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Factorizations for a class of multivariate polynomial matrices

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Abstract
This paper investigates how to factorize a class of multivariate polynomial matrices. We prove that an $l \times m$ multivariate polynomial matrix admits a matrix factorization with respect to a given polynomial if the polynomial and all the $(l-1) \times (l-1)$ reduced minors of the matrix generate a unit ideal. This result is a generalization of a theorem in Liu et al. (Circuits Syst Signal Process 30(3):553–566, 2011). Based on three main theorems presented in the paper and a constructive algorithm proposed by Lin et al. (Circuits Syst Signal Process 20(6):601–618, 2001), we give an algorithm which can be used to factorize more multivariate polynomial matrices. In addition, an illustrative example is given to show the effectiveness of the proposed algorithm.

Keywords Multivariate polynomial matrices · Matrix factorization · Reduced minors · Reduced Gröbner basis

1 Introduction

The study of matrix factorizations for multivariate polynomial matrices began with the development of multidimensional systems theory in the late 1970s (Youla and Gnavi 1979), and the existence problem of matrix factorizations for multivariate polynomial matrices was con-

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sidered to be one of the basic problems of this subject. Since then, great progress has been achieved.

Bose (1982) introduced some basic concepts of multivariate polynomial matrices. After that, Bose et al. (2003) proposed several algorithms to factorize bivariate polynomial matrices, and introduced the latest research trends of matrix factorizations with three or more variables. The existence problem of matrix factorizations for bivariate polynomial matrices has been completely solved in Guiver and Bose (1982), Liu and Wang (2013), Morf et al. (1977), but for the case of more than two variables is still open.

Charoenlarpnopparut and Bose (1999) used Gröbner bases of modules to compute zero prime matrix factorizations of multivariate polynomial matrices. For some multivariate polynomial matrices with special properties, Lin (1999a, 2001) proposed some methods to compute zero prime matrix factorizations of the matrices. Meanwhile, Lin and Bose (2001) presented the Lin-Bose's conjecture: a multivariate polynomial matrix admits a zero prime matrix factorization if its all maximal reduced minors generate a unit ideal. This conjecture was proved in Liu et al. (2014), Pommaret (2001), Wang and Feng (2004), so the existence problem of zero prime matrix factorizations has been solved. Subsequently, Wang and Kwong (2005) put forward an effective algorithm based on module theory to solve the existence problem of minor prime matrix factorizations. Guan et al. (2019) studied the existence problem of minor prime matrix factorizations under the condition that matrices are not of full rank, and they generalized the main results in Wang and Kwong (2005). So far, some achievements have been made on the existence problem of factor prime matrix factorizations (Guan et al. 2018; Liu and Wang 2010, 2015; Wang 2007). However, the general case is unsolved.

As far as we know, there is a class of multivariate polynomial matrices that has always attracted attention. That is,

$$\mathcal{S} = \{\mathbf{F} \in k[\mathbf{z}]^{l\times m} : d = z_1 - f(\mathbf{z}_2) \text{ is a divisor of } d_l(\mathbf{F}) \text{ with } f(\mathbf{z}_2) \in k[\mathbf{z}_2]\},$$

where $\mathbf{z}_2 = \{z_2, \ldots, z_n\}$ and $d_l(\mathbf{F})$ is the GCD of all the $l \times l$ minors of $\mathbf{F}$. For the existence problem of matrix factorizations for multivariate polynomial matrices in $\mathcal{S}$, many people tried to solve it.

From the 1990s to the present, Lin and coauthors have done a lot of basic work on the existence problem of a matrix factorization for $\mathbf{F} \in \mathcal{S}$. Lin first studied the case for $\mathbf{F} \in \mathcal{S}$ with $n = 3$, and then he successfully extended the main results in Lin (1992) to the case of $n > 3$ (Lin 1993). Let $d_l(\mathbf{F}) = d$, Lin et al. (2005) obtained a necessary and sufficient condition for $\mathbf{F} \in \mathcal{S}$ to admit a minor prime matrix factorization. Moreover, Lin et al. (2006) showed that a square matrix $\mathbf{F} \in k[\mathbf{z}]^{l\times l}$ with $d_l(\mathbf{F}) = d$ is equivalent to $\mathrm{diag}(1, \ldots, 1, d)$. For the general case of $d \mid d_l(\mathbf{F})$, Lin et al. (2001) proved that $\mathbf{F} \in \mathcal{S}$ admits a matrix factorization w.r.t. d if the rank of $\mathbf{F}(f, \mathbf{z}_2)$ is $(l-1)$ for every $(\mathbf{z}_2) \in k^{1\times(n-1)}$. Furthermore, they proposed a constructive algorithm to factorize $\mathbf{F}$.

After that, Liu et al. (2011) obtained that $\mathbf{F} \in \mathcal{S}$ has a matrix factorization w.r.t. d if d and all the $(l-1)\times(l-1)$ minors of $\mathbf{F}$ generate $k[\mathbf{z}]$. This result is a generalization of the previous result in Lin et al. (2001). However, we find that there are still many of multivariate polynomial matrices in $\mathcal{S}$ that can be factorized without satisfying the result in Liu et al. (2011). So, in this paper we continue to study the existence problem of a matrix factorization for $\mathbf{F} \in \mathcal{S}$.

This paper is organized as follows. In Sect. 2, we fist outline some knowledge about multivariate polynomial matrices, and then propose two problems that we shall consider. Main results and some generalizations are presented in Sect. 3. A multivariate polynomial matrix factorization algorithm is given in Sect. 4, and we use an example to illustrate the calculation process of the algorithm. Further remarks are provided in Sect. 5.

2 Preliminaries and problems

In this section we first introduce some basic notions which will be used in the following sections, and then present two problems we are going to consider.

2.1 Basic notions

We denote by k an algebraically closed field, $\mathbf{z}$ the n variables $z_1, z_2, \ldots, z_n$, $\mathbf{z}_2$ the $(n-1)$ variables $z_2, \ldots, z_n$, where $n \geq 3$. Let $k[\mathbf{z}]$ and $k[\mathbf{z}_2]$ be the ring of polynomials in variables $\mathbf{z}$ and $\mathbf{z}_2$ with coefficients in k, respectively. Let $k[\mathbf{z}]^{l\times m}$ be the set of $l \times m$ matrices with entries in $k[\mathbf{z}]$. Without loss of generality, we assume that $l \leq m$, and for convenience we use uppercase bold letters to denote polynomial matrices.

Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$. Assume that $f \in k[\mathbf{z}_2]$, then $\mathbf{F}(f, \mathbf{z}_2)$ denotes a polynomial matrix in $k[\mathbf{z}_2]^{l\times m}$ which is formed by transforming z_1 in $\mathbf{F}$ into f. If $l = m$, we denote by $\det(\mathbf{F})$ the determinant of $\mathbf{F}$, and if $\mathbf{F}$ is of full rank, we use $\mathbf{F}^{-1}$ to represent the invertible matrix of $\mathbf{F}$. Moreover, $\mathbf{F}^{\mathrm{T}}$ represents the transposed matrix of $\mathbf{F}$.

Assume that $f_1, \ldots, f_s \in k[\mathbf{z}]$, we use $\langle f_1, \ldots, f_s \rangle$ to denote the ideal generated by $f_1, \ldots, f_s$ in $k[\mathbf{z}]$. Let $g, h \in k[\mathbf{z}]$, then $g \mid h$ means that g is a divisor of h. In addition, "w.r.t." and "GCD" stand for "with respect to" and "greatest common divisor", respectively.

We first introduce two basic concepts in matrix theory.

Definition 1 Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$, and given $2r$ positive integers arbitrarily such that $1 \leq i_1 < \cdots < i_r \leq l$ and $1 \leq j_1 < \cdots < j_r \leq m$. Let $\mathbf{F}\begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix}$ denotes an $r \times r$ matrix consisting of the $i_1, \ldots, i_r$ rows and $j_1, \ldots, j_r$ columns of $\mathbf{F}$, then $\det\left(\mathbf{F}\begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix}\right)$ is called an $r \times r$ **minor** of $\mathbf{F}$.

Definition 2 Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$, then the rank of $\mathbf{F}$ is r $(1 \leq r \leq l)$ if there exists a nonzero $r \times r$ minor of $\mathbf{F}$, and all the $i \times i$ $(i > r)$ minors of $\mathbf{F}$ vanish identically. For convenience, we denote the rank of $\mathbf{F}$ by $\mathrm{rank}(\mathbf{F})$.

Then, we recall the concept of zero left prime matrix from multidimensional systems theory.

