A Property of Modules Over a Polynomial Ring With an Application in Multivariate Polynomial Matrix Factorizations

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ABSTRACT

This paper is concerned with a property of modules over a polynomial ring and its application in multivariate polynomial matrix factorizations. We construct a specific polynomial such that the product of the polynomial and a nonzero vector in a module over a polynomial ring can be represented by the elements in a maximum linearly independent vector set of the module over the polynomial ring. Based on this property, a relationship between a rank-deficient matrix and any of its full row rank submatrices is presented. By this result, we show that the problem for general factorizations of rank-deficient matrices can be translated into that of any of their full row rank submatrices in the regular case. Then many results on factorizations of full row rank matrices, such as zero prime factorizations, minor prime factorizations and factor prime factorizations, can be extended to the rank-deficient case. We implement the algorithm of general factorizations for rank-deficient matrices on the computer algebra system Maple, and two examples are given to illustrate the algorithm.

CCS CONCEPTS

• **Computing methodologies** → **Symbolic and algebraic algorithms**; **Algebraic algorithms**.

KEYWORDS

Multivariate polynomial matrices, Rank-deficient matrices, General factorization, Free module

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1 INTRODUCTION

It is well known that every maximum linearly independent vector set of a vector space over a field is a basis of the vector space. For instance, let $\mathcal{V}$ be a subspace of k^m, and $\mathcal{G}$ be a maximum k-linearly independent vector set of $\mathcal{V}$, where k is a field and m is a positive integer. Then $\mathcal{G}$ is a basis of $\mathcal{V}$, and each vector $\vec{g} \in \mathcal{V}$ can be expressed as linear combinations of the elements in $\mathcal{G}$ over k. However, this fact may not hold for the case of modules over a polynomial ring.

Example 1.1. Let $\mathcal{M}$ be a submodule of $k[z_1, z_2]^3$ generated by $\vec{w}_1, \vec{w}_2, \vec{w}_3$, where $\vec{w}_1 = (1-z_2, -z_1-1, 0)$, $\vec{w}_2 = (z_1+z_1z_2, 0, -z_1-1)$ and $\vec{w}_3 = (0, z_1 + z_1z_2, z_2 - 1)$. Since

$$(z_1 + z_1z_2)\vec{w}_1 + (z_2 - 1)\vec{w}_2 + (z_1 + 1)\vec{w}_3 = \vec{0},$$

$\vec{w}_1, \vec{w}_2, \vec{w}_3$ are $k[z_1, z_2]$-linearly dependent. Setting $\mathcal{G} = \{\vec{w}_1, \vec{w}_2\}$. It is easy to check that $\vec{w}_1$ and $\vec{w}_2$ are $k[z_1, z_2]$-linearly independent. This implies that both the rank of $\mathcal{M}$ and the module generated by the elements in $\mathcal{G}$ are 2. It follows that $\mathcal{G}$ is a maximum $k[z_1, z_2]$-linearly independent vector set of $\mathcal{M}$. However, $\vec{w}_3 \in \mathcal{M}$ cannot be represented by the elements in $\mathcal{G}$ over $k[z_1, z_2]$. That is, there are no polynomials $h_1, h_2 \in k[z_1, z_2]$ such that

$$\vec{w}_3 = h_1\vec{w}_1 + h_2\vec{w}_2.$$

Obviously, $\mathcal{G}$ is not a basis of $\mathcal{M}$.

It can be seen from the above example that a maximum linearly independent vector set of a module over the multivariate polynomial ring $k[\mathbf{z}]$ may not be a basis, which leads to some elements in the module that cannot be represented by the elements in the set over $k[\mathbf{z}]$. This phenomenon cannot occur in any vector space over k.

If a module $\mathcal{M} \subset k[\mathbf{z}]^m$ has a basis $\mathcal{G}$ (that is, $\mathcal{M}$ generated by $\mathcal{G}$ and the elements in $\mathcal{G}$ are $k[\mathbf{z}]$-linearly independent), then $\mathcal{M}$ is said to be a free module. The traditional method of checking that a module is a free module is to calculate Fitting ideals of the module, we refer to [4, 6, 9] for more details. Lu et al. [24] proposed a new simpler verification method by computing column reduced minors of the module. If the ideal generated by column reduced minors of a module is the unit ideal $k[\mathbf{z}]$, then the module is a free module. It is easy to check that the 2×2 column reduced minors of $\mathcal{M}$ in Example 1.1 generate $k[z_1, z_2]$, then $\mathcal{M}$ is a free module. Nevertheless, it can be difficult to compute a basis of a free module. In 2007, Fabiańska and Quadrat [7] designed an algorithm, which is quite complicated, to compute bases of free modules over polynomial rings. The algorithm was implemented on a *Maple* package, called QUILLENSUSLIN.

Let $\mathcal{M}$ be a module over $k[\mathbf{z}]$. We can judge whether $\mathcal{M}$ is a free module by calculating Fitting ideals or column reduced minors of $\mathcal{M}$. If $\mathcal{M}$ is a free module, we can get a basis $\mathcal{G}$ of $\mathcal{M}$ by using the package QUILLENSUSLIN. Then, every vector in $\mathcal{M}$ can be represented by the elements in $\mathcal{G}$ over $k[\mathbf{z}]$. Otherwise, we take $\mathcal{G}$ as a maximum $k[\mathbf{z}]$-linearly independent vector set of $\mathcal{M}$. Obviously, there exist some vectors in $\mathcal{M}$ that cannot be expressed as linear combinations of the elements in $\mathcal{G}$ over $k[\mathbf{z}]$. Based on this situation, we consider the following problem.

Problem 1.2. *Let $\mathcal{M}$ be a module over $k[\mathbf{z}]$ and $\mathcal{G}$ be a maximum $k[\mathbf{z}]$-linearly independent vector set of $\mathcal{M}$. For each $\vec{w} \in \mathcal{M} \setminus \mathcal{G}$, how to give an explicit construction of $h \in k[\mathbf{z}]$ such that $h \cdot \vec{w}$ can be represented by the elements in $\mathcal{G}$ over $k[\mathbf{z}]$? Furthermore, by the explicit construction, we want to get a general and simple formula to describe the "minimal" h for any $\vec{w}$ and $\mathcal{G}$ without complicated calculation.*

The above problem will play an important role in multivariate polynomial matrix factorizations. Multivariate polynomial matrix factorizations have been widely investigated during the past years due to the wide range of applications in multidimensional (n-D) circuits, systems, controls, signal processing, and other related areas (see e.g., [1, 2]). For univariate and bivariate cases, polynomial matrix factorizations have been completely solved in [12, 25] using the so called matrix primitive factorizations over three decades ago. Deng et al. [5] presented an efficient algorithm to compute the μ-basis of a rational surface by using the matrix factorization algorithm proposed in [12]. However, when $n \geq 3$, the corresponding problems seem to be rather difficult. Since the pioneering work of Youla and Gnavi [36] on the basic structure of n-D systems theory was published, much attention has been directed to develop n-D systems theory for $n \geq 3$ [3, 8, 14–18, 22, 23, 27, 28, 30–34]. Among them, multivariate polynomial matrix factorizations based on the approach of modules over a polynomial ring have made great progress.

In 2005, Wang and Kwong [33] first established a one-to-one correspondence between the existence of an minor prime factorization of a multivariate polynomial matrix with full rank and the freeness of a certain module over a polynomial ring. And then, this module theoretic approach is further extended by Wang [30] to the study of existence of factor prime factorizations of multivariate polynomial matrices with full rank. Form our personal viewpoint, all these methods proposed in [30, 33] provide solid foundation for further research on this topic. For example, some meaningful results [20, 21, 31] on multivariate polynomial matrix factorizations have been obtained by using the methods.

