On General Factorization Problems of n-D Polynomial Matrices

Dong Lu, Dingkang Wang, Fanghui Xiao, and Xiaopeng Zheng

***Abstract*—Different from previous perspective, this brief studies relationships between the modules $\rho(\mathbf{F_1})$ and $\rho(\mathbf{F})$ generated by the rows of $\mathbf{F_1}$ and $\mathbf{F}$ respectively under the assumption of $\mathbf{F}$ admitting a general factorization $\mathbf{F} = \mathbf{G_1F_1}$. Two necessary and sufficient conditions for the equivalence of $\rho(\mathbf{F_1})$ and quotient modules of $\rho(\mathbf{F})$ with respect to some polynomials are obtained for two different situations. As a by-product, some general factorizations of $\mathbf{F}$ by computing related quotient modules can be got.**

***Index Terms*—*n*-D polynomial matrix, general factorization, quotient module, regular divisor, free basis.**

I. Introduction

An important and classical subject in multidimensional (n-D) systems is to factorize n-D polynomial matrices, because of the wide applications for n-D polynomial matrix general factorizations in n-D systems, circuits, controls, signal processing, and other related areas (see, e.g., [1], [2], [3], [4], [5], [6] and references therein). This subject has been studied for decades by engineers and mathematicians in n-D systems and computer algebra. Meantime, various methods have been proposed and great progress has been made [7], [8], [9], [10], [11], [12], [13], [14], [15], [16].

On the existence problem for zero prime factorizations of n-D polynomial matrices, Lin and Bose [17] raised the famous Lin-Bose conjecture. It describes the thing that a n-D polynomial matrix admits a zero prime factorization if all its maximal reduced minors generate a unit ideal. This conjecture has been solved by Pommaret [18], Srinivas [19], Wang and Feng [20], Liu et al. [21], respectively. From our personal viewpoint, the methods proposed in [18], [19], [20], [21] may only be considered as a proof of the existence for zero prime factorizations, rather than as constructive methods for carrying out the actual matrix factorizations.

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A constructive criterion was presented by Wang and Kwong [22] for the existence problem of minor prime factorizations of n-D polynomial matrices. They established a one-to-one correspondence between the existence for a minor prime factorization of a n-D polynomial matrix and the freeness of a certain module by using the modular approach. Moreover, they proposed an effective factorization algorithm which can be applicable to both zero and minor prime factorizations. Motivated by the work in [22], Wang [23] proposed another scheme for solving the existence problem for factor prime factorizations of a large class of n-D polynomial matrices.

At present, most of literatures focus on the existence problem for general factorizations of n-D polynomial matrices. In this brief, we will study general factorizations of n-D polynomial matrices from another perspective. When a n-D polynomial matrix $\mathbf{F}$ has a general factorization $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$, can we get some other new relationships between $\mathbf{F}$ and $\mathbf{F}_1$? This brief attempts to give some new results.

The rest of this brief is organized as follows. In Section II, we will introduce some important concepts and theories, and propose two problems that we shall consider. The main results are presented in Section III. Some conclusions are drawn in Section IV.

II. Preliminaries and Problems

Let k be a field, and n an integer. Let $k[\mathbf{z}]$ be the polynomial ring in variables $z_1, \ldots, z_n$ over k, and $k[\mathbf{z}]^{l\times m}$ the set of $l \times m$ matrices with entries in $k[\mathbf{z}]$. Without loss of generality, we assume that $l \le m$. Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, then we use $\rho(\mathbf{F})$ to denote the submodule of $k[\mathbf{z}]^{1\times m}$ generated by the rows of $\mathbf{F}$, $\mathrm{Syz}(\mathbf{F})$ to represent the syzygy module $\{\vec{u} \in k[\mathbf{z}]^{1\times l} : \vec{u}\mathbf{F} = \mathbf{0}_{1\times m}\}$ of $\mathbf{F}$, and $d_i(\mathbf{F})$ to signify the greatest common divisor of all the $i \times i$ minors of $\mathbf{F}$. Let $h \in k[\mathbf{z}]$, then $[h\mathbf{I}_l, \mathbf{F}]$ stands for the matrix concatenating $h\mathbf{I}_l$ and $\mathbf{F}$, where $\mathbf{I}_l$ is the $l \times l$ identity matrix. In addition, we use "w.r.t." to represent "with respect to".