Definition 3 (Bose 1982; Youla and Gnavi 1979) Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank. If all the $l \times l$ minors of $\mathbf{F}$ generate $k[\mathbf{z}]$, then $\mathbf{F}$ is said to be a zero left prime (ZLP) matrix.

Let I be an ideal generated by all the $l \times l$ minors of $\mathbf{F}$, then we can compute a reduced Gröbner basis $\mathcal{G}$ of I w.r.t. a monomial order to check $I = k[\mathbf{z}]$. That is, if $\mathcal{G} = \{1\}$, then $I = k[\mathbf{z}]$. The definition of the reduced Gröbner basis and how to compute a reduced Gröbner basis of an ideal can be found in Buchberger (1965), Cox et al. (2007).

Serre (1955) raised the question whether any finitely generated projective module over a polynomial ring is free. This question was solved positively and independently by Quillen (1976) and Suslin (1976), and the result is called Quillen-Suslin theorem. For Quillen-Suslin theorem, there are two equivalent descriptions as follows.

Lemma 1 *Let $\mathbf{w} \in k[\mathbf{z}]^{1\times l}$ be a* ZLP *vector and $\mathbf{M}$ be a module generated by all vectors in $\{\mathbf{q} \in k[\mathbf{z}]^{l\times 1} : \mathbf{wq} = 0\}$, then $\mathbf{M}$ is free.*

Lemma 2 *Let $\mathbf{w} \in k[\mathbf{z}]^{1\times l}$ be a* ZLP *vector, then a unimodular matrix $\mathbf{U} \in k[\mathbf{z}]^{l\times l}$ can be constructed such that $\mathbf{w}$ is its first row.*

In Lemma 1, $\mathbf{M}$ is called the syzygy module of $\mathbf{w}$. Fabiańska and Quadrat (2006) gave an algorithm to compute free bases of free modules over polynomial rings, and the algorithm was implemented in QuillenSuslin package (Fabiańska et al. 2007). In Lemma 2, $\mathbf{U}$ is a unimodular matrix if and only if $\det(\mathbf{U})$ is a nonzero constant in k. There are many methods to construct $\mathbf{U}$ such that $\mathbf{w}$ is its first row, we refer to Logar and Sturmfels (1992), Lu et al. (2017), Park (1995), Youla and Pickel (1984) for more details.

Throughout the paper, we use the following notation conventions.

Convention 1 *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$. We use $d_l(\mathbf{F})$ and $d_{l-1}(\mathbf{F})$ to denote the GCD of all the $l \times l$ minors and all the $(l-1) \times (l-1)$ minors of $\mathbf{F}$, respectively.*

Convention 2 *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$. We use $a_1, \ldots, a_\eta$ to denote all the $l \times l$ minors of $\mathbf{F}$, where $\eta = \binom{m}{l}$. Extracting $d_l(\mathbf{F})$ from $a_1, \ldots, a_\eta$ yields*

$$a_i = d_l(\mathbf{F}) \cdot b_i, \ \ i = 1, \ldots, \eta,$$

then $b_1, \ldots, b_\eta$ are called the $l \times l$ **reduced minors** *of $\mathbf{F}$.*

Convention 3 *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$. We use $c_1, \ldots, c_\gamma$ to denote all the $(l-1) \times (l-1)$ minors of $\mathbf{F}$, where $\gamma = \binom{l}{l-1} \cdot \binom{m}{l-1}$. Extracting $d_{l-1}(\mathbf{F})$ from $c_1, \ldots, c_\gamma$ yields*

$$c_i = d_{l-1}(\mathbf{F}) \cdot h_i, \ \ i = 1, \ldots, \gamma,$$

then $h_1, \ldots, h_\gamma$ are called the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}$.

Remark 1 The ideal generated by all the $i \times i$ minors of $\mathbf{F}$ is called the i-th determinantal ideal of $\mathbf{F}$ for $i = 1, \ldots, l$ (this notion is from Brown 1993). Moreover, it is related to the i-th Fitting ideal of a $k[\mathbf{z}]$-module. We refer to Eisenbud (2013) for more details. In this paper we focus on reduced minors rather than minors of $\mathbf{F}$.

Next, we introduce three important lemmas in matrix theory.

Lemma 3 (Binet-Cauchy formula, Strang 2010) *Let $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$, where $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$. Then an $r \times r$ $(r \leq l)$ minor of $\mathbf{F}$ is*

$$\det\left(\mathbf{F}\left(\begin{smallmatrix} i_1\cdots i_r \\ j_1\cdots j_r \end{smallmatrix}\right)\right) = \sum_{1\leq s_1<\cdots<s_r\leq l} \det\left(\mathbf{G}_1\left(\begin{smallmatrix} i_1\cdots i_r \\ s_1\cdots s_r \end{smallmatrix}\right)\right) \cdot \det\left(\mathbf{F}_1\left(\begin{smallmatrix} s_1\cdots s_r \\ j_1\cdots j_r \end{smallmatrix}\right)\right). \tag{1}$$

In particular, when $r = l$, we have

$$\det\left(\mathbf{F}\left(\begin{smallmatrix} 1\cdots l \\ j_1\cdots j_l \end{smallmatrix}\right)\right) = \det(\mathbf{G}_1) \cdot \det\left(\mathbf{F}_1\left(\begin{smallmatrix} 1\cdots l \\ j_1\cdots j_l \end{smallmatrix}\right)\right). \tag{2}$$

Lemma 4 (Lin 1993, 1999b) *Let $\mathbf{F}_1 = [\mathbf{F}_{11}, \mathbf{F}_{12}] \in k[\mathbf{z}]^{l\times(m+l)}$ be of full row rank and $\mathbf{F}_2 = [\mathbf{F}_{21}^{\mathrm{T}}, -\mathbf{F}_{22}^{\mathrm{T}}]^{\mathrm{T}} \in k[\mathbf{z}]^{(m+l)\times m}$ be of full column rank, where $\mathbf{F}_{11}, \mathbf{F}_{22} \in k[\mathbf{z}]^{l\times m}$, $\mathbf{F}_{12} \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{F}_{21} \in k[\mathbf{z}]^{m\times m}$. If $\mathbf{F}_1\mathbf{F}_2 = \mathbf{0}_{l\times m}$, then $\det(\mathbf{F}_{12}) \neq 0$ if and only if $\det(\mathbf{F}_{21}) \neq 0$.*

Lemma 5 (Lin 1988) *Assume that $\mathbf{F}_{12}^{-1}\mathbf{F}_{11} = \mathbf{F}_{22}\mathbf{F}_{21}^{-1}$, where $\mathbf{F}_{11}, \mathbf{F}_{22} \in k[\mathbf{z}]^{l\times m}$, $\mathbf{F}_{12}^{-1} \in k(\mathbf{z})^{l\times l}$ and $\mathbf{F}_{21}^{-1} \in k(\mathbf{z})^{m\times m}$. Let $\bar{p}_1, \ldots, \bar{p}_{\xi_1}$ be all the $l \times l$ reduced minors of $[\mathbf{F}_{11}, \mathbf{F}_{12}]$, and $p_1, \ldots, p_{\xi_2}$ be all the $m \times m$ reduced minors of $[\mathbf{F}_{21}^{\mathrm{T}}, -\mathbf{F}_{22}^{\mathrm{T}}]^{\mathrm{T}}$, where $\xi_1 = \binom{m+l}{l} = \xi_2 = \binom{m+l}{m}$. Then, $\bar{p}_i = \pm p_i$ for $i = 1, \ldots, \xi_1$, and the sign depends on the index i.*

2.2 Problems

Now, the definition of the matrix factorization is as follows.

Definition 4 Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ and $d \mid d_l(\mathbf{F})$. We say that $\mathbf{F}$ admits a matrix factorization w.r.t. d if $\mathbf{F}$ can be factorized as

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$$

such that $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$, $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$, and $\det(\mathbf{G}_1) = d$.

As we mentioned in Sect. 1, we consider the existence problem of a matrix factorization for $\mathbf{F} \in \mathcal{S}$ w.r.t. $d = z_1 - f(\mathbf{z}_2)$, where $f(\mathbf{z}_2) \in k[\mathbf{z}_2]$. For this problem, Liu et al. (2011) got an important result.

Lemma 6 (Liu et al. 2011) *Let* $\mathbf{F} \in \mathcal{S}$. *If* $\langle d, c_1, \ldots, c_\gamma\rangle = k[\mathbf{z}]$, *then* $\mathbf{F}$ *admits a matrix factorization w.r.t.* d.