Recently, Guan et al. [10, 11] focused on factorization problems of rank-deficient matrices. Actually, factorization problems of rank-deficient matrices are also important in linear and multilinear algebra and other related areas. In matrix theory over any field, a rank-deficient matrix always can decompose into a product of two full rank matrices, which is called *full rank factorization* [26] (in this paper called *general factorization*). However, for a multivariate polynomial matrix with rank-deficiency the full rank factorization does not necessarily exist. A counter-example can be found in [36]. Guan et al. [11] generalized the main theorem in [33] and presented a criterion on the existence of minor prime factorizations for rank-deficient matrices. As for factor prime factorizations, a necessary condition for the existence of rank-deficient matrix factorizations in a special case was also given by Guan et al. in [10]. Subsequently, Lu et al. [24] improved the above result and obtained a necessary and sufficient condition.

In the paper, we pay close attention to general factorizations of rank-deficient matrices and try to propose a new method to study this topic by using the module theoretic approach. The main idea is as follows. Let $\mathcal{M}$ be a module over $k[\mathbf{z}]$ generated by the rows of a rank-deficient matrix $\mathbf{F}$. If $\mathcal{M}$ is a free module, then we compute a basis $\mathcal{G}'$ of $\mathcal{M}$ and obtain a polynomial matrix $\mathbf{F}_1$ with full row rank composed of the elements in $\mathcal{G}'$. In this case, we get a general factorization of $\mathbf{F}$ by factorizing $\mathbf{F}_1$. Otherwise, we have another polynomial matrix $\mathbf{F}_2$ with full row rank consisting of elements in a maximum $k[\mathbf{z}]$-linearly independent vector set $\mathcal{G}''$ of $\mathcal{M}$. If we solve Problem 1.2, then we can establish a connection between $\mathbf{F}$ and $\mathbf{F}_2$. By this way, we translate the factorization problem of a rank-deficient matrix into that of any of its full row rank submatrices. At this moment, we get a general factorization of $\mathbf{F}$ by factorizing $\mathbf{F}_2$.

The rest of the paper is organized as follows. In Section 2, we will introduce some concepts and basic theories of modules and matrix factorizations. Then in Section 3, we give a positive answer to Problem 1.2, and a relationship between a rank-deficient matrix and any of its full row rank submatrices is provided. In Section 4, we study the application of Problem 1.2 in multivariate polynomial matrix factorizations. We end with some concluding remarks in Section 5.

2 PRELIMINARIES

Let k be a field, and n an integer. Let $k[\mathbf{z}]$ be a polynomial ring in variables $z_1, \ldots, z_n$ over k, $k(\mathbf{z})$ the rational function field and $k[\mathbf{z}]^{l \times m}$ the set of $l \times m$ matrices with entries in $k[\mathbf{z}]$. Without loss of generality, we assume that $l \leq m$. We write $k[\mathbf{z}]^{1 \times m}$ as $k[\mathbf{z}]^m$ which is a free module over $k[\mathbf{z}]$. In addition, we use uppercase bold letter to denote polynomial matrices, and "w.r.t." to represent "with respect to".

Definition 2.1. Let $\mathcal{M} \subset k[\mathbf{z}]^m$, and $\vec{w}_1, \ldots, \vec{w}_t$ be nonzero vectors in $\mathcal{M}$. $\{\vec{w}_1, \ldots, \vec{w}_t\}$ is called a maximum $k[\mathbf{z}]$-linearly independent vector set of $\mathcal{M}$, if $\vec{w}_1, \ldots, \vec{w}_t$ are $k[\mathbf{z}]$-linearly independent, and for each vector $\vec{w} \in \mathcal{M} \setminus \{\vec{w}_1, \ldots, \vec{w}_t\}$ we have that $\vec{w}, \vec{w}_1, \ldots, \vec{w}_t$ are $k[\mathbf{z}]$-linearly dependent.

Definition 2.2. Let $\vec{w}, \vec{w}_1, \ldots, \vec{w}_t$ be vectors in $k[\mathbf{z}]^m$. Then $\vec{w}$ is said to be represented by the elements in $\{\vec{w}_1, \ldots, \vec{w}_t\}$ over $k[\mathbf{z}]$ if there exist $h_1, \ldots, h_t \in k[\mathbf{z}]$ such that $\vec{w} = h_1\vec{w}_1 + \cdots + h_t\vec{w}_t$.

Definition 2.3. Suppose $\mathcal{M} \subset k[\mathbf{z}]^m$ generated by $\vec{w}_1, \ldots, \vec{w}_s$ and the rows of $\mathbf{F} \in k[\mathbf{z}]^{s\times m}$ are $\vec{w}_1, \ldots, \vec{w}_s$. Then the rank of $\mathcal{M}$ is defined as the rank of $\mathbf{F}$.

Remark 2.4. *It is easy to prove that the rank of $\mathcal{M}$ in the above definition does not depend on the choice of the system of generators of $\mathcal{M}$. Furthermore, the rank of $\mathcal{M}$ is also equal to the number of elements in $\mathcal{G}$, where $\mathcal{G}$ is a maximum $k[\mathbf{z}]$-linearly independent vector set of $\mathcal{M}$.*

Definition 2.5. Let $\mathcal{K}$ be a submodule of $k[\mathbf{z}]^m$, and $J \subseteq k[\mathbf{z}]$ be a nonzero ideal. We define

$$\mathcal{K} : J = \{\vec{u} \in k[\mathbf{z}]^m \mid J\vec{u} \subseteq \mathcal{K}\},$$

where $J\vec{u}$ is the set $\{f\vec{u} \mid f \in J\}$.

It is easy to show that $\mathcal{K} : J$ is a submodule of $k[\mathbf{z}]^m$. Obviously, $\mathcal{K} \subseteq \mathcal{K} : J$. For the sake of simplicity, we write $\mathcal{K} : \langle f \rangle$ as $\mathcal{K} : f$ for any $f \in k[\mathbf{z}]$.

Given a matrix $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$, let $\mathbf{F}^{\mathrm{T}}$ be the transposed matrix of $\mathbf{F}$, rank($\mathbf{F}$) be the rank of $\mathbf{F}$, $d_i(\mathbf{F})$ be the greatest common divisor of all the $i \times i$ minors of $\mathbf{F}$, $\rho(\mathbf{F})$ be the submodule of $k[\mathbf{z}]^m$ generated by the rows of $\mathbf{F}$, Syz($\mathbf{F}$) be the syzygy module of $\mathbf{F}$. In addition, we use det($\mathbf{F}$) to denote the determinant of $\mathbf{F}$ if $l = m$.

The following three concepts, which were first proposed in [36], play an important role in n-D systems.

Definition 2.6 ([36]). Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank.

(1) If all the $l \times l$ minors of $\mathbf{F}$ generate $k[\mathbf{z}]$, then $\mathbf{F}$ is said to be a zero left prime (ZLP) matrix.
(2) If all the $l \times l$ minors of $\mathbf{F}$ are relatively prime, i.e., $d_l(\mathbf{F})$ is a nonzero constant, then F is said to be a minor left prime (MLP) matrix.
(3) If for all polynomial matrix factorization $\mathbf{F} = \mathbf{F}_1\mathbf{F}_2$ in which $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times l}, \mathbf{F}_1$ is necessarily a unimodular matrix, i.e., det($\mathbf{F}_1$) is a nonzero constant, then $\mathbf{F}$ is said to be a factor left prime (FLP) matrix.