A. Basic Notions

We first introduce an important concept, which characterize conditions for the existence of general factorizations of n-D polynomial matrices in some cases.

Definition 1: Let $\mathcal{K}$ be a submodule of $k[\mathbf{z}]^{1\times m}$, and f be a nonzero polynomial in $k[\mathbf{z}]$. We define

$$\mathcal{K} : f = \{\vec{v} \in k[\mathbf{z}]^{1\times m} : f\vec{v} \in \mathcal{K}\},$$

and it is called the **quotient module** of $\mathcal{K}$ w.r.t. f.

It is easy to show that $\mathcal{K} : f$ is a submodule of $k[\mathbf{z}]^{1\times m}$. Obviously, $\mathcal{K} \subseteq \mathcal{K} : f$. Wang and Kwong [22] established an explicit relationship between $\mathcal{K} : f$ and a syzygy module of a certain n-D polynomial matrix.

Lemma 1: Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, and f be a divisor of $d_l(\mathbf{F})$. Let $\mathbf{A} = \begin{bmatrix} \mathbf{F} \\ -f\mathbf{I}_m \end{bmatrix}$, then $\rho(\mathbf{F}) : f$ is isomorphic to Syz($\mathbf{A}$). Moreover, if Syz($\mathbf{A}$) is generated by the rows of $[\mathbf{B}, \mathbf{C}] \in k[\mathbf{z}]^{s\times(l+m)}$ with $\mathbf{B} \in k[\mathbf{z}]^{s\times l}$ and $\mathbf{C} \in k[\mathbf{z}]^{s\times m}$, then $\rho(\mathbf{C}) = \rho(\mathbf{F}) : f$.

It is easy to see that Lemma 1 provides a method for computing a system of generators of $\rho(\mathbf{F}) : f$. We now recall the following concept in [5], which plays an important role in n-D systems.

Definition 2: Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank. Then $\mathbf{F}$ is said to be a minor left prime (MLP) matrix if all the $l \times l$ minors of $\mathbf{F}$ are relatively prime, that is, $d_l(\mathbf{F})$ is a nonzero constant.

Please refer to [5] for the definitions of zero left prime (ZLP) and factor left prime (FLP), and the relationships among ZLP, MLP and FLP.

Lin [24] proposed a necessary and sufficient condition for a syzygy module to be a free module.

Lemma 2: Let $\mathbf{A} \in k[\mathbf{z}]^{s\times t}$ be of full column rank with $s > t$, and $r = s - t$. Then Syz($\mathbf{A}$) is a free $k[\mathbf{z}]$-module of rank r if and only if there is an MLP matrix $\mathbf{B} \in k[\mathbf{z}]^{r\times s}$ such that $\mathbf{BA} = \mathbf{0}_{r\times t}$. Moreover, Syz($\mathbf{A}$) is generated by the rows of $\mathbf{B}$, i.e., Syz($\mathbf{A}$) $= \rho(\mathbf{B})$.

There are many other ways, such as calculating Fitting ideals or column reduced minors, to check whether a submodule of $k[\mathbf{z}]^{1\times s}$ is a free $k[\mathbf{z}]$-module. We refer to [25], [26] for more details. Nevertheless, it can be difficult to compute a free basis of a free module. Based on the famous Quillen-Suslin theorem [27], [28], it was not until 2007 that Fabiańska and Quadrat [29] constructed an algorithm, which is quite complicated, to compute free bases of free modules over polynomial rings. The algorithm was implemented on a Maple package, called QUILLENSUSLIN.

A general factorization of a n-D polynomial matrix is now formulated as follows [9].