Obviously, $\langle d, c_1, \ldots, c_\gamma\rangle \subseteq \langle d, d_{l-1}(\mathbf{F})\rangle \subseteq k[\mathbf{z}]$. If $d \mid d_{l-1}(\mathbf{F})$, then $\langle d, c_1, \ldots, c_\gamma\rangle \subseteq \langle d\rangle \neq k[\mathbf{z}]$ and Lemma 6 is invalid. Therefore, the prerequisite for the establishment of Lemma 6 is that $d \nmid d_{l-1}(\mathbf{F})$. Based on this phenomenon, we now construct a subset of $\mathcal{S}$:

$$\mathcal{S}_1 = \{\mathbf{F} \in \mathcal{S} : d \nmid d_{l-1}(\mathbf{F})\}.$$

Then, $\mathbf{F} \in \mathcal{S}_1$ admits a matrix factorization w.r.t. d if $\langle d, c_1, \ldots, c_\gamma\rangle = k[\mathbf{z}]$, and Lemma 6 is invalid for any $\mathbf{F} \in \mathcal{S} \setminus \mathcal{S}_1$.

Although $d \nmid d_{l-1}(\mathbf{F})$ for any $\mathbf{F} \in \mathcal{S}_1$, d and $d_{l-1}(\mathbf{F})$ may have common zeros. It follows that Lemma 6 is invalid if $\langle d, d_{l-1}(\mathbf{F})\rangle \neq k[\mathbf{z}]$. However, we find that there exists $\mathbf{F} \in \mathcal{S}_1$ which satisfies $\langle d, d_{l-1}(\mathbf{F})\rangle \neq k[\mathbf{z}]$, still admits a matrix factorization w.r.t. d.

Example 1 Let

$$\mathbf{F} = \begin{bmatrix} z_1z_2 - z_1 - z_2^2 - z_2z_3 & z_1z_3 + z_1 - z_2z_3 - z_2 - z_3^2 - z_3 & \mathbf{F}[1,3] \\ -z_1z_2 - z_1z_3 + z_2 + z_3 & z_2 + z_3 & z_1z_2 + z_1z_3 \\ z_1 & -z_1 + z_2 + z_3 & -2z_1 + z_2 + z_3 + 1 \end{bmatrix}$$

be a multivariate polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ with $z_1 > z_2 > z_3$, where $\mathbf{F}[1,3] = -z_1z_2 + z_1z_3 + 2z_1 + z_2^2 - z_2 - z_3^2 - 2z_3 - 1$ and $\mathbb{C}$ is a complex field.

It is easy to compute that $d_3(\mathbf{F}) = (z_1 - z_2)(z_2 + z_3)^2$ and $d_2(\mathbf{F}) = z_2 + z_3$. Let $d = z_1 - z_2$, then $d \mid d_3(\mathbf{F})$ but $d \nmid d_2(\mathbf{F})$. This implies that $\mathbf{F} \in \mathcal{S}_1$. Since the reduced Gröbner basis of $\langle d, d_2(\mathbf{F})\rangle$ w.r.t. the lexicographic order is $\{z_1 + z_3, z_2 + z_3\}$, we have $\langle d, c_1, \ldots, c_9\rangle \subseteq \langle d, d_2(\mathbf{F})\rangle \neq k[\mathbf{z}]$, where $c_1, \ldots, c_9$ are all the 2×2 minors of $\mathbf{F}$. Then, Lemma 6 is invalid.

However, we can get a matrix factorization of $\mathbf{F}$ w.r.t. d:

$$\mathbf{F} = \begin{bmatrix} d & 0 & -z_3 - 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_2 + z_3 & 0 & -z_2 - z_3 \\ -z_1z_2 - z_1z_3 + z_2 + z_3 & z_2 + z_3 & z_1z_2 + z_1z_3 \\ z_1 & -z_1 + z_2 + z_3 & -2z_1 + z_2 + z_3 + 1 \end{bmatrix}.$$

Example 1 shows that Lemma 6 is only a sufficient condition for $\mathbf{F} \in \mathcal{S}_1$ admitting a matrix factorization w.r.t. d. This implies that it would be significant to propose a new criterion for factorizing $\mathbf{F} \in \mathcal{S}_1$ w.r.t. d. In Example 1, we find that the reduced Gröbner basis of $\langle d, h_1, \ldots, h_9\rangle$ w.r.t. the lexicographic order is $\{1\}$. In spire of it, we consider the following problem.

Problem 1 Let $\mathbf{F} \in \mathcal{S}_1$. If $\langle d, h_1, \ldots, h_\gamma \rangle = k[\mathbf{z}]$, does $\mathbf{F}$ have a matrix factorization w.r.t. d?

As we mentioned above, Lemma 6 is invalid for any $\mathbf{F} \in \mathcal{S} \backslash \mathcal{S}_1$. Then, we consider another problem as follows.

Problem 2 Let $\mathbf{F} \in \mathcal{S} \setminus \mathcal{S}_1$. Is there a way to solve the existence problem of a matrix factorization for $\mathbf{F}$?

3 New criteria for factorizing polynomial matrices in $\mathcal{S}$

The main objective of this section is to solve Problems 1 and 2.

3.1 The case for $\mathbf{F} \in \mathcal{S}_1$

Before giving the main theorem, we introduce three important lemmas.

Lemma 7 (Lin et al. 2001) *Let $g \in k[\mathbf{z}]$ and $f(\mathbf{z}_2) \in k[\mathbf{z}_2]$. If $g(f, \mathbf{z}_2)$ is a zero polynomial in $k[\mathbf{z}_2]$, then $(z_1 - f(\mathbf{z}_2))$ is a divisor of g.*

Lemma 8 *Let $\mathbf{F} \in \mathcal{S}_1$, then* $\mathrm{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l - 1$.

Proof For simplicity of presentation, let $\hat{\mathbf{F}} = \mathbf{F}(f, \mathbf{z}_2)$. Then, $\hat{\mathbf{F}} \in k[\mathbf{z}_2]^{l \times m}$. Let $\hat{a}_1, \ldots, \hat{a}_\eta \in k[\mathbf{z}_2]$ and $\hat{c}_1, \ldots, \hat{c}_\gamma \in k[\mathbf{z}_2]$ be all the $l \times l$ minors and $(l-1) \times (l-1)$ minors of $\hat{\mathbf{F}}$, respectively. Then,

$$\hat{a}_i = a_i(f, \mathbf{z}_2) \text{ and } \hat{c}_j = c_j(f, \mathbf{z}_2), \text{ where } i = 1, \ldots, \eta \text{ and } j = 1, \ldots, \gamma.$$

Since $d \mid d_l(\mathbf{F})$, we have $\hat{a}_i = 0$ for $i = 1, \ldots, \eta$. This implies that $\mathrm{rank}(\hat{\mathbf{F}}) \leq l - 1$. If $\mathrm{rank}(\hat{\mathbf{F}}) < l - 1$, then $c_j(f, \mathbf{z}_2) = 0$ for $j = 1, \ldots, \gamma$. It follows from Lemma 7 that d is a common divisor of $c_1, \ldots, c_\gamma$. Thus $d \mid d_{l-1}(\mathbf{F})$, which contradicts the fact that $d \nmid d_{l-1}(\mathbf{F})$ for $\mathbf{F} \in \mathcal{S}_1$. Therefore, $\mathrm{rank}(\hat{\mathbf{F}}) = l - 1$. □

The following lemma is a non-trivial generalization of Lemma 2 in Lin et al. (2001). Although the proof of the lemma is similar to that of Theorem 1 in Lin (1993), for the sake of the rigor of the argument and the ease of understanding we still give a detailed proof here.

Lemma 9 *Let $\mathbf{F} \in k[\mathbf{z}]^{l \times m}$ with* $\mathrm{rank}(\mathbf{F}) = l - 1$. *If $\langle h_1, \ldots, h_\gamma \rangle = k[\mathbf{z}]$, then there is a* ZLP *vector $\mathbf{w} \in k[\mathbf{z}]^{1 \times l}$ such that $\mathbf{w}\mathbf{F} = \mathbf{0}_{1 \times m}$.*

Proof In view of $\mathrm{rank}(\mathbf{F}) = l - 1$, we could assume that the first $(l-1)$ row vectors $\mathbf{f}_1, \ldots, \mathbf{f}_{l-1}$ of $\mathbf{F}$ are $k[\mathbf{z}]$-linearly independent. This implies that $\mathbf{f}_1, \ldots, \mathbf{f}_{l-1}$ and $\mathbf{f}_l$ are $k[\mathbf{z}]$-linearly dependent. Thus $\mathbf{w}\mathbf{F} = \mathbf{0}_{1 \times m}$ for some nonzero row vector $\mathbf{w} = [w_1, \ldots, w_l] \in k[\mathbf{z}]^{1 \times l}$, where $w_l \neq 0$ and $\mathrm{GCD}(w_1, \ldots, w_l) = 1$. Obviously, $w_1, \ldots, w_l$ are all the 1×1 reduced minors of $\mathbf{w}$.