Let $\mathbf{F} \in k[\mathbf{z}]^{m\times l}$ with $m \geq l$, then a ZRP (MRP, FRP) matrix can be similarly defined. Please refer to [36] for the relationships among ZLP, MLP and FLP matrices.

A general factorization of a multivariate polynomial matrix is now formulated as follows.

Definition 2.7. Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with rank r, and f (f') be a divisor of $d_r(\mathbf{F})$. $\mathbf{F}$ is said to admit a left (right) factorization w.r.t. f (f') if $\mathbf{F}$ can be factorized as

$$\mathbf{F} = \mathbf{G}\mathbf{F}' \tag{1}$$

such that $\mathbf{G} \in k[\mathbf{z}]^{l\times r}$, $\mathbf{F}' \in k[\mathbf{z}]^{r\times m}$ with $d_r(\mathbf{G}) = f$ ($d_r(\mathbf{F}') = f'$).

In the above definition, Equation (1) is said to be a ZLP (MLP, FLP) factorization of $\mathbf{F}$ w.r.t f if $\mathbf{F}'$ is a ZLP (MLP, FLP) matrix.

We introduce a lemma, which can be easily proved by Binet-Cauchy formula. Thus, the proof is omitted here.

Lemma 2.8. *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with rank r. If $\mathbf{F} = \mathbf{G}\mathbf{F}'$ with $\mathbf{G} \in k[\mathbf{z}]^{l\times r}$ and $\mathbf{F}' \in k[\mathbf{z}]^{r\times m}$, then $d_r(\mathbf{F}) = d_r(\mathbf{G})d_r(\mathbf{F}')$.*

Remark 2.9. *By Lemma 2.8, $\mathbf{F}$ admits a left factorization w.r.t. f if and only if $\mathbf{F}$ admits a right factorization w.r.t. $f' = \frac{d_r(\mathbf{F})}{f}$.*

Definition 2.10 ([13, 29]). Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with rank r. For any given integer i with $1 \leq i \leq r$, let $a_1, \ldots, a_\beta$ denote all the $i \times i$ minors of $\mathbf{F}$, where $\beta = \binom{l}{i}\binom{m}{i}$. Extracting $d_i(\mathbf{F})$ from $a_1, \ldots, a_\beta$ yields $a_j = d_i(\mathbf{F}) \cdot b_j$, where $j = 1, \ldots, \beta$. Then $b_1, \ldots, b_\beta$ are called all the $i \times i$ reduced minors of $\mathbf{F}$.

Reduced minors of a polynomial matrix can be computed by the following way, which was proposed by Lin et al. in [16].

Lemma 2.11 ([16]). *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with rank r, $\mathbf{E} \in k[\mathbf{z}]^{r\times m}$ be a full row rank submatrix of $\mathbf{F}$, and $\mathbf{B} \in \mathbf{k}[\mathbf{z}]^{l\times r}$ be a full column rank submatrix of $\mathbf{F}$. Let $e_1, \ldots, e_\eta$ be the $r \times r$ reduced minors of $\mathbf{E}$, and $b_1, \ldots, b_\beta$ be the $r \times r$ reduced minors of $\mathbf{B}$, where $\beta = \binom{l}{r}$ and $\eta = \binom{m}{r}$. Then the $r \times r$ reduced minors of $\mathbf{F}$ are given by*

$$b_1e_1, \ldots, b_1e_\eta, \ldots, b_\beta e_1, \ldots, b_\beta e_\eta.$$

In [30], some progress about FLP factorizations of multivariate polynomial matrices with full rank were obtained. Wang [30] first presented a concept called *regular*.

Definition 2.12 ([30]). Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, and $f \in k[\mathbf{z}]$. f is said to be regular w.r.t. $\mathbf{F}$ if $d_l([f\mathbf{I}_l,\ \mathbf{F}]) = f$ up to multiplication by a nonzero constant, where $\mathbf{I}_l$ is the $l \times l$ identity matrix.

From Definition 2.12, it is easy to see that $f \mid d_l(\mathbf{F})$ if f is regular w.r.t. $\mathbf{F}$. Liu and Wang [19] proposed an equivalent condition, which can be easily verified, to describe regular.

Theorem 2.13 ([19]). *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank and $f \mid d_l(F)$, then f is regular w.r.t. F if and only if*

$$\gcd\left(d_{l-1}(F), \frac{d_l(F)}{f}, f\right) = 1.$$

Under the constraint of regularity, Wang [30] gave a necessary and sufficient condition for the existence of matrix factorizations.

Theorem 2.14 ([30]). *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank and f be regular w.r.t. $\mathbf{F}$. Then the following conditions are equivalent.*

(1) *$\mathbf{F}$ admits a left factorization w.r.t. f, i.e., $\mathbf{F}$ can be factorized as $\mathbf{F} = \mathbf{G}\mathbf{F}'$ such that $\mathbf{F}' \in k[\mathbf{z}]^{l\times m}$ and $\mathbf{G} \in k[\mathbf{z}]^{l\times l}$ with* $\det(\mathbf{G}) = f$.
(2) *$\rho(\mathbf{F}) : f$ is a free $k[\mathbf{z}]$-module of rank l.*

Moreover, if one of above conditions holds, then $\rho(\mathbf{F}') = \rho(\mathbf{F}) : f$.

Suppose $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ is a rank-deficient matrix with rank r and $f \mid d_r(\mathbf{F})$. If $\rho(\mathbf{F})$ is a free module, then we can obtain a basis $\mathcal{G}$ of $\rho(\mathbf{F})$. Let $\mathbf{F}_1 \in k[\mathbf{z}]^{r\times m}$ be composed of the elements in $\mathcal{G}$, then $\rho(\mathbf{F}_1) = \rho(\mathbf{F})$. It is easy to prove that $d_r(\mathbf{F}_1) = d_r(\mathbf{F})$. At this moment, $\mathbf{F}$ admits a left factorization w.r.t. f if and only if $\mathbf{F}_1$ admits a left factorization w.r.t. f. By Theorem 2.14, we can verify whether $\mathbf{F}_1$ admits a left factorization w.r.t. f in the regular case. For this reason, in the following we only need to consider the case where $\rho(\mathbf{F})$ is not a free module.

Now, we discuss the problem for general factorizations of multivariate polynomial matrices with rank-deficiency.

Problem 2.15. *Let* $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ *with rank* r *and* $f' \mid d_r(\mathbf{F})$*. Determine whether* $\mathbf{F}$ *admits a right factorization w.r.t.* f' *and design an algorithm to compute the factorization if it exists.*

We will solve this problem in Section 4. The main idea is to establish a connection between a rank-deficient matrix and any of its full row rank submatrices by using the module theoretic approach, which will be discussed in the next section.

3 PROPERTY OF MODULES

As mentioned in Introduction, a vector $\vec{w}$ in $\mathcal{M} \subset k[\mathbf{z}]^m$ may not be represented by the elements in $\mathcal{G}$ over $k[\mathbf{z}]$, where $\mathcal{G}$ is a maximum $k[\mathbf{z}]$-linearly independent vector set of $\mathcal{M}$. However, the product of a certain polynomial and $\vec{w}$ can be expressed as a linear combination of the elements in $\mathcal{G}$ over $k[\mathbf{z}]$. Combined with some simple theories of matrices, we now give the following theorem to describe this property of modules over $k[\mathbf{z}]$.