Definition 3: Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, and f be a divisor of $d_l(\mathbf{F})$. $\mathbf{F}$ is said to admit a **general factorization** w.r.t. f if $\mathbf{F}$ can be factorized as

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 \tag{1}$$

with $\det(\mathbf{G}_1) = f$, where $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$.

In order to study the existence problem for general factorizations of n-D polynomial matrices, Wang [23] proposed a concept called *regular* divisor.

Definition 4: Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, and f be a divisor of $d_l(\mathbf{F})$. Then f is said to be **regular** w.r.t. $\mathbf{F}$ if $d_l([f\mathbf{I}_l, \mathbf{F}]) = f$ up to multiplication by a nonzero constant.

Liu and Wang [30] further extended the above concept and presented the following definition of *weakly regular* divisor.

Definition 5: Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, and f be a divisor of $d_l(\mathbf{F})$. Then f is said to be weakly regular w.r.t. $\mathbf{F}$ if there exists $h \in k[\mathbf{z}]$ such that $d_l([h\mathbf{I}_l, \mathbf{F}]) = f$ up to multiplication by a nonzero constant. In this case, f is said to be **weakly regular** w.r.t. $\mathbf{F}$ and h.

B. Problems

Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, and f be a divisor of $d_l(\mathbf{F})$. Assume $\mathbf{F}$ can be factorized as $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ such that $\det(\mathbf{G}_1) = f$. Liu and Wang [31] obtained that

$$\rho(\mathbf{F}_1) : \frac{d_l(\mathbf{F})}{f} = \rho(\mathbf{F}) : d_l(\mathbf{F}). \tag{2}$$

The above equation establishes a connection between $\rho(\mathbf{F}_1)$ and $\rho(\mathbf{F})$. In this brief we will further discuss relationships between them.

As $\rho(\mathbf{F}) : d_l(\mathbf{F}) = (\rho(\mathbf{F}) : f) : \frac{d_l(\mathbf{F})}{f}$, a problem can be naturally raised from Equation (2). That is, under what condition is $\rho(\mathbf{F}_1)$ equal to $\rho(\mathbf{F}) : f$? Wang [23] proved that $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : f$ if f is regular w.r.t. $\mathbf{F}$. Generally speaking, f may not be regular w.r.t. $\mathbf{F}$. In this case, is $\rho(\mathbf{F}_1)$ still equal to $\rho(\mathbf{F}) : f$? Thus, we are very interested in the following problem.

Problem 1: What is the necessary and sufficient condition for $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : f$?

If $\rho(\mathbf{F}_1) \neq \rho(\mathbf{F}) : f$, then the form of $\rho(\mathbf{F}_1)$ needs to be further studied. We next consider another problem.

Problem 2: Is there another divisor h of $d_l(\mathbf{F})$ such that $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : h$?

In this brief, we will give positive solutions to the above two problems.

III. Main Results

In this section, we solve the problems raised in the above section one by one.

A. Solution to Problem 1

We first give the solution to Problem 1.

Theorem 1: Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, and f be a divisor of $d_l(\mathbf{F})$. Assume $\mathbf{F}$ can be factorized as $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ such that $\det(\mathbf{G}_1) = f$. Then $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : f$ if and only if f is regular w.r.t. $\mathbf{F}$.

Proof: Sufficiency: We refer to [23, Lemma 3.5] for more details.

Necessity: Since $\mathbf{F}_1$ is a full row rank matrix and $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : f$, we have that $\rho(\mathbf{F}) : f$ is a free $k[\mathbf{z}]$-module of rank l. Let $\mathbf{A} = \begin{bmatrix} \mathbf{F} \\ -f\mathbf{I}_m \end{bmatrix}$, then Syz($\mathbf{A}$) is a free $k[\mathbf{z}]$-module of rank l by Lemma 1. According to Lemma 2, there exists an MLP matrix $\mathbf{B} = [\mathbf{C}, \mathbf{D}] \in k[\mathbf{z}]^{l\times(l+m)}$ such that $\rho(\mathbf{B}) =$ Syz($\mathbf{A}$), where $\mathbf{C} \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{D} \in k[\mathbf{z}]^{l\times m}$. Moreover, $\rho(\mathbf{D}) = \rho(\mathbf{F}) : f$. It follows that there is a unimodular matrix $\mathbf{U} \in k[\mathbf{z}]^{l\times l}$ such that