The next thing is to prove that $w_1, \ldots, w_l$ generate $k[\mathbf{z}]$. Let $\mathbf{F}_1, \ldots, \mathbf{F}_\beta$ be all the $l \times (l-1)$ sub-matrices of $\mathbf{F}$, where $\beta = \binom{m}{l-1}$. For each $1 \leq i \leq \beta$, let $c_{i1}, \ldots, c_{il}$ and $h_{i1}, \ldots, h_{il}$ be all the $(l-1) \times (l-1)$ minors and all the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}_i$, respectively. Then $c_{ij} = d_{l-1}(\mathbf{F}_i) \cdot h_{ij}$, where $i = 1, \ldots, \beta$ and $j = 1, \ldots, l$. Let $\mathbf{w} = [\mathbf{w}_1, w_l]$ and $\mathbf{F}_i = [\mathbf{F}_{i1}^{\mathrm{T}}, -\mathbf{F}_{i2}^{\mathrm{T}}]^{\mathrm{T}}$, where $\mathbf{w}_1 = [w_1, \ldots, w_{l-1}]$, $\mathbf{F}_{i1} \in k[\mathbf{z}]^{(l-1) \times (l-1)}$ and $\mathbf{F}_{i2} \in k[\mathbf{z}]^{1 \times (l-1)}$. If

$\mathbf{F}_i$ is not of full column rank, then $c_{ij} = 0$ and $h_{ij} = 0$, $j = 1, \ldots, l$. If $\mathbf{F}_i$ is of full column rank, then it follows from $\mathbf{wF} = \mathbf{0}_{1\times m}$ that

$$\left[\mathbf{w}_1, w_l\right] \begin{bmatrix} \mathbf{F}_{i1} \\ -\mathbf{F}_{i2} \end{bmatrix} = \mathbf{0}_{1\times(l-1)}. \tag{3}$$

Since $w_l \neq 0$, $\det(\mathbf{F}_{i1}) \neq 0$ by Lemma 4. From Eq. (3) we have

$$w_l^{-1}\mathbf{w}_1 = \mathbf{F}_{i2}\mathbf{F}_{i1}^{-1}. \tag{4}$$

According to Lemma 5, all the 1×1 reduced minors of $\mathbf{w}$ are equal to all the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}_i$ without considering the sign, i.e., $w_j = h_{ij}$ for $j = 1, \ldots, l$. Therefore, all the $(l-1) \times (l-1)$ minors of $\mathbf{F}$ are as follows:

$$d_{l-1}(\mathbf{F}_1) \cdot w_1, \ldots, d_{l-1}(\mathbf{F}_1) \cdot w_l, \cdots, d_{l-1}(\mathbf{F}_\beta) \cdot w_1, \ldots, d_{l-1}(\mathbf{F}_\beta) \cdot w_l.$$

Let $\bar{d} \in k[\mathbf{z}]$ be the GCD of $d_{l-1}(\mathbf{F}_1), \ldots, d_{l-1}(\mathbf{F}_\beta)$, then there exists $\bar{d}_i \in k[\mathbf{z}]$ such that $d_{l-1}(\mathbf{F}_i) = \bar{d} \cdot \bar{d}_i$, where $i = 1, \ldots, \beta$. In the following we prove that the polynomials

$$\begin{array}{cccc} \bar{d}_1 w_1, & \bar{d}_1 w_2, & \cdots & \bar{d}_1 w_l, \\ \bar{d}_2 w_1, & \bar{d}_2 w_2, & \cdots & \bar{d}_2 w_l, \\ \vdots & \vdots & \ddots & \vdots \\ \bar{d}_\beta w_1, & \bar{d}_\beta w_2, & \cdots & \bar{d}_\beta w_l, \end{array}$$

are all the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}$. It follows from $\mathrm{GCD}(w_1, \ldots, w_l) = 1$ and $\mathrm{GCD}(\bar{d}_1, \cdots, \bar{d}_\beta) = 1$ that

$$\begin{aligned} &\mathrm{GCD}(\bar{d}_1 w_1, \ldots, \bar{d}_1 w_l, \cdots, \bar{d}_\beta w_1, \ldots, \bar{d}_\beta w_l) \\ &\quad = \mathrm{GCD}(\mathrm{GCD}(\bar{d}_1 w_1, \ldots, \bar{d}_1 w_l), \cdots, \mathrm{GCD}(\bar{d}_\beta w_1, \ldots, \bar{d}_\beta w_l)) \\ &\quad = \mathrm{GCD}(\bar{d}_1, \cdots, \bar{d}_\beta) \\ &\quad = 1. \end{aligned}$$

Therefore, $\bar{d}_1 w_1, \ldots, \bar{d}_1 w_l, \cdots, \bar{d}_\beta w_1, \ldots, \bar{d}_\beta w_l$ are all the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}$, i.e., they are equal to $h_1, \ldots, h_\gamma$. Since $\langle h_1, \ldots, h_\gamma \rangle = k[\mathbf{z}]$, $w_1, \ldots, w_l$ generate $k[\mathbf{z}]$. □

Combining the above three lemmas, we can solve Problem 1.

Theorem 4 *Let* $\mathbf{F} \in \mathcal{S}_1$. *If* $\langle d, h_1, \ldots, h_\gamma \rangle = k[\mathbf{z}]$, *then* $\mathbf{F}$ *admits a matrix factorization w.r.t.* d.

Proof Let $\hat{\mathbf{F}} = \mathbf{F}(f, \mathbf{z}_2)$, then $\mathrm{rank}(\hat{\mathbf{F}}) = l - 1$ by Lemma 8. We first prove that all the $(l-1) \times (l-1)$ reduced minors of $\hat{\mathbf{F}}$ generate $k[\mathbf{z}_2]$. Let $\bar{h} \in k[\mathbf{z}_2]$ be the GCD of $h_1(f, \mathbf{z}_2), \ldots, h_\gamma(f, \mathbf{z}_2)$, then there exist $\hat{h}_1, \ldots, \hat{h}_\gamma \in k[\mathbf{z}_2]$ such that

$$h_j(f, \mathbf{z}_2) = \bar{h} \cdot \hat{h}_j \text{ for } j = 1, \ldots, \gamma \text{ and } \mathrm{GCD}(\hat{h}_1, \ldots, \hat{h}_\gamma) = 1.$$

Let $g = d_{l-1}(\mathbf{F})$ and $\hat{c}_1, \ldots, \hat{c}_\gamma \in k[\mathbf{z}_2]$ be all the $(l-1) \times (l-1)$ minors of $\hat{\mathbf{F}}$. It follows from $c_j = d_{l-1}(\mathbf{F}) \cdot h_j$ that

$$\hat{c}_j = g(f, \mathbf{z}_2) \cdot h_j(f, \mathbf{z}_2),$$

where $j = 1, \ldots, \gamma$. Thus $d_{l-1}(\hat{\mathbf{F}}) = g(f, \mathbf{z}_2) \cdot \bar{h}$ and $\hat{h}_1, \ldots, \hat{h}_\gamma$ are all the $(l-1) \times (l-1)$ reduced minors of $\hat{\mathbf{F}}$. Assume that $\langle \hat{h}_1, \ldots, \hat{h}_\gamma \rangle \neq k[\mathbf{z}_2]$, then there exists $(\alpha_2, \ldots, \alpha_n) \in k^{1\times(n-1)}$ such that $\hat{h}_j(\alpha_2, \ldots, \alpha_n) = 0$, where $j = 1, \ldots, \gamma$. Let $\alpha_1 = f(\alpha_2, \ldots, \alpha_n)$, then

$$h_j(\alpha_1, \alpha_2, \ldots, \alpha_n) = \bar{h}(\alpha_2, \ldots, \alpha_n) \cdot \hat{h}_j(\alpha_2, \ldots, \alpha_n) = 0, \ j = 1, \ldots, \gamma.$$

This implies that $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in k^{1\times n}$ is a common zero of $d, h_1, \ldots, h_\gamma$, which contradicts the fact that $\langle d, h_1, \ldots, h_\gamma \rangle = k[\mathbf{z}]$.