Theorem 3.1. *Let* $\mathcal{M}$ *be a submodule of* $k[\mathbf{z}]^m$ *with rank* r*, where* $r < m$*,* $\mathcal{G}$ *be a maximum* $k[\mathbf{z}]$*-linearly independent vector set of* $\mathcal{M}$*, and* $\vec{w} \in \mathcal{M} \setminus \mathcal{G}$ *be a nonzero vector. Suppose* $\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \vec{w} \end{bmatrix} \in k[\mathbf{z}]^{(r+1)\times m}$*, where* $\mathbf{F}_1 \in k[\mathbf{z}]^{r\times m}$ *is composed of the elements in* $\mathcal{G}$*. Let* $d = d_r(\mathbf{F})$ *and* $d_1 = d_r(\mathbf{F}_1)$*, then*

$$\vec{w} \in \rho(\mathbf{F}_1) : \frac{d_1}{d}.$$

Proof. Notice that the rows of $\mathbf{F}_1$ are the elements in $\mathcal{G}$, so $\mathbf{F}_1$ is of full row rank. Since the rank of $\mathcal{M}$ is r and $\vec{w} \in \mathcal{M}$, we have $\text{rank}(\mathbf{F}) = r$. Moreover, there exists a nonzero vector $\vec{v} \in k(\mathbf{z})^r$ such that

$$\vec{w} = \vec{v} \cdot \mathbf{F}_1.$$

Without loss of generality, we assume that the first r columns of $\mathbf{F}_1$ are $k[\mathbf{z}]$-linearly independent. Let $\mathbf{F}_1 = [\mathbf{F}_{11}, \mathbf{F}_{12}]$, where $\mathbf{F}_{11} \in k[\mathbf{z}]^{r\times r}$ and $\mathbf{F}_{12} \in k[\mathbf{z}]^{r\times(m-r)}$. We denote $\vec{w}$ by $(w_1, \ldots, w_r, w_{r+1}, \ldots, w_m)$, where $w_j \in k[\mathbf{z}]$ for $j = 1, \ldots, m$. Then,

$$(w_1, \ldots, w_r) = \vec{v} \cdot \mathbf{F}_{11}.$$

By Cramer's Rule, we have

$$\vec{v} = \frac{1}{\det(\mathbf{F}_{11})} \cdot (D_1, \ldots, D_r),$$

where D_i is the determinant of the matrix formed by replacing the i-th row of $\mathbf{F}_{11}$ with $(w_1, \ldots, w_r)$, and $i = 1, \ldots, r$.

Let $\widetilde{\mathbf{F}} = \begin{bmatrix} \mathbf{F}_{11} \\ (w_1, \ldots, w_r) \end{bmatrix} \in k[\mathbf{z}]^{(r+1)\times r}$. It follows that $\det(\mathbf{F}_{11})$, $D_1, \ldots, D_r$ are all the $r \times r$ minors of $\widetilde{\mathbf{F}}$ (up to a sign). Setting

$$b_1 = \frac{\det(\mathbf{F}_{11})}{d_r(\widetilde{\mathbf{F}})} \text{ and } b_j = \frac{D_{j-1}}{d_r(\widetilde{\mathbf{F}})},$$

where $j = 2, \ldots, r+1$. Therefore,

$$\vec{v} = \frac{1}{\det(\mathbf{F}_{11})} \cdot (D_1, \ldots, D_r) = \frac{(b_2, \ldots, b_{r+1})}{b_1},$$

and $b_1, \ldots, b_{r+1}$ are all $r \times r$ reduced minors of $\widetilde{\mathbf{F}}$ (up to a sign). Let $e_1, \ldots, e_\eta$ be all $r \times r$ reduced minors of $\mathbf{F}_1$, where $\eta = \binom{m}{r}$. By Lemma 2.11, the $r \times r$ reduced minors of $\mathbf{F}$ are given by

$$\begin{matrix} b_1e_1, & b_1e_2, & \cdots & b_1e_\eta, \\ b_2e_1, & b_2e_2, & \cdots & b_2e_\eta, \\ \vdots & \vdots & \ddots & \vdots \\ b_{r+1}e_1, & b_{r+1}e_2, & \cdots & b_{r+1}e_\eta. \end{matrix}$$

Then the $r \times r$ minors of $\mathbf{F}$ are as follows:

$$\begin{matrix} b_1e_1d, & b_1e_2d, & \cdots & b_1e_\eta d, \\ b_2e_1d, & b_2e_2d, & \cdots & b_2e_\eta d, \\ \vdots & \vdots & \ddots & \vdots \\ b_{r+1}e_1d, & b_{r+1}e_2d, & \cdots & b_{r+1}e_\eta d. \end{matrix}$$

It follows that $d_1 = b_1 d$. Thus,

$$\frac{d_1}{d} \cdot \vec{v} = b_1 \cdot \frac{(b_2, \ldots, b_{r+1})}{b_1} = (b_2, \ldots, b_{r+1}) \in k[\mathbf{z}]^r.$$

This implies that

$$\frac{d_1}{d} \cdot \vec{w} = \frac{d_1}{d} \cdot \vec{v} \cdot \mathbf{F}_1 = (b_2, \ldots, b_{r+1}) \cdot \mathbf{F}_1 \in \rho(\mathbf{F}_1).$$

That is, $\vec{w} \in \rho(\mathbf{F}_1) : \frac{d_1}{d}$.

The proof is completed. □

Based on the above theorem, we solve Problem 1.2. Before Proceeding further, let us remark on Theorem 3.1.

Remark 3.2. *If* $\vec{w} \in \mathcal{G}$*, it is easy to check that* $d = d_1$*. This is because* $\vec{w}$ *is a row of* $\mathbf{F}_1$*. Thus, we have* $\vec{w} \in \rho(\mathbf{F}_1)$*. If* $\mathcal{M}$ *is a free module and* $\mathcal{G}$ *is a basis of* $\mathcal{M}$*, then* $d = d_1$*. This is because* $\vec{w}$ *is a linear combination of the elements in* $\mathcal{G}$ *over* $k[\mathbf{z}]$*. So we also get* $\vec{w} \in \rho(\mathbf{F}_1)$*. When* $r = m$*, the theorem still holds.*

From Theorem 3.1, we know that $\frac{d_1}{d} \cdot \vec{w} \in \rho(\mathbf{F}_1)$. If there exists a nonzero polynomial $h \in k[\mathbf{z}]$ such that $h \cdot \vec{w} \in \rho(\mathbf{F}_1)$, then it is interesting to know what is the relationship between $\frac{d_1}{d}$ and h. The answer is as follows.

Corollary 3.3. *Let* $\mathbf{F}_1 \in k[\mathbf{z}]^{r\times m}$ *be of full row rank,* $\vec{w} \in k[\mathbf{z}]^m$ *be a nonzero vector and* $h \in k[\mathbf{z}]$ *be a nonzero polynomial. Suppose the rank of* $\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \vec{w} \end{bmatrix} \in k[\mathbf{z}]^{(r+1)\times m}$ *is* r*, where* $r < m$*. Let* $d = d_r(\mathbf{F})$ *and* $d_1 = d_r(\mathbf{F}_1)$*, then* $h \cdot \vec{w} \in \rho(\mathbf{F}_1)$ *if and only if* $\frac{d_1}{d} \mid h$*.*

Proof. *Sufficiency.* If $\frac{d_1}{d} \mid h$, then $h \cdot \vec{w} \in \rho(\mathbf{F}_1)$ by Theorem 3.1.