$$\mathbf{D} = \mathbf{U}\mathbf{F}_1. \tag{3}$$

Since $[f\mathbf{I}_l, \mathbf{F}]\mathbf{A} = \mathbf{0}_{l\times m}$, all the rows of $[f\mathbf{I}_l, \mathbf{F}]$ belong to $\rho(\mathbf{B})$. Then there is a matrix $\mathbf{E} \in k[\mathbf{z}]^{l\times l}$ such that

$$[f\mathbf{I}_l, \mathbf{F}] = \mathbf{E}[\mathbf{C}, \mathbf{D}]. \tag{4}$$

Combining Equations (3) and (4) we have $\mathbf{F} = \mathbf{EUF}_1$. It follows from $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ that $(\mathbf{G}_1 - \mathbf{EU})\mathbf{F}_1 = \mathbf{0}_{l\times m}$. Based on the fact that $\mathbf{F}_1$ is a full row rank matrix, we get $\mathbf{G}_1 = \mathbf{EU}$. Since $\det(\mathbf{G}_1) = f$ and $\mathbf{U}$ is a unimodular matrix, we obtain $\det(\mathbf{E}) = f$. By Equation (4), we have

$$d_l([f\mathbf{I}_l, \mathbf{F}]) = \det(\mathbf{E})d_l(\mathbf{B}). \tag{5}$$

It follows from $\mathbf{B}$ being an MLP matrix that $d_l([f\mathbf{I}_l, \mathbf{F}]) = f$. Thus, f is regular w.r.t. $\mathbf{F}$. ■

According to the above theorem, $\rho(\mathbf{F}_1) \neq \rho(\mathbf{F}) : f$ if f is not regular w.r.t. $\mathbf{F}$, and we can deduce a similar conclusion as follows.

Corollary 1: Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, and f be a divisor of $d_l(\mathbf{F})$. Assume that $\rho(\mathbf{F}) : f$ is a free $k[\mathbf{z}]$-module of rank l and the rows of $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$ are composed of any free basis of $\rho(\mathbf{F}) : f$. Then the following conditions are equivalent.

1) $d_l(\mathbf{F}_1) = \frac{d_l(\mathbf{F})}{f}$.
2) f is regular w.r.t. $\mathbf{F}$.

From a computational point of view, Corollary 1 gives a more efficient way to judge whether $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ is a general factorization of $\mathbf{F}$ w.r.t. f. The reason is as follows.

If $\rho(\mathbf{F}) : f$ is a free $k[\mathbf{z}]$-module of rank l, then by $\rho(\mathbf{F}) \subseteq \rho(\mathbf{F}) : f$ we can obtain a general factorization of $\mathbf{F}$ w.r.t. g, where $g \in k[\mathbf{z}]$ needs to be further solved. To confirm what is g equal to, the traditional method is first to compute a free basis $\mathcal{G}$ of $\rho(\mathbf{F}) : f$, and then use the elements in $\mathcal{G}$ to form the matrix $\mathbf{F}_1$, and finally compute $d_l(\mathbf{F}_1)$ and obtain $g = \frac{d_l(\mathbf{F})}{d_l(\mathbf{F}_1)}$. Nevertheless, the complexity of the algorithm for computing free bases is double exponential. This implies that it will take a lot of time to compute g. If $g = f$, then $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ is a general factorization of $\mathbf{F}$ w.r.t. f; otherwise not. Compared with the traditional method, we only need to verify whether f is regular w.r.t. $\mathbf{F}$ by Corollary 1 and solve the above problem. Thus, heavy calculations for a basis of a free module are avoided.

B. Solution to Problem 2

We first use the following example to describe the difficulty of Problem 2.