Now, we remark that $\mathbf{F}$ admits a matrix factorization w.r.t. d. Using Lemma 9, we get $\mathbf{w}\hat{\mathbf{F}} = \mathbf{0}_{1\times m}$, where $\mathbf{w} \in k[\mathbf{z}_2]^{1\times l}$ is a ZLP vector. Meanwhile, according to Lemma 2, a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{l\times l}$ can be constructed such that $\mathbf{w}$ is its first row. Let $\mathbf{F}_0 = \mathbf{UF}$, then the first row of $\mathbf{F}_0(f, \mathbf{z}_2) = \mathbf{U}\hat{\mathbf{F}}$ is a zero vector. By Lemma 7, d is a common divisor of the polynomials in the first row of $\mathbf{F}_0$, thus

$$\mathbf{F}_0 = \mathbf{UF} = \mathbf{DF}_1 = \begin{bmatrix} d & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \bar{f}_{11} & \cdots & \bar{f}_{1m} \\ f_{21} & \cdots & f_{2m} \\ \vdots & \ddots & \vdots \\ f_{l1} & \cdots & f_{lm} \end{bmatrix}.$$

Consequently, we can now derive the matrix factorization of $\mathbf{F}$ w.r.t. d:

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1,$$

where $\mathbf{G}_1 = \mathbf{U}^{-1}\mathbf{D} \in k[\mathbf{z}]^{l\times l}$, $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$ and $\det(\mathbf{G}_1) = d$. □

Theorem 4 presents a new criterion to factorize $\mathbf{F} \in \mathcal{S}_1$ w.r.t. d, and it is a generalization of Lemma 6.

Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ and $d_0 = \prod_{t=1}^{s}(z_1 - f_t(\mathbf{z}_2))$ be a divisor of $d_l(\mathbf{F})$, where $f_1(\mathbf{z}_2), \ldots, f_s(\mathbf{z}_2) \in k[\mathbf{z}_2]$. Liu et al. (2011) proved that if $\langle d_0, c_1, \ldots, c_\gamma \rangle = k[\mathbf{z}]$, then $\mathbf{F}$ admits a matrix factorization w.r.t. d_0. It would be interesting to know whether Theorem 4 can be generalized to the case for $s > 1$. Without loss of generality, we consider the case for $s = 2$.

Theorem 5 *Let* $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ *and* $d_0 = (z_1 - f_1(\mathbf{z}_2))(z_1 - f_2(\mathbf{z}_2))$ *be a divisor of* $d_l(\mathbf{F})$. *If* $\mathrm{GCD}(d_0, d_{l-1}(\mathbf{F})) = 1$ *and* $\langle d_0, h_1, \ldots, h_\gamma \rangle = k[\mathbf{z}]$, *then* $\mathbf{F}$ *admits a matrix factorization w.r.t.* d_0.

Proof Let $d_1 = z_1 - f_1(\mathbf{z}_2)$ and $d_2 = z_1 - f_2(\mathbf{z}_2)$. Obviously, $d_1 \nmid d_{l-1}(\mathbf{F})$ and $\langle d_1, h_1, \ldots, h_\gamma \rangle = k[\mathbf{z}]$. By Theorem 4, there exist $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$ such that

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1,$$

where $\mathbf{G}_1 = \mathbf{U}_1^{-1}\mathbf{D}_1$ with $\mathbf{D}_1 = \mathrm{diag}(d_1, 1, \ldots, 1)$, and $\mathbf{U}_1 \in k[\mathbf{z}_2]^{l\times l}$ is a unimodular matrix. According the Eq. (2) in Lemma 3, $d_2 = z_1 - f_2(\mathbf{z}_2)$ is a divisor of $d_l(\mathbf{F}_1)$. Next we prove that $\mathbf{F}_1$ admits a matrix factorization w.r.t. d_2.

We first prove that $d_2 \nmid d_{l-1}(\mathbf{F}_1)$. Otherwise, it follows from $d_{l-1}(\mathbf{F}_1) \mid d_{l-1}(\mathbf{F})$ that $d_2 \mid d_{l-1}(\mathbf{F})$, which contradicts the fact that $\mathrm{GCD}(d_0, d_{l-1}(\mathbf{F})) = 1$. This implies that $\mathbf{F}_1 \in \mathcal{S}_1$.

Second, we prove that d_2 and all the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}_1$ generate $k[\mathbf{z}]$. Let $\mathbf{F}_{i1} \in k[\mathbf{z}]^{(l-1)\times m}$ be a sub-matrix obtained by removing the i-th row of $\mathbf{F}_1$, and $\bar{c}_{i1}, \ldots, \bar{c}_{i\beta}$ be all the $(l-1) \times (l-1)$ minors of $\mathbf{F}_{i1}$, where $i = 1, \ldots, l$. Then,

$\bar{c}_{11}, \ldots, \bar{c}_{1\beta}, \ldots, \bar{c}_{l1}, \ldots, \bar{c}_{l\beta}$ are all the $(l-1) \times (l-1)$ minors of $\mathbf{F}_1$. Extracting $d_{l-1}(\mathbf{F}_1)$ from each $\bar{c}_{ij}$ yields $\bar{c}_{ij} = d_{l-1}(\mathbf{F}_1) \cdot \bar{h}_{ij}$, then $\bar{h}_{11}, \ldots, \bar{h}_{1\beta}, \ldots, \bar{h}_{l1}, \ldots, \bar{h}_{l\beta}$ are all the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}_1$. Hence, we only need to prove that $\langle d_2, \bar{h}_{11}, \ldots, \bar{h}_{l\beta}\rangle = k[\mathbf{z}]$.

Since $\mathbf{D}_1 = \mathrm{diag}(d_1, 1, \ldots, 1)$, all the $(l-1) \times (l-1)$ minors of $\mathbf{D}_1\mathbf{F}_1$ are

$$\bar{c}_{11}, \ldots, \bar{c}_{1\beta}, d_1\bar{c}_{21}, \ldots, d_1\bar{c}_{2\beta}, \ldots, d_1\bar{c}_{l1}, \ldots, d_1\bar{c}_{l\beta}.$$

Then, there is at least one integer $j \in \{1, \ldots, \beta\}$ such that $d_1 \nmid \bar{c}_{1j}$. Otherwise, $d_1 \mid d_{l-1}(\mathbf{D}_1\mathbf{F}_1)$. It follows form $\mathbf{F} = \mathbf{U}_1^{-1}\mathbf{D}_1\mathbf{F}_1$ and the Eq. (1) in Lemma 3 that $d_{l-1}(\mathbf{D}_1\mathbf{F}_1) \mid d_{l-1}(\mathbf{F})$. Thus $d_1 \mid d_{l-1}(\mathbf{F})$, which leads to a contradiction. Since $d_1 = z_1 - f_1(\mathbf{z}_2)$ is an irreducible polynomial, we have

$$\begin{aligned}
&\mathrm{GCD}(\bar{c}_{11}, \ldots, \bar{c}_{1\beta}, d_1\bar{c}_{21}, \ldots, d_1\bar{c}_{2\beta}, \ldots, d_1\bar{c}_{l1}, \ldots, d_1\bar{c}_{l\beta}) \\
&\quad = \mathrm{GCD}(\bar{c}_{11}, \ldots, \bar{c}_{1\beta}, \bar{c}_{21}, \ldots, \bar{c}_{2\beta}, \ldots, \bar{c}_{l1}, \ldots, \bar{c}_{l\beta}).
\end{aligned}$$

Therefore, $d_{l-1}(\mathbf{D}_1\mathbf{F}_1) = d_{l-1}(\mathbf{F}_1)$. It follows from $\mathbf{U}_1\mathbf{F} = \mathbf{D}_1\mathbf{F}_1$ that $d_{l-1}(\mathbf{F}) \mid d_{l-1}(\mathbf{D}_1\mathbf{F}_1)$. This implies that $d_{l-1}(\mathbf{F}) = d_{l-1}(\mathbf{F}_1)$.

The Eq. (1) in Lemma 3 implies that each c_{i_0} is a $k[\mathbf{z}]$-linear combination of $\bar{c}_{11}, \ldots, \bar{c}_{l\beta}$, where $1 \le i_0 \le \gamma$. Since $d_{l-1}(\mathbf{F}) = d_{l-1}(\mathbf{F}_1)$, we get that each h_{i_0} is a $k[\mathbf{z}]$-linear combination of $\bar{h}_{11}, \ldots, \bar{h}_{l\beta}$, where $1 \le i_0 \le \gamma$. Since $\langle d_0, h_1, \ldots, h_\gamma\rangle = k[\mathbf{z}]$, we have $\langle d_2, h_1, \ldots, h_\gamma\rangle = k[\mathbf{z}]$. If $\langle d_2, \bar{h}_{11}, \ldots, \bar{h}_{l\beta}\rangle \neq k[\mathbf{z}]$, then there exists $(\alpha_1, \ldots, \alpha_n) \in k^{1\times n}$ such that $\alpha_1 = f_2(\alpha_2, \ldots, \alpha_n)$ and $\bar{h}_{ij}(\alpha_1, \ldots, \alpha_n) = 0$, where $i = 1, \ldots, l$ and $j = 1, \ldots, \beta$. This implies that $(\alpha_1, \ldots, \alpha_n)$ is a common zero of $d_2, h_1, \ldots, h_\gamma$, which leads to a contradiction.