Necessity. Since $\text{rank}(\mathbf{F}) = r$, there exists a nonzero vector $\vec{v} \in k(\mathbf{z})^r$ such that

$$\vec{w} = \vec{v} \cdot \mathbf{F}_1.$$

As we see in Theorem 3.1,

$$\vec{v} = \frac{1}{b_1} \cdot (b_2, \ldots, b_{r+1}) \text{ and } b_1 = \frac{d_1}{d},$$

where $b_1, b_2 \ldots, b_{r+1}$ are all $r \times r$ reduced minors of $\widetilde{\mathbf{F}}$ (up to a sign). At this moment, we have

$$\gcd(b_1, b_2 \ldots, b_{r+1}) = 1.$$

If $h \cdot \vec{w} \in \rho(\mathbf{F}_1)$, then there is a nonzero vector $\vec{u} \in k[\mathbf{z}]^r$ such that

$$h \cdot \vec{w} = \vec{u} \cdot \mathbf{F}_1.$$

Since $\mathbf{F}_1$ is a full row rank matrix, we have

$$\vec{v} = \frac{1}{h} \cdot \vec{u}.$$

Assume that $\vec{u} = (u_1, \ldots, u_r)$, where $u_1, \ldots, u_r$ are polynomials in $k[\mathbf{z}]$. Then, we get

$$\frac{b_{i+1}}{b_1} = \frac{u_i}{h},\ i = 1, \ldots, r.$$

This implies that $b_1 \mid hb_{i+1}$ for each integer i with $1 \le i \le r$. Suppose $\gcd(b_2, \ldots, b_{r+1}) = g$, where $g \in k[\mathbf{z}] \setminus \{0\}$. Then $\gcd(hb_2, \ldots, hb_{r+1}) = hg$ and $b_1 \mid hg$. As $\gcd(b_1, b_2 \ldots, b_{r+1}) = 1$, we obtain $\gcd(b_1, g) = 1$. It follows that $b_1 \mid h$. Combined with the equation $b_1 = \frac{d_1}{d}$, we get

$$\frac{d_1}{d} \mid h.$$

The proof is completed. □

Remark 3.4. *Corollary 3.3 indicates that $\frac{d_1}{d}$ is the minimal polynomial which makes $\frac{d_1}{d} \cdot \vec{w} \in \rho(\mathbf{F}_1)$.*

With the help of Theorem 3.1, we can now obtain a relationship between a rank-deficient matrix and any of its full row rank submatrices.

Theorem 3.5. *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with rank r, and $\mathbf{F}_1 \in k[\mathbf{z}]^{r\times m}$ be any of its full row rank submatrices, where $r < l$. Let $d = d_r(\mathbf{F})$ and $d_1 = d_r(\mathbf{F}_1)$, then*

$$\rho(\mathbf{F}_1) \subseteq \rho(\mathbf{F}) \subseteq \rho(\mathbf{F}_1) : \frac{d_1}{d}.$$

Proof. It is easy to see that $\rho(\mathbf{F}_1) \subseteq \rho(\mathbf{F})$, so we only need to prove $\rho(\mathbf{F}) \subseteq \rho(\mathbf{F}_1) : \frac{d_1}{d}$.

Without loss of generality, let $\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \vec{w}_{r+1} \\ \vdots \\ \vec{w}_l \end{bmatrix}$. It suffices to prove that $\vec{w}_i \in \rho(\mathbf{F}_1) : \frac{d_1}{d}$ for $i = r+1, \ldots, l$.

For each vector $\vec{w}_i$, set $\mathbf{F}^{(i)} = \begin{bmatrix} \mathbf{F}_1 \\ \vec{w}_i \end{bmatrix}$, where $i = r+1, \ldots, l$. By Theorem 3.1, we have

$$\vec{w}_i \in \rho(\mathbf{F}_1) : \frac{d_1}{d_r(\mathbf{F}^{(i)})}.$$

It follows from $d \mid d_r(\mathbf{F}^{(i)})$ that $\frac{d_1}{d_r(\mathbf{F}^{(i)})} \mid \frac{d_1}{d}$. Then for each integer i with $r+1 \le i \le l$ we get

$$\vec{w}_i \in \rho(\mathbf{F}_1) : \frac{d_1}{d_r(\mathbf{F}^{(i)})} \subseteq \rho(\mathbf{F}_1) : \frac{d_1}{d}.$$

It is obvious that $\rho(\mathbf{F}_1) \subseteq \rho(\mathbf{F}_1) : \frac{d_1}{d}$. Thus,

$$\rho(\mathbf{F}) \subseteq \rho(\mathbf{F}_1) : \frac{d_1}{d}.$$

The proof is completed. □

4 APPLICATION IN POLYNOMIAL MATRIX FACTORIZATIONS

In the previous section, we discussed a property of a module over $k[\mathbf{z}]$. In this section, we discuss its application in multivariate polynomial matrix factorizations.

4.1 Factorization Theorem and Algorithm

According to Theorem 3.5, we now give a partial answer to Problem 2.15.

Theorem 4.1. *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with rank r and $f' \mid d_r(\mathbf{F})$, $\mathbf{F}_1 \in k[\mathbf{z}]^{r\times m}$ be a full row rank submatrix of $\mathbf{F}$ which satisfies the condition that f' is regular w.r.t. $\mathbf{F}_1$. Then $\mathbf{F}$ admits a right factorization w.r.t. f' if and only if $\mathbf{F}_1$ admits a right factorization w.r.t. f'.*

Proof. *Necessity.* If $\mathbf{F}$ admits a right factorization w.r.t. f', then there are two matrices $\mathbf{G} \in k[\mathbf{z}]^{l\times r}$ and $\mathbf{F}' \in k[\mathbf{z}]^{r\times m}$ such that $\mathbf{F} = \mathbf{G}\mathbf{F}'$ with $d_r(\mathbf{F}') = f'$. Without loss of generality, we assume that $\mathbf{F}_1$ and $\mathbf{G}_1 \in k[\mathbf{z}]^{r\times r}$ are composed of the first r rows of $\mathbf{F}$ and $\mathbf{G}$, respectively. It follows from $\mathbf{F} = \mathbf{G}\mathbf{F}'$ that

$$\mathbf{F}_1 = \mathbf{G}_1\mathbf{F}'.$$

Then, $\mathbf{F}_1$ admits a right factorization w.r.t. f'.

Sufficiency. Assume that $\mathbf{F}_1$ admits a right factorization w.r.t. f'. Let $f_1 = \frac{d_r(\mathbf{F}_1)}{f'}$. By Remark 2.9, $\mathbf{F}_1$ admits a left factorization w.r.t. f_1, i.e., $\mathbf{F}_1$ can be factorized as $\mathbf{F}_1 = \mathbf{G}_1\mathbf{F}'$, where $\mathbf{F}' \in k[\mathbf{z}]^{r\times m}$, and $\mathbf{G}_1 \in k[\mathbf{z}]^{r\times r}$ with $\det(\mathbf{G}_1) = f_1$. Obviously, $d_r(\mathbf{F}') = f'$. Since f' is regular w.r.t. $\mathbf{F}_1$, we have that f_1 is regular w.r.t. $\mathbf{F}_1$ by Theorem 2.13. According to Theorem 2.14, we get

$$\rho(\mathbf{F}') = \rho(\mathbf{F}_1) : f_1.$$

Since $f' \mid d_r(\mathbf{F})$, we have $\frac{d_r(\mathbf{F}_1)}{d_r(\mathbf{F})} \mid \frac{d_r(\mathbf{F}_1)}{f'}$. It follows that

$$\rho(\mathbf{F}_1) : \frac{d_r(\mathbf{F}_1)}{d_r(\mathbf{F})} \subseteq \rho(\mathbf{F}_1) : \frac{d_r(\mathbf{F}_1)}{f'}.$$

Using Theorem 3.5, we obtain

$$\rho(\mathbf{F}) \subseteq \rho(\mathbf{F}_1) : \frac{d_r(\mathbf{F}_1)}{d_r(\mathbf{F})} \subseteq \rho(\mathbf{F}_1) : \frac{d_r(\mathbf{F}_1)}{f'} = \rho(\mathbf{F}_1) : f_1 = \rho(\mathbf{F}').$$

Thus, there exists a matrix $\mathbf{G} \in k[\mathbf{z}]^{l\times r}$ such that $\mathbf{F} = \mathbf{G}\mathbf{F}'$. It follows from $d_r(\mathbf{F}') = f'$ that $\mathbf{F}$ admits a right factorization w.r.t. f'.