Example 1: Let

$$\mathbf{F} = \begin{bmatrix} z_1z_2^3z_3 & z_1z_3^2 & z_3^2 + z_2^3z_3 \\ z_1z_2z_3 & 0 & z_2z_3 \end{bmatrix}$$

be a polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{2\times 3}$, where $\mathbb{C}$ is the complex field.

It is easy to compute that $d_2(\mathbf{F}) = z_1z_2z_3^3$. Let $f = z_1z_2z_3^2$, then f is a divisor of $d_2(\mathbf{F})$. By calculation, $\mathbf{F}$ has a general factorization w.r.t. f, i.e.,

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 = \begin{bmatrix} -z_1z_3 & z_3(z_2^3 + z_3) \\ 0 & z_2z_3 \end{bmatrix}\begin{bmatrix} z_3 & -z_3 & 0 \\ z_1 & 0 & 1 \end{bmatrix},$$

where $\det(\mathbf{G}_1) = f$. It is easy to check that f is not regular w.r.t. $\mathbf{F}$. According to Theorem 1, we have $\rho(\mathbf{F}_1) \neq \rho(\mathbf{F}) : f$. Let $h = z_1z_2z_3$, then h is a divisor of $d_2(\mathbf{F})$. Through verification, $\rho(\mathbf{F}) : h$ is a free $\mathbb{C}[z_1, z_2, z_3]$-module of rank 2 and $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : h$.

However, we obtain another general factorization of $\mathbf{F}$ w.r.t. f, i.e.,

$$\mathbf{F} = \mathbf{G}_1'\mathbf{F}_1' = \begin{bmatrix} z_1z_2^3z_3 & z_2^3 + z_3 \\ z_1z_2z_3 & z_2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & z_1z_3 & z_3 \end{bmatrix},$$

where $\det(\mathbf{G}_1') = f$. By calculation, it is easy to check that there is no divisor h' of $d_2(\mathbf{F})$ such that $\rho(\mathbf{F}_1') = \rho(\mathbf{F}) : h'$.

It follows from Example 1 that the solution to Problem 2 relies on general factorizations of $\mathbf{F}$ w.r.t. f. In other words, the general factorization $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ of $\mathbf{F}$ w.r.t. f must satisfy some special conditions, and it is possible to have a divisor h of $d_l(\mathbf{F})$ such that $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : h$. From this point of view, we get the following result.

Theorem 2: Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, f and h be two divisors of $d_l(\mathbf{F})$. Assume $\mathbf{F}$ can be factorized as $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ such that $\det(\mathbf{G}_1) = f$. Then the following conditions are equivalent.

1) $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : h$.
2) f is weakly regular w.r.t. $\mathbf{F}$ and h, and $f \mid hd_{l-1}(\mathbf{G}_1)$.

Proof: 1) $\to$ 2). The proof of f being weakly regular w.r.t. $\mathbf{F}$ and h is similar to that of the necessity of Theorem 1, and is omitted here. We next need to prove $f \mid hd_{l-1}(\mathbf{G}_1)$. By the previous proof, there is a polynomial matrix $\mathbf{C}' \in k[\mathbf{z}]^{l\times l}$ such that $[h\mathbf{I}_l, \mathbf{F}] = \mathbf{G}_1[\mathbf{C}', \mathbf{F}_1]$. This implies that

$$h\mathbf{I}_l = \mathbf{G}_1\mathbf{C}'. \tag{6}$$

Let $\mathbf{G}_1^* \in k[\mathbf{z}]^{l\times l}$ be the adjoint matrix of $\mathbf{G}_1$, then $f\mathbf{I}_l = \mathbf{G}_1\mathbf{G}_1^*$. It follows that

$$h\mathbf{I}_l = \mathbf{G}_1 \cdot \frac{h\mathbf{G}_1^*}{f}. \tag{7}$$

Combining Equations (6) and (7), we have

$$\mathbf{C}' = \frac{h\mathbf{G}_1^*}{f} \tag{8}$$

by the fact that $\det(\mathbf{G}_1) \neq 0$. It follows from Equation (8) that $\frac{h\mathbf{G}_1^*}{f}$ is a polynomial matrix. So, f is a common divisor of all the entries in $h\mathbf{G}_1^*$. Notice that the entries in $\mathbf{G}_1^*$ are all the $(l-1)\times(l-1)$ minors of $\mathbf{G}_1$ up to multiplication by one sign. Then, we get $f \mid hd_{l-1}(\mathbf{G}_1)$.

