c_2c_6 - c_3c_4 - c_3c_6 + 3c_4c_6,$$
$$D_3^{(2)} = c_1^2c_2^2 - c_1^2c_2c_3 - c_1^2c_2c_6 + c_1^2c_3^2 - c_1^2c_3c_6 + c_1^2c_6^2 - c_1c_2^2c_3$$
$$- c_1c_2^2c_6 - c_1c_2c_3^2 + 6c_1c_2c_3c_6 - c_1c_2c_6^2 - c_1c_3^2c_6 - c_1c_3c_6^2 + c_2^2c_3^2 - c_2^2c_3c_6$$
$$+ c_2^2c_6^2 - c_2c_3^2c_6 - c_2c_3c_6^2 + c_3^2c_6^2.$$

Here, we regard c_1, c_2, c_6 as independent variables. From $D_4^{(2)} = 0$ and $D_3^{(2)} = 0$, for fixed c_1, c_2, c_6 we can only have two possible c_3, c_4. This proves Case 2).

For Case 3), we use

$$wsolve([\gamma_4, \gamma_6], [c_6, c_3], (c_1 + c_2 + c_3)^2 - 3(c_1c_2 + c_1c_3 + c_2c_3), -c_1 - c_2 - c_3 + 3c_4) = [D_6^{(3)}, D_3^{(3)}],$$

where

$$D_6^{(3)} = c_1c_2 + c_1c_3 - c_1c_4 - c_1c_6 + c_2c_3 - c_2c_4 - c_2c_6 - c_3c_4 - c_3c_6 + 3c_4c_6,$$
$$D_3^{(3)} = c_1^2c_2^2 - c_1^2c_2c_3 - c_1^2c_2c_4 + c_1^2c_3^2 - c_1^2c_3c_4 + c_1^2c_4^2 - c_1c_2^2c_3$$
$$- c_1c_2^2c_4 - c_1c_2c_3^2 + 6c_1c_2c_3c_4 - c_1c_2c_4^2 - c_1c_3^2c_4 - c_1c_3c_4^2 + c_2^2c_3^2$$
$$- c_2^2c_3c_4 + c_2^2c_4^2 - c_2c_3^2c_4 - c_2c_3c_4^2 + c_3^2c_4^2.$$

Here, we regard c_1, c_2, c_4 as independent variables. From $D_6^{(3)} = 0$ and $D_3^{(3)} = 0$, for fixed c_1, c_2, c_4 we can only have two possible c_3, c_6. This proves Case 3).

For Case 4), we use $wsolve([\gamma_4, \gamma_6], [c_3, c_2], (c_1 + c_2 + c_3)^2 - 3(c_1c_2 + c_1c_3 + c_2c_3), -c_1 - c_2 -c_3 + 3c_4) = [D_3^{(4)}, D_2^{(4)}]$, where

$$\begin{aligned}
D_3^{(4)} &= c_1c_2 + c_1c_3 - c_1c_4 - c_1c_6 + c_2c_3 - c_2c_4 - c_2c_6 - c_3c_4 - c_3c_6 + 3c_4c_6,\\
D_2^{(4)} &= 3c_1^2c_2^2 - 3c_1^2c_2c_4 - 3c_1^2c_2c_6 + c_1^2c_4^2 + c_1^2c_4c_6 + c_1^2c_6^2 - 3c_1c_2^2c_4\\
&\quad - 3c_1c_2^2c_6 + c_1c_2c_4^2 + 10c_1c_2c_4c_6 + c_1c_2c_6^2 - 3c_1c_4^2c_6 - 3c_1c_4c_6^2 + c_2^2c_4^2\\
&\quad + c_2^2c_4c_6 + c_2^2c_6^2 - 3c_2c_4^2c_6 - 3c_2c_4c_6^2 + 3c_4^2c_6^2.
\end{aligned}$$

Here, we regard c_1, c_4, c_6 as independent variables. From $D_3^{(4)} = 0$ and $D_2^{(4)} = 0$, for fixed c_1, c_4, c_6 we can only have two possible c_2, c_3. However, there is only one possibility for the unordered pair $\{c_1, c_2\}$. This proves Case 4). ∎

Figure 3 shows an example of this curve, where $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 1, a_5 = 0, a_6 = 1, a_7 = 1$ and the curve $f = 0$ is $x^3 + xy^2 + y^3 + y^2 = 0$. The real part of the curve is shown by the curve with label 1. The origin is the cusp point. The y-axis is the straight line connecting the cusp point and the inflexion point. The straight line with label 2 is one of the tangent lines passing through S. The line with label 3 is the straight line connecting S and the intersection point of the inflexion tangent line and the tangent line at the cusp point. The straight line with label 4 is the inflexion tangent line. Two other tangent lines passing through S are $x = \frac{9}{62}\sqrt[3]{4} - \frac{3}{31}\sqrt[3]{2} - \frac{4}{31} + \frac{1}{2}\sqrt{3}(-\frac{9}{31}\sqrt[3]{4} - \frac{6}{31}\sqrt[3]{2})\mathrm{i}$ and $x = \frac{9}{62}\sqrt[3]{4} - \frac{3}{31}\sqrt[3]{2} - \frac{4}{31} - \frac{1}{2}\sqrt{3}(-\frac{9}{31}\sqrt[3]{4} - \frac{6}{31}\sqrt[3]{2})\mathrm{i}$, which are not real lines.

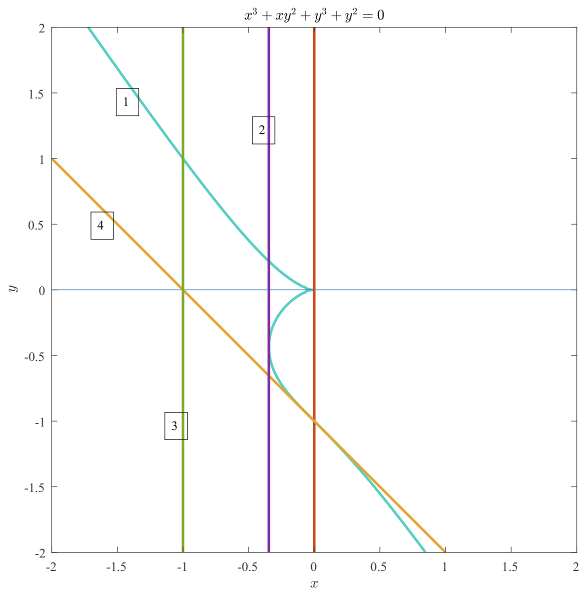

Figure 3

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