The proof is completed. □

With the help of Theorem 2.14, Theorem 4.1 shows that a general factorization of a rank-deficient matrix can be constructed. The special algorithm is as follows.

We first give some explanations to Algorithm 1.

- Theorem 4.1 guarantees the correctness of Algorithm 1.
- In step 4, there are some strategies to choose $\mathbf{F}_1 \in S$ which satisfies the regularity assumption of Theorem 4.1. For instance, if $d_r(\mathbf{F}_1)$ is a square-free polynomial or $\gcd(f', \frac{d_r(\mathbf{F}_1)}{f'}) = 1$, then f' is regular w.r.t. $\mathbf{F}_1$ by Theorem 2.13.
- In step 7, Wang and Kwong [33] proved that there are one-to-one correspondence between $\mathrm{Syz}([\mathbf{F}_1^{\mathrm{T}},\ -f_1\mathbf{I}_m]^{\mathrm{T}})$ and $\rho(\mathbf{F}_1) : f_1$. That is, we compute a Gröbner basis $\{[\vec{g}_1, \vec{h}_1], \ldots, [\vec{g}_s, \vec{h}_s]\}$ of $\mathrm{Syz}([\mathbf{F}_1^{\mathrm{T}},\ -f_1\mathbf{I}_m]^{\mathrm{T}})$, then $\{\vec{h}_1, \ldots, \vec{h}_s\}$ is a system of generators of $\rho(\mathbf{F}_1) : f_1$, where $[\vec{g}_i, \vec{h}_i] \in k[\mathbf{z}]^{r+m}$ for $i = 1, \ldots, s$.

Algorithm 1: General factorization

Input : $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$, the rank r of $\mathbf{F}$ and a divisor f' of $d_r(\mathbf{F})$.
Output: a right factorization of $\mathbf{F}$ w.r.t. f'.

1 **begin**
2 $S := \{$ all the full row rank submatrices of $\mathbf{F}$ $\}$;
3 **while** $S \neq \emptyset$ **do**
4 choose $\mathbf{F}_1 \in S$ and $S := S \setminus \{\mathbf{F}_1\}$;
5 **if** f' *is regular w.r.t.* $\mathbf{F}_1$ **then**
6 compute $d_r(\mathbf{F}_1)$ and $f_1 := \frac{d_r(\mathbf{F}_1)}{f'}$;
7 compute a system of generators of $\rho(\mathbf{F}_1) : f_1$ and use it to constitute a matrix $\mathbf{F}_1'$;
8 compute the reduced Gröbner basis $\mathcal{G}$ of all the $r \times r$ column reduced minors of $\mathbf{F}_1'$;
9 **if** $\mathcal{G} = \{1\}$ **then**
10 compute a free basis of $\rho(\mathbf{F}_1')$ and use it to make up $\mathbf{F}' \in k[\mathbf{z}]^{r\times m}$;
11 compute $\mathbf{G} \in k[\mathbf{z}]^{l\times r}$ such that $\mathbf{F} = \mathbf{G}\mathbf{F}'$;
12 **return** $\mathbf{G}$ and $\mathbf{F}'$;
13 **else**
14 **return** "$\mathbf{F}$ has no right factorizations w.r.t. f'";
15 **return** "fail".

- The purpose of step 8 is to check whether $\rho(\mathbf{F}_1) : f_1$ is a free module of rank r. By the computation in step 7, if $\mathbf{F}_1'$ is a full row rank matrix, then $\rho(\mathbf{F}_1') = \rho(\mathbf{F}_1) : f_1$ is free and $\mathbf{F}' = \mathbf{F}_1'$ in step 10, which implies that we do not need to compute a reduced Gröbner basis $\mathcal{G}$ of all the $r \times r$ column reduced minors for $\mathbf{F}_1'$ and a free basis of $\rho(\mathbf{F}_1')$, i.e., without the computation of step 8 and step 10; otherwise, we need to use step 8 to determine whether $\rho(\mathbf{F}_1) : f_1$ is a free module of rank r (see Proposition 2 in [24]).
- In step 11, we need to compute $\mathbf{G} \in k[\mathbf{z}]^{l\times r}$ such that $\mathbf{F} = \mathbf{G}\mathbf{F}'$. Xiao et al. [35] designed a *Maple* package, called poly-matrix-equation, for solving multivariate polynomial matrix Diophantine equations, which can be used to compute $\mathbf{G}$. Nevertheless, we ensure that there is a unique solution to the equation, so in the algorithm implementation we can use the right inverse computation to advantageously obtain $\mathbf{G}$. That is to compute the right inverse $(\mathbf{F}')^{-1}$ of $\mathbf{F}'$ over $k(\mathbf{z})$, then $\mathbf{G} = \mathbf{F}(\mathbf{F}')^{-1}$.
- Step 12 means that $\mathbf{F}$ has a right factorization w.r.t. f', that is, $\mathbf{F} = \mathbf{G}\mathbf{F}'$ with $d_r(\mathbf{F}') = f'$.
- In step 14, Theorem 4.1 can guarantee that $\mathbf{F}$ has no right factorizations w.r.t. f'.
- In step 15, "fail" implies that we currently cannot judge whether $\mathbf{F}$ admits a right factorization w.r.t. f'. Since there is no full row rank matrix $\mathbf{F}_1 \in S$ such that f' is regular w.r.t. $\mathbf{F}_1$, it follows that Theorem 4.1 cannot be applied. This is a problem that we will solve in the future.

We now use an example to illustrate Algorithm 1.

Example 4.2. Let

$$\mathbf{F} = \begin{bmatrix} z_1z_2^3 & z_1z_3^2 & z_2^3z_3 + z_3^3 \\ z_1z_2^2 & 0 & z_2^2z_3 \\ 0 & z_1^2z_3 & z_1z_3^2 \end{bmatrix}$$

be a polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{3\times 3}$, where $\mathbb{C}$ is the complex field.

It is easy to compute that rank($\mathbf{F}$)=2 and $d_2(\mathbf{F}) = z_1z_2^2z_3$. Let $f' = z_2$, then $f' \mid d_2(\mathbf{F})$. We want to check whether $\mathbf{F}$ admits a right factorization w.r.t. f'.

We first choose a full row rank matrix

$$\mathbf{F}_1 = \begin{bmatrix} z_1z_2^3 & z_1z_3^2 & z_2^3z_3 + z_3^3 \\ z_1z_2^2 & 0 & z_2^2z_3 \end{bmatrix}$$

from all full row rank submatrices of $\mathbf{F}$. Based on Theorem 2.13, it is easy to check that f' is regular w.r.t. $\mathbf{F}_1$. Then by Theorem 4.1, $\mathbf{F}$ admits a right factorization w.r.t. f' if and only if $\mathbf{F}_1$ admits a right factorization w.r.t. f'.