2) $\to$ 1). Let $\mathbf{G}_1^* \in k[\mathbf{z}]^{l\times l}$ be the adjoint matrix of $\mathbf{G}_1$. It follows from $f \mid hd_{l-1}(\mathbf{G}_1)$ that $\frac{h\mathbf{G}_1^*}{f}$ is a polynomial matrix. As $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$, we have

$$[h\mathbf{I}_l, \mathbf{F}] = \mathbf{G}_1 \cdot [\frac{h\mathbf{G}_1^*}{f}, \mathbf{F}_1]. \tag{9}$$

Since f is weakly regular w.r.t. $\mathbf{F}$ and h, we can derive that $d_l([\frac{h\mathbf{G}_1^*}{f}, \mathbf{F}_1])$ is a nonzero constant, i.e., $[\frac{h\mathbf{G}_1^*}{f}, \mathbf{F}_1]$ is an MLP matrix. Let $\mathbf{A} = \begin{bmatrix} \mathbf{F} \\ -h\mathbf{I}_m \end{bmatrix}$, then $[h\mathbf{I}_l, \mathbf{F}]\mathbf{A} = \mathbf{0}_{l\times m}$. It follows from $\det(\mathbf{G}_1) \neq 0$ that

$$[\frac{h\mathbf{G}_1^*}{f}, \mathbf{F}_1]\mathbf{A} = \mathbf{0}_{l\times m}. \tag{10}$$

According to Lemma 2, Syz($\mathbf{A}$) is a free $k[\mathbf{z}]$-module of rank l and $\mathrm{Syz}(\mathbf{A}) = \rho([\frac{h\mathbf{G}_1^*}{f}, \mathbf{F}_1])$. Then, $\rho(\mathbf{F}) : h$ is a free $k[\mathbf{z}]$-module of rank l and $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : h$.

The proof is completed. ∎

The above theorem gives the solution to Problem 2. It is easy to see that Theorem 2 is the same as Theorem 1 in the special case of $h = f$.

For the first general factorization $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ of $\mathbf{F}$ w.r.t. f in Example 1, it is easy to check that f is weakly regular w.r.t. $\mathbf{F}$ and h, and $f \mid hd_1(\mathbf{G}_1)$. However, for another general factorization $\mathbf{F} = \mathbf{G}_1'\mathbf{F}_1'$ of $\mathbf{F}$ w.r.t. f, there is no divisor h' of $d_2(\mathbf{F})$ such that the two conditions, that f is weakly regular w.r.t. $\mathbf{F}$ and h', and $f \mid h'd_1(\mathbf{G}_1')$, hold simultaneously. Therefore, it follows from Theorem 2 that there is no divisor h' of $d_2(\mathbf{F})$ such that $\rho(\mathbf{F}_1') = \rho(\mathbf{F}) : h'$.

Based on Theorem 2, we can obtain the following necessary and sufficient condition for factorizing $\mathbf{F}$ w.r.t. f, and it is a nontrivial generalization of [30, Proposition 2].

Corollary 2: Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank, and f be weakly regular w.r.t. $\mathbf{F}$ and h. Then the following conditions are equivalent.

1) There exists a general factorization $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ of $\mathbf{F}$ w.r.t. f such that $f \mid hd_{l-1}(\mathbf{G}_1)$.
2) $\rho(\mathbf{F}) : h$ is a free $k[\mathbf{z}]$-module of rank l.

Moreover, if one of the above conditions holds, then we have $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : h$.