According to Theorem 4 again, there exist $\mathbf{G}_2 \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{F}_2 \in k[\mathbf{z}]^{l\times m}$ such that $\mathbf{F}_1 = \mathbf{G}_2\mathbf{F}_2$, where $\mathbf{G}_2 = \mathbf{U}_2^{-1}\mathbf{D}_2$ with $\mathbf{D}_2 = \mathrm{diag}(d_2, 1, \ldots, 1)$, and $\mathbf{U}_2 \in k[\mathbf{z}_2]^{l\times l}$ is a unimodular matrix.

Finally, we get a matrix factorization of $\mathbf{F}$ w.r.t. d_0:

$$\mathbf{F} = \mathbf{G}_0\mathbf{F}_2,$$

where $\mathbf{G}_0 = \mathbf{G}_1\mathbf{G}_2$ with $\det(\mathbf{G}_0) = d_0$. □

Remark 2 In Theorem 5, $\mathrm{GCD}(d_0, d_{l-1}(\mathbf{F})) = 1$ implies that $d_1 \nmid d_{l-1}(\mathbf{F})$ and $d_2 \nmid d_{l-1}(\mathbf{F})$. The most important thing is that we can factorize $\mathbf{F}_1$ w.r.t. d_2 without checking $d_2 \nmid d_{l-1}(\mathbf{F}_1)$ and the ideal generated by d_2 and all the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}_1$ is equal to $k[\mathbf{z}]$, which can help us improve the computational efficiency of matrix factorizations.

It is worth noting that if $f_1(\mathbf{z}_2) = f_2(\mathbf{z}_2)$ in Theorem 5, we have the following corollary.

Corollary 1 *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ and $d_0 = (z_1 - f_1(\mathbf{z}_2))^r$ be a divisor of $d_l(\mathbf{F})$. If $\mathrm{GCD}(d_0, d_{l-1}(\mathbf{F})) = 1$ and $\langle d_0, h_1, \ldots, h_\gamma\rangle = k[\mathbf{z}]$, then $\mathbf{F}$ admits a matrix factorization w.r.t. d_0.*

Further, if $f_1(\mathbf{z}_2) \neq f_2(\mathbf{z}_2)$ in Theorem 5, we have another corollary.

Corollary 2 *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ and $d_0 = \prod_{t=1}^{s}(z_1 - f_j(\mathbf{z}_2))^{q_t}$ be a divisor of $d_l(\mathbf{F})$. If $\mathrm{GCD}(d_0, d_{l-1}(\mathbf{F})) = 1$ and $\langle d_0, h_1, \ldots, h_\gamma\rangle = k[\mathbf{z}]$, then $\mathbf{F}$ admits a matrix factorization w.r.t. d_0.*

3.2 The case for $\mathbf{F} \in \mathcal{S} \setminus \mathcal{S}_1$

Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$, we denote by $d_i(\mathbf{F})$ the GCD of all the $i \times i$ minors of $\mathbf{F}$, where $i = 1, \ldots, l$. For convenience, let $d_0(\mathbf{F}) = 1$. According to the basic property of matrix theory, we have $d_0(\mathbf{F}) \mid d_1(\mathbf{F}) \mid \cdots \mid d_{l-1}(\mathbf{F}) \mid d_l(\mathbf{F})$.

Let $\mathbf{F} \in \mathcal{S} \setminus \mathcal{S}_1$. Since d is a divisor of $d_{l-1}(\mathbf{F})$, there is a unique integer i such that $d \mid d_{l-i+1}(\mathbf{F})$ but $d \nmid d_{l-i}(\mathbf{F})$, where $2 \le i \le l$. Based on this phenomenon, we construct $l-1$ subsets of $\mathcal{S} \setminus \mathcal{S}_1$:

$$\mathcal{S}_i = \{\mathbf{F} \in \mathcal{S} \setminus \mathcal{S}_1 : d \mid d_{l-i+1}(\mathbf{F}) \text{ but } d \nmid d_{l-i}(\mathbf{F})\},\ i = 2, \ldots, l.$$

Then, For any given $\mathbf{F} \in \mathcal{S} \setminus \mathcal{S}_1$ and $d = z_1 - f(\mathbf{z}_2)$, there exists a unique integer i such that $\mathbf{F} \in \mathcal{S}_i$ with $2 \le i \le l$. Therefore, Problem 2 is equivalent to the problem: is there a way to solve the existence problem of a matrix factorization for $\mathbf{F} \in \mathcal{S}_i$ with $2 \le i \le l$?

We first solve the case of $\mathbf{F} \in \mathcal{S}_l$.

Theorem 6 *Let $\mathbf{F} \in \mathcal{S}_l$, then $\mathbf{F}$ admits a matrix factorization w.r.t. d^l.*

Proof Let $\mathbf{F} = \left[f_{ij}\right]_{l\times m}$, where $f_{ij} \in k[\mathbf{z}]$. It follows from $\mathbf{F} \in \mathcal{S}_l$ that $d \mid f_{ij}$, where $i = 1, \ldots, l$ and $j = 1, \ldots, m$. Then, we can extract d from each f_{ij} and obtain a matrix factorization of $\mathbf{F}$ w.r.t. d^l. That is, there exists $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$ such that $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ with $\mathbf{G}_1 = \mathrm{diag}(d, \ldots, d)$. □

In the following, we consider the case of $\mathbf{F} \in \mathcal{S}_i$ with $2 \le i < l$. We can generalize Theorem 4 to the case of $\mathbf{F} \in \mathcal{S}_i$. A proof can be given similarly as the proof for Theorem 4, thus it is omitted here.

Theorem 7 *Let $\mathbf{F} \in \mathcal{S}_i$ with $2 \le i < l$ and $h_{(l-i),1}, \ldots, h_{(l-i),\gamma_{(l-i)}}$ be all the $(l-i) \times (l-i)$ reduced minors of $\mathbf{F}$. If $\langle d, h_{(l-i),1}, \ldots, h_{(l-i),\gamma_{(l-i)}}\rangle = k[\mathbf{z}]$, then $\mathbf{F}$ admits a matrix factorization w.r.t. d^i.*

Remark 3 The proof of Theorem 7 can show that we extract d^i by only once matrix factorization. Moreover, Theorem 7 is a generalization of Theorem 3.3 in Lu et al. (2017).

In summary, for each i with $2 \le i \le l$, we propose a new criterion to factorize $\mathbf{F} \in \mathcal{S}_i$ w.r.t. d^i. This implies that there is a way to solve the existence problem of a matrix factorization for $\mathbf{F} \in \mathcal{S} \setminus \mathcal{S}_1$. Therefore, we solve Problem 2.

3.3 Generalizations of the type of polynomial matrices

Let $f(\mathbf{z}\backslash z_j)$ be a polynomial in $k[z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n]$, where $1 \le j \le n$. Then, we can get the following corollaries.

Corollary 3 *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ and $d_0 = (z_j - f(\mathbf{z}\backslash z_j))^r$ be a divisor of $d_l(\mathbf{F})$. If $\mathrm{GCD}(d_0, d_{l-1}(\mathbf{F})) = 1$ and $\langle d_0, h_1, \ldots, h_\gamma\rangle = k[\mathbf{z}]$, then $\mathbf{F}$ admits a matrix factorization w.r.t. d_0.*

Corollary 4 *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ and $d_0 = \prod_{j=1}^{n}\prod_{t=1}^{s_j}(z_j - f_t(\mathbf{z}\backslash z_j))^{q_{jt}}$ be a divisor of $d_l(\mathbf{F})$. If $\mathrm{GCD}(d_0, d_{l-1}(\mathbf{F})) = 1$ and $\langle d_0, h_1, \ldots, h_\gamma\rangle = k[\mathbf{z}]$, then $\mathbf{F}$ admits a matrix factorization w.r.t. d_0.*

Corollary 5 *Let* $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ *and* $d_j = z_j - f(\mathbf{z}\backslash z_j)$ *be a divisor of* $d_l(\mathbf{F})$. *For any given* i *with* $1 \le i < l$, *let* $h_{(l-i),1}, \ldots, h_{(l-i),\gamma_{(l-i)}}$ *be all the* $(l-i)\times(l-i)$ *reduced minors of* $\mathbf{F}$. *Assume that* $d_j \mid d_{l-i+1}(\mathbf{F})$ *but* $d_j \nmid d_{l-i}(\mathbf{F})$. *Then* $\mathbf{F}$ *admits a matrix factorization w.r.t.* d_j^i *if* $\langle d_j, h_{(l-i),1}, \ldots, h_{(l-i),\gamma_{(l-i)}}\rangle = k[\mathbf{z}]$.