It is easy to compute that $d_2(\mathbf{F}_1) = z_1z_2^2z_3^2$ and $f_1 = \frac{d_2(\mathbf{F}_1)}{f'} = z_1z_2z_3^2$. By computation, a system of generators of $\rho(\mathbf{F}_1) : f_1$ is $\{(0, z_1, z_3), (z_2, -z_2, 0), (z_1z_2, 0, z_2z_3), (z_1z_2^3, z_1z_3^2, z_2^3z_3+z_3^3)\}$. Hence, we have

$$\mathbf{F}_1' = \begin{bmatrix} 0 & z_1 & z_3 \\ z_2 & -z_2 & 0 \\ z_1z_2 & 0 & z_2z_3 \\ z_1z_2^3 & z_1z_3^2 & z_2^3z_3 + z_3^3 \end{bmatrix}.$$

Then we compute the reduced Gröbner basis $\mathcal{G}$ of all the 2×2 column reduced minors of $\mathbf{F}_1'$ w.r.t. the degree reverse lexicographic order with $z_1 > z_2 > z_3$ and obtain $\mathcal{G} = \{1\}$, which means $\rho(\mathbf{F}_1') = \rho(\mathbf{F}_1) : f_1$ is free. Therefore, $\mathbf{F}_1$ admits a right factorization w.r.t. f'.

By using the *Maple* package QUILLENSUSLIN, we get a free basis $\{(0, z_1, z_3), (z_2, -z_2, 0)\}$ of $\rho(\mathbf{F}_1) : f_1$ and construct

$$\mathbf{F}' = \begin{bmatrix} 0 & z_1 & z_3 \\ z_2 & -z_2 & 0 \end{bmatrix}.$$

Finally, by computing the right inverse of $\mathbf{F}'$ over $k(\mathbf{z})$ to solve the polynomial matrix equation

$$\mathbf{F} = \mathbf{G}\mathbf{F}', \text{ where } \mathbf{G} \in k[\mathbf{z}]^{3\times 2} \text{ is unknown.}$$

We obtain

$$\mathbf{G} = \mathbf{F}(\mathbf{F}')^{-1} = \mathbf{F} \begin{bmatrix} \frac{1}{z_1} & \frac{1}{z_2} \\ \frac{1}{z_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} z_2^3 + z_3^2 & z_1z_2^2 \\ z_2^2 & z_1z_2 \\ z_1z_3 & 0 \end{bmatrix}.$$

Thus, we have a right factorization of $\mathbf{F}$ w.r.t. f', i.e.,

$$\mathbf{F} = \mathbf{G}\mathbf{F}' = \begin{bmatrix} z_2^3 + z_3^2 & z_1z_2^2 \\ z_2^2 & z_1z_2 \\ z_1z_3 & 0 \end{bmatrix} \begin{bmatrix} 0 & z_1 & z_3 \\ z_2 & -z_2 & 0 \end{bmatrix}$$

with $d_2(\mathbf{F}') = f'$.

Remark 4.3. *We can further verify that* $\mathbf{F}'$ *has a right factorization w.r.t.* 1, *i.e.,* $\mathbf{F}'$ *has an MLP factorization. We compute a system of generators of* $\rho(\mathbf{F}') : d_2(\mathbf{F}')$, *and obtain* $\{(1, -1, 0), (0, z_1, z_3)\}$. *This implies that* $\rho(\mathbf{F}') : d_2(\mathbf{F}')$ *is a free module of rank* 2. *Then,* $\mathbf{F}'$ *has a left factorization w.r.t.* $d_2(\mathbf{F}')$ *by Theorem 2.14. Furthermore, this implies that* $\mathbf{F}'$ *is not a FLP matrix.*

We next use Algorithm 1 to give an alternative solution of the following example, which is completely different from the algebraic method used by Youla and Gnavi in [36].

Example 4.4 ([36]). Let

$$\mathbf{F} = \begin{bmatrix} -z_1^3 + z_2 z_3 & -z_1 z_3 + z_2^2 & 0 \\ -z_1^2 z_2 + z_3^2 & -z_1^3 + z_2 z_3 & z_1^5 - 3z_1^2 z_2 z_3 + z_1 z_2^3 + z_3^3 \\ 0 & z_1 & -z_1^3 + z_2 z_3 \end{bmatrix}$$

be a polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{3\times 3}$, where $\mathbb{C}$ is the complex field.

It is easy to compute that $\text{rank}(\mathbf{F}) = 2$ and $d_2(\mathbf{F}) = 1$. Setting $f' = 1$ and

$$\mathbf{F}_1 = \begin{bmatrix} -z_1^3 + z_2 z_3 & -z_1 z_3 + z_2^2 & 0 \\ -z_1^2 z_2 + z_3^2 & -z_1^3 + z_2 z_3 & z_1^5 - 3z_1^2 z_2 z_3 + z_1 z_2^3 + z_3^3 \end{bmatrix}.$$

It is easy to check that $\mathbf{F}_1$ is a full row rank submatrix of $\mathbf{F}$, and f' is regular w.r.t. $\mathbf{F}_1$. By computation, $f_1 = \frac{d_2(\mathbf{F}_1)}{f'} = z_1^5 - 3z_1^2 z_2 z_3 + z_1 z_2^3 + z_3^3$ and a system of generators of $\rho(\mathbf{F}_1) : f_1$ is

$$\{(-z_3, -z_2, z_1^2 z_2 - z_3^2), (0, -z_1, z_1^3 - z_2 z_3), (-z_1, 0, -z_1 z_3 + z_2^2)\}.$$

That is,

$$\mathbf{F}_1' = \begin{bmatrix} -z_3 & -z_2 & z_1^2 z_2 - z_3^2 \\ 0 & -z_1 & z_1^3 - z_2 z_3 \\ -z_1 & 0 & -z_1 z_3 + z_2^2 \end{bmatrix}.$$

The reduced Gröbner basis of all the 2×2 column reduced minors of $\mathbf{F}_1'$ w.r.t. the degree reverse lexicographic order with $z_1 > z_2 > z_3$ is $\mathcal{G} = \{z_1, z_2, z_3\}$, which means that $\rho(\mathbf{F}_1') = \rho(\mathbf{F}_1) : f_1$ is not free. Thus, $\mathbf{F}_1$ does not admit right factorizations w.r.t. f' by Theorem 2.14. Based on Theorem 4.1, $\mathbf{F}$ does not admit right factorizations w.r.t. f'.

It follows that $\mathbf{F}$ can not be factorized as $\mathbf{F} = \mathbf{G}\mathbf{F}'$ such that $\mathbf{G} \in k[\mathbf{z}]^{3\times 2}$ and $\mathbf{F}' \in k[\mathbf{z}]^{2\times 3}$ with $d_2(\mathbf{F}') = 1$.

Remark 4.5. *Compared with the solution proposed by Youla and Gnavi in [36], our method is more straightforward.*

4.2 Comparison of Existing Factorization Criteria

In the following, we will consider ZLP, MLP and FLP factorizations of multivariate polynomial matrices with rank-deficiency, which are also studied in [10, 11, 24].

The ZLP and MLP factorization problems for full row rank matrices have been completely solved in [32, 33]. For the ZLP and MLP factorization problems of rank-deficient matrices, they can be translated into the problems of ZLP and MLP factorizations for any of their full row rank submatrices immediately by Theorem 3.5.

Corollary 4.6. *Let* $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ *with rank* r*, and* $\mathbf{F}_1 \in k[\mathbf{z}]^{r\times m}$ *be any of its full row rank submatrices. Then* $\mathbf{F}$ *admits a ZLP factorization if and only if* $\mathbf{F}_1$ *admits a ZLP factorization.*

Proof. The proof of necessity is similar to that of Theorem 4.1. Here we only prove the sufficiency. Assume that $\mathbf{F}_1$ admits a ZLP factorization, i.e., $\mathbf{F}_1$ can be factorized as $\mathbf{F}_1 = \mathbf{G}_1\mathbf{F}'$ such that $\mathbf{F}'$ is a ZLP matrix, where $\mathbf{G}_1 \in k[\mathbf{z}]^{r\times r}$ and $\mathbf{F}' \in k[\mathbf{z}]^{r\times m}$. By the definition of ZLP, we have $d_r(\mathbf{F}') = 1$. Based on Theorems 3.5 and 2.14, we get

$$\rho(\mathbf{F}) \subseteq \rho(\mathbf{F}_1) : \frac{d_r(\mathbf{F}_1)}{d_r(\mathbf{F})} \subseteq \rho(\mathbf{F}_1) : d_r(\mathbf{F}_1) = \rho(\mathbf{F}_1) : \det(\mathbf{G}_1) = \rho(\mathbf{F}').$$

Therefore, there exists $\mathbf{G} \in k[\mathbf{z}]^{l\times r}$ such that $\mathbf{F} = \mathbf{G}\mathbf{F}'$. Since $\mathbf{F}'$ is ZLP, $\mathbf{F}$ admits a ZLP factorization.

The proof is completed. □

MLP factorizations of rank-deficient matrices can be obtained similarly.

Corollary 4.7. *Let* $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ *with rank* r*, and* $\mathbf{F}_1 \in k[\mathbf{z}]^{r\times m}$ *be any of its full row rank submatrices. Then* $\mathbf{F}$ *admits an MLP factorization if and only if* $\mathbf{F}_1$ *admits an MLP factorization.*

Based on Corollaries 4.6 and 4.7, we completely solve the ZLP and MLP factorization problems of rank-deficient matrices.

Combining Theorem 5 in [33] with Corollary 4.7, we obtain that $\mathbf{F}$ admits an MLP factorization if and only if $\rho(\mathbf{F}_1) : d_r(\mathbf{F}_1)$ is a free module of rank r. Guan et al. [11] proposed another necessary and sufficient condition for an MLP factorization of a rank-deficient matrix. They pointed out that $\mathbf{F}$ admits an MLP factorization if and only if $\rho(\mathbf{F}) : I_r(\mathbf{F})$ is a free module of rank r, where $I_r(\mathbf{F})$ is the ideal in $k[\mathbf{z}]$ generated by all the $r \times r$ minors of $\mathbf{F}$. Computing a system of generators of $\rho(\mathbf{F}) : I_r(\mathbf{F})$ requires more computation than ours. The reason is as follows. Assume that $\{f_1, \ldots, f_s\}$ is a reduced Gröbner basis of $I_r(\mathbf{F})$, then

$$\rho(\mathbf{F}) : I_r(\mathbf{F}) = \bigcap_{i=1}^{s} (\rho(\mathbf{F}) : f_i).$$

Obviously, our method does not need to compute s systems of generators and $s - 1$ intersections of modules.

As for the FLP factorization problem of rank-deficient matrices, we can translate it into the FLP factorization problem of full row rank matrices in the following case.

Corollary 4.8. *Let* $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ *with rank* r*, and* $\mathbf{F}_1 \in k[\mathbf{z}]^{r\times m}$ *be a full row rank submatrix of* $\mathbf{F}$*. Let* $d = d_r(\mathbf{F})$ *and* $d_1 = d_r(\mathbf{F}_1)$*. Assume that* $f \mid d$ *and* $\frac{d}{f}$ *is regular w.r.t.* $\mathbf{F}_1$*. Then* $\mathbf{F}$ *admits a FLP factorization w.r.t.* f *if and only if* $\mathbf{F}_1$ *admits a FLP factorization w.r.t.* $\frac{d_1 f}{d}$*.*

The proof of the above corollary is similar to that of Theorem 4.1 and Corollary 4.6, and omitted here.

Lu et al. [24] extended the results proposed by Guan et al. [10] and obtained a necessary and sufficient condition for a FLP factorization of a rank-deficient matrix when some divisors of $d_r(\mathbf{F})$ satisfy a certain condition. According to Theorem 5 in [24], the authors required that the ideal generated by all the $r \times r$ column reduced minors of $\mathbf{F}$ is $k[\mathbf{z}]$. This implies that $\rho(\mathbf{F})$ is a free module of rank r. We compute a free basis of $\rho(\mathbf{F})$ and use the basis to constitute a full row rank matrix $\tilde{\mathbf{F}} \in k[\mathbf{z}]^{r\times m}$, then $\rho(\mathbf{F}) = \rho(\tilde{\mathbf{F}})$. Thus, $\mathbf{F}$ admits a FLP factorization w.r.t. f if and only if $\tilde{\mathbf{F}}$ admits a FLP factorization w.r.t. f. Compared with Corollary 4.8, the essence of Theorem 5 in [24] is to consider the FLP factorization problem of full row rank matrices.

5 CONCLUDING REMARKS

In this paper, we have studied a property of modules over a polynomial ring and applied it to multivariate polynomial matrix factorizations.

Modules over a polynomial ring $k[\mathbf{z}]$ are quite different from vector spaces over a field k. For instance, there exist many vectors in $\mathcal{M} \subset k[\mathbf{z}]^m$ that cannot be expressed as linear combinations of the elements in a maximum $k[\mathbf{z}]$-linearly independent vector set of $\mathcal{M}$. The essential reason is that the polynomial division algorithm cannot be done arbitrarily in $k[\mathbf{z}]$. Let $\vec{w} \in \mathcal{M}$ be a nonzero vector and $\mathcal{G}$ be a maximum $k[\mathbf{z}]$-linearly independent vector set of $\mathcal{M}$, then Theorem 3.1 tells us that h can be obtained from the greatest common divisors of maximal minors of the relevant matrices composed of $\vec{w}$ and $\mathcal{G}$. Furthermore, Corollary 3.3 shows that the polynomial constructed in Theorem 3.1 is minimal.

With the help of the above property, a connection (Theorem 3.5) between a rank-deficient matrix and any of its full row rank submatrices was established. Based on this relationship, the problem of general factorizations of rank-deficient matrices can be translated into that of full row rank matrices in regular case (Theorem 4.1). Then, a constructive algorithm (Algorithm 1) for computing general factorizations of rank-deficient matrices has been proposed. Moreover, the previous results, such as ZLP, MLP, FLP factorizations of full row rank matrices, can be extended to the rank-deficient case.

In order to enable everyone to understand Algorithm 1 more intuitively, we deliberately avoid tricks and optimizations, such as the check of regularity for f' w.r.t. some matrices in the algorithm. We have implemented the algorithm on *Maple* with k of characteristic 0. For interested readers, more examples can be generated by the codes at: http://www.mmrc.iss.ac.cn/~dwang/software.html.

Although we have proposed a new method for studying general factorizations of rank-deficient matrices, there is a requirement: f' is regular w.r.t. $\mathbf{F}_1$ in Theorem 4.1. This implies that Theorem 4.1 depends on Theorem 2.14. Therefore, when the condition of regularity in Theorem 2.14 fails, how can we construct a general factorization of a full row rank matrix? If we can solve this problem, then the general factorizations of rank-deficient matrices will be further developed.

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