Corollary 6 *Let* $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ *and* $d_j = z_j - f(\mathbf{z}\backslash z_j)$ *be a divisor of* $d_1(\mathbf{F})$, *where* $1 \le j \le n$. *Then* $\mathbf{F}$ *admits a matrix factorization w.r.t.* d_j^l.

4 Algorithm and example

According to the main results presented in Sect. 3, there is a way to solve the existence problem of a matrix factorization for $\mathbf{F} \in \mathcal{S}$. Combining the constructive algorithm proposed by Lin et al. (2001), we get the following algorithm for computing a matrix factorization of $\mathbf{F} \in \mathcal{S}$.

Algorithm 1: multivariate polynomial matrix factorization

Input : $\mathbf{F} \in \mathcal{S}$, $d = z_1 - f(\mathbf{z}_2)$ and a monomial order $\prec_{\mathbf{z}}$.
Output: a matrix factorization of $\mathbf{F}$ w.r.t. d^r, where r is an integer with $1 \le r \le l$.

1 **begin**
2   **for** i *from* 1 *to* l **do**
3   compute $d_{l-i}(\mathbf{F})$ and $(l-i)\times(l-i)$ reduced minors $h_{(l-i),1}, \ldots, h_{(l-i),\gamma_{(l-i)}}$;
4   **if** $d \nmid d_{l-i}(\mathbf{F})$ **then**
5       $r := i$;
6   **break**;
7   **if** $r = l$ **then**
8   extract d from each row of $\mathbf{F}$, i.e., $\mathbf{F} = \mathrm{diag}(d, \ldots, d)\cdot \mathbf{F}_1$;
9   **return** $\mathrm{diag}(d, \ldots, d)$ and $\mathbf{F}_1$.
10   compute a reduced Gröbner basis $\mathcal{G}$ of $\langle d, h_{(l-r),1}, \ldots, h_{(l-r),\gamma_{(l-r)}}\rangle$ w.r.t. $\prec_{\mathbf{z}}$;
11   **if** $\mathcal{G} = \{1\}$ **then**
12   compute a ZLP matrix $\mathbf{w} \in k[\mathbf{z}_2]^{r\times l}$ such that $\mathbf{w}\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{r\times m}$;
13   construct a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{l\times l}$ such that $\mathbf{w}$ is its first r rows;
14   extract d from the first r rows of $\mathbf{UF}$, i.e., $\mathbf{UF} = \mathrm{diag}(d, \ldots, d, 1, \ldots, 1)\cdot \mathbf{F}_1$;
15   **return** $\mathbf{U}^{-1}\cdot \mathrm{diag}(d, \ldots, d, 1, \ldots, 1)$ and $\mathbf{F}_1$.
16   **else**
17   **return** unable to judge.

In Algorithm 1, we use Step 4 to verify $\mathbf{F} \in \mathcal{S}_r$. Step 7 shows that $\mathbf{F} \in \mathcal{S}_l$ and we can extract d directly from each row of $\mathbf{F}$ by Theorem 6. Then, it is easy to obtain a matrix factorization of $\mathbf{F}$ w.r.t. d^l. When $1 \le r < l$, we use Theorems 4 and 7 to verify whether there is a matrix factorization of $\mathbf{F} \in \mathcal{S}_r$ w.r.t. d^r. Step 16 tells us that $\mathcal{G} \ne \{1\}$. This implies that Theorems 4 and 7 are invalid at this time. Thus, we return "unable to judge". In the following, we show how to compute Step 12 and Step 13 for the case of $\mathbf{F} \in \mathcal{S}_1$.

Let $\hat{\mathbf{F}} = \mathbf{F}(f, \mathbf{z}_2)$ and $\mathrm{Syz}_1(\hat{\mathbf{F}})$ be a module generated by all vectors in $\{\mathbf{p} \in k[\mathbf{z}_2]^{1\times l} : \mathbf{p}\hat{\mathbf{F}} = \mathbf{0}_{1\times m}\}$. Since $\mathrm{rank}(\hat{\mathbf{F}}) = l - 1$, we have $\mathrm{rank}(\mathrm{Syz}_1(\hat{\mathbf{F}})) = 1$. Then, we compute a reduced Gröbner basis of $\mathrm{Syz}_1(\hat{\mathbf{F}})$ and select a nonzero vector from the Gröbner basis.

Let $\mathbf{w}_1 = [w_{11}, \dots, w_{1l}] \in k[\mathbf{z}_2]^{1\times l}$ be the nonzero vector and $w \in k[\mathbf{z}_2]$ be the GCD of $w_{11}, \dots, w_{1l}$, then $\mathbf{w} = \frac{\mathbf{w}_1}{w}$.

According to Theorem 4, $\mathbf{w}$ is a ZLP vector. Then there exists a column vector $\mathbf{q}_1 \in k[\mathbf{z}_2]^{l\times 1}$ such that $\mathbf{w}\mathbf{q}_1 = 1$. This calculation problem is equivalent to a lifting homomorphism problem in Decker and Lossen (2006) (see Problem 4.1, page 129), and the command "lift" of the computer algebra system *Singular* in Decker et al. (2016) can help us compute $\mathbf{q}_1$. Let $\mathrm{Syz}_2(\mathbf{w})$ be a module generated by all vectors in $\{\mathbf{q} \in k[\mathbf{z}_2]^{l\times 1} \mid \mathbf{w}\mathbf{q} = 0\}$, then $\mathrm{Syz}_2(\mathbf{w})$ is a free module with $\mathrm{rank}(\mathrm{Syz}_2(\mathbf{w})) = l - 1$ by Lemma 1. Let $\mathbf{q}_2, \dots, \mathbf{q}_l \in k[\mathbf{z}_2]^{l\times 1}$ be a free basis of $\mathrm{Syz}_2(\mathbf{w})$, then $\mathbf{V} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_l] \in k[\mathbf{z}_2]^{l\times l}$ is a unimodular matrix and $\mathbf{U} = \mathbf{V}^{-1}$ is one that we want by Theorem 4.4 in Lu et al. (2017).

According to the above calculations, we obtain $\mathbf{w}$ and $\mathbf{U}$ in Step 12 and Step 13. Since the calculation process of Step 12 and Step 13 for the case of $\mathbf{F} \in \mathcal{S}_r$ with $1 < r < l$ is similar to that of $\mathbf{F} \in \mathcal{S}_1$, we refer to Lu et al. (2017) for more details.

Now, we use an example to illustrate the calculation process of Algorithm 1. We return to Example 1.

Example 2 Let

$$\mathbf{F} = \begin{bmatrix} z_1z_2 - z_1 - z_2^2 - z_2z_3 & z_1z_3 + z_1 - z_2z_3 - z_2 - z_3^2 - z_3 & \mathbf{F}[1,3] \\ -z_1z_2 - z_1z_3 + z_2 + z_3 & z_2 + z_3 & z_1z_2 + z_1z_3 \\ z_1 & -z_1 + z_2 + z_3 & -2z_1 + z_2 + z_3 + 1 \end{bmatrix}$$

be a multivariate polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ and $\prec_{\mathbf{z}}$ be the lexicographic order with $z_1 > z_2 > z_3$, where $\mathbf{F}[1,3] = -z_1z_2 + z_1z_3 + 2z_1 + z_2^2 - z_2 - z_3^2 - 2z_3 - 1$ and $\mathbb{C}$ is a complex field.

It is easy to compute that $d_3(\mathbf{F}) = (z_1 - z_2)(z_2 + z_3)^2$. Let $d_1 = z_1 - z_2$ and $d_2 = z_2 + z_3$. In the following, we first compute a matrix factorization of $\mathbf{F}$ w.r.t. d_1 and obtain $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ with $\det(\mathbf{G}_1) = d_1$. Second, we compute a matrix factorization of $\mathbf{F}_1$ w.r.t. d_2^2. Finally, we have a matrix factorization of $\mathbf{F}$ w.r.t. $d_3(\mathbf{F})$.

Now, the input of Algorithm 1 are $\mathbf{F}$, d_1 and $\prec_{\mathbf{z}}$.

As already noted in Example 1, $\mathbf{F} \in \mathcal{S}_1$ and $\langle d_1, h_1, \dots, h_9\rangle = \mathbb{C}[z_1, z_2, z_3]$, where $h_1, \dots, h_9$ are all the 2×2 reduced minors of $\mathbf{F}$. This implies that $\mathbf{F}$ admits a matrix factorization w.r.t. d_1.

Step 1: Let $\hat{\mathbf{F}} = \mathbf{F}(z_2, z_2, z_3)$. We compute a ZLP vector $\mathbf{w} \in k[z_2, z_3]^{1\times 3}$ such that $\mathbf{w}\hat{\mathbf{F}} = \mathbf{0}_{1\times 3}$, where

$$\hat{\mathbf{F}} = \begin{bmatrix} -z_2(z_3+1) & -z_3(z_3+1) & -(z_3 - z_2 + 1)(z_3 + 1) \\ (1 - z_2)(z_2 + z_3) & z_2 + z_3 & z_2(z_2 + z_3) \\ z_2 & z_3 & z_3 - z_2 + 1 \end{bmatrix}.$$

Then, we use *Singular* command "syz" to compute a reduced Gröbner basis of $\mathrm{Syz}_1(\hat{\mathbf{F}})$ and obtain $\mathbf{w} = [1, 0, z_3 + 1]$.

Step 2: Construct a unimodular matrix $\mathbf{U} \in k[z_2, z_3]^{3\times 3}$ such that $\mathbf{w}$ is its first row. According to the instruction of the construction for $\mathbf{U}$ below Algorithm 1, we divide it into three small steps.

Step 2.1: Using *Singular* command "lift" to compute $\mathbf{q}_1 \in k[z_2, z_3]^{3\times 1}$ such that $\mathbf{w}\mathbf{q}_1 = 1$, we get $\mathbf{q}_1 = [1, 0, 0]^{\mathrm{T}}$.

Step 2.2: Using QuillenSuslin package to compute a free basis of $\mathrm{Syz}_2(\mathbf{w})$, we have $\mathbf{q}_2 = [0, 1, 0]^{\mathrm{T}}$ and $\mathbf{q}_3 = [-(z_3 + 1), 0, 1]^{\mathrm{T}}$.

Step 2.3: Let $\mathbf{V} = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]$, then

$$\mathbf{U} = \mathbf{V}^{-1} = \begin{bmatrix} 1 & 0 & z_3+1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 3: Extracting d_1 from the first row of $\mathbf{UF}$, we get $\mathbf{UF} = \mathbf{DF}_1$, where $\mathbf{D} = \mathrm{diag}(d_1, 1, 1)$ and

$$\mathbf{F}_1 = \begin{bmatrix} z_2+z_3 & 0 & -z_2-z_3 \\ -z_1z_2-z_1z_3+z_2+z_3 & z_2+z_3 & z_1z_2+z_1z_3 \\ z_1 & -z_1+z_2+z_3 & -2z_1+z_2+z_3+1 \end{bmatrix}.$$

Then, we obtain a matrix factorization of $\mathbf{F}$ w.r.t. d_1:

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 = \begin{bmatrix} d_1 & 0 & -z_3-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_2+z_3 & 0 & -z_2-z_3 \\ -z_1z_2-z_1z_3+z_2+z_3 & z_2+z_3 & z_1z_2+z_1z_3 \\ z_1 & -z_1+z_2+z_3 & -2z_1+z_2+z_3+1 \end{bmatrix},$$

where $\mathbf{G}_1 = \mathbf{U}^{-1}\mathbf{D}$ and $\det(\mathbf{G}_1) = d_1$.

At this point, the input of Algorithm 1 are $\mathbf{F}_1$, d_2 and $\prec_{\mathbf{z}}$.

It is easy to compute that $d_3(\mathbf{F}_1) = (z_2+z_3)^2$, $d_2(\mathbf{F}_1) = z_2+z_3$ and $d_1(\mathbf{F}_1) = 1$. Since $d_2 \mid d_2(\mathbf{F}_1)$ and $d_2 \nmid d_1(\mathbf{F}_1)$, we have $\mathbf{F}_1 \in \mathcal{S}_2$. It follows from $d_1(\mathbf{F}_1) = 1$ that the entries in $\mathbf{F}_1$ are all the 1×1 reduced minors of $\mathbf{F}_1$. Let $h_{1,1}, \ldots, h_{1,9}$ be all the 1×1 reduced minors of $\mathbf{F}_1$, then a reduced Gröbner basis of $\langle d_2, h_{1,1}, \ldots, h_{1,9} \rangle$ w.r.t. $\prec_{\mathbf{z}}$ is $\mathcal{G} = \{1\}$. Thus, $\mathbf{F}_1$ admits a matrix factorization w.r.t. d_2^2.

Similarly, we obtain a matrix factorization of $\mathbf{F}_1$ w.r.t. d_2^2:

$$\mathbf{F}_1 = \mathbf{G}_2\mathbf{F}_2 = \begin{bmatrix} d_2 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -z_1+1 & 1 & z_1 \\ z_1 & -z_1+z_2+z_3 & -2z_1+z_2+z_3+1 \end{bmatrix},$$

where $\det(\mathbf{G}_2) = d_2^2$.

In summary, we have a matrix factorization of $\mathbf{F}$ w.r.t. $d_3(\mathbf{F})$:

$$\mathbf{F} = \mathbf{GF}_2,$$

where $\mathbf{G} = \mathbf{G}_1\mathbf{G}_2$ and $\det(\mathbf{G}) = d_3(\mathbf{F})$.

Remark 4 In Example 2, we can first inputs $\mathbf{F}$, d_2 and $\prec_{\mathbf{z}}$. Then we can verify that $\mathbf{F} \in \mathcal{S}_2$ and $\mathbf{F}$ admits a matrix factorization w.r.t. d_2^2. Assume that $\mathbf{F} = \mathbf{G}_1'\mathbf{F}_1'$ with $\det(\mathbf{G}_1') = d_2^2$. Second, we inputs $\mathbf{F}_1'$, d_1 and $\prec_{\mathbf{z}}$. Then we can check that $\mathbf{F}_1' \in \mathcal{S}_1$ and $\mathbf{F}_1'$ admits a matrix factorization w.r.t. d_1. Assume that $\mathbf{F}_1' = \mathbf{G}_2'\mathbf{F}_2'$ with $\det(\mathbf{G}_2') = d_1$. Therefore, we obtain a matrix factorization of $\mathbf{F}$ w.r.t. $d_3(\mathbf{F})$: $\mathbf{F} = \mathbf{G}'\mathbf{F}_2'$, where $\mathbf{G}' = \mathbf{G}_1'\mathbf{G}_2'$ and $\det(\mathbf{G}') = d_3(\mathbf{F})$. The detailed calculation process is similar to that of Example 2, thus it is omitted here.

5 Conclusions

In this paper, it is shown that polynomial matrices in $\mathcal{S}$ can be factorized under satisfying some new criteria. Based on the condition $d \nmid d_{l-1}(\mathbf{F})$, we divide $\mathcal{S}$ into two parts: $\mathcal{S}_1$ and $\mathcal{S} \backslash \mathcal{S}_1$. When $\mathbf{F} \in \mathcal{S}_1$, we focus on the relationship among d and all the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}$, and get a new criterion to judge whether $\mathbf{F}$ admits a matrix factorization w.r.t. d. Then, we successfully extend this result to the case of $\mathbf{F} \in \mathcal{S} \setminus \mathcal{S}_1$. Some generalizations about the type of polynomial matrices have been presented, and the implementation of our algorithm has been illustrated by a non-trivial example.

The main contributions of this paper include: (1) three main theorems (Theorems 4, 6 and 7) propose some new criteria to factorize polynomial matrices in $\mathcal{S}$, as a consequence, the application range of the constructive algorithm in Lin et al. (2001) has been greatly extended; (2) for the case of $\mathbf{F} \in \mathcal{S}_1$, a nice property about reducing the amount of calculation has been presented (Theorem 5).

If $\mathcal{G} \neq \{1\}$, then Algorithm 1 returns "unable to judge". At this moment, how to establish a necessary and sufficient condition for $\mathbf{F} \in \mathcal{S}_i$ $(1 \leq i < l)$ admitting a matrix factorization w.r.t. d^i is the question that remain for further investigation.

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