Assume that f is weakly regular w.r.t. $\mathbf{F}$ and h, and $\rho(\mathbf{F}) : h$ is a free $k[\mathbf{z}]$-module of rank l. We next consider two different cases. If $h = f$, then for any general factorization $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ of $\mathbf{F}$ w.r.t. f, the condition $f \mid hd_{l-1}(\mathbf{G}_1)$ naturally holds. In this case, $\rho(\mathbf{F}_1)$ is uniquely determined by $\rho(\mathbf{F}) : h$ using Corollary 2. If $h \neq f$, then the establishment of the condition $f \mid hd_{l-1}(\mathbf{G}_1)$ depends on the specific form of a general factorization $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ of $\mathbf{F}$ w.r.t. f. For this case, $\rho(\mathbf{F}_1)$ may not be uniquely determined by $\rho(\mathbf{F}) : h$. As we can see in Example 1, it is easy to check that $f \nmid hd_1(\mathbf{G}_1')$ for the second general factorization. Thus, $\rho(\mathbf{F}_1') \neq \rho(\mathbf{F}) : h$. This tells us why general factorizations of $\mathbf{F}$ w.r.t. f in Example 1 are not unique when the assumptions are satisfied.

C. Algorithm and Example

Up to now, one of the most effective methods of general factorizations is to compute quotient modules of n-D polynomial matrices. Based on the above main results, we propose an algorithm to judge some possible existing general factorizations for n-D polynomial matrices.

Before proceeding further, we give some explanations to Algorithm 1.

- According to Corollary 1, $(\rho(\mathbf{F}) : f, f)$ in Step 8 means that $\mathbf{F}$ has a general factorization w.r.t. f.
- By Corollary 2, $(\rho(\mathbf{F}) : f, f')$ in Step 11 means that $\mathbf{F}$ can be factorized as $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ such that $\det(\mathbf{G}_1) = f'$ and $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : f$, where $f' \neq f$. If f' is regular w.r.t. $\mathbf{F}$, then $\rho(\mathbf{F}_1) = \rho(\mathbf{F}) : f'$ by Theorem 1. Thus, we can remove f' from S in Step 13, and the calculation for verifying that $\rho(\mathbf{F}) : f'$ is free can be avoided.

Algorithm 1: General Factorization Algorithm

Input : $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with full row rank.
Output: general factorizations of $\mathbf{F}$.

1 **begin**
2 $S^* := \{$all non-constant divisors of $d_l(\mathbf{F})\}$;
3 $W := \emptyset$ and $S := S^*$;
4 **while** $S \neq \emptyset$ **do**
5 choose f from S and $S := S \setminus \{f\}$;
6 **if** $\rho(\mathbf{F}) : f$ *is free* **then**
7 **if** f *is regular w.r.t.* $\mathbf{F}$ **then**
8 $W := W \cup \{(\rho(\mathbf{F}) : f, f)\}$;
9 **else**
10 **if** *there is* $f' \in S^*$ *such that* f' *is weakly regular w.r.t.* $\mathbf{F}$ *and* f **then**
11 $W := W \cup \{(\rho(\mathbf{F}) : f, f')\}$;
12 **if** f' *is regular w.r.t.* $\mathbf{F}$ **then**
13 $S := S \setminus \{f'\}$;
14 **else**
15 $W := W \cup \{(\rho(\mathbf{F}) : f, f'')\}$;
16 **return** W.

- To confirm what is f'' equal to in Step 15, we need to compute a free basis of $\rho(\mathbf{F}) : f$.
- W may be the empty set $\emptyset$ in Step 16. This implies that there is no divisor f of $d_l(\mathbf{F})$ such that $\rho(\mathbf{F}) : f$ is free. We currently have no further ways to deal with this situation to obtain general factorizations of $\mathbf{F}$. This is a problem that we will solve in the future.

We now use an example to illustrate Algorithm 1.

Example 2: Let

$$\mathbf{F} = \begin{bmatrix} z_2^2(z_1-1) & z_2^2 & z_2^2(z_3-2) \\ 0 & z_2^2 z_3 & z_2 z_3 \end{bmatrix}$$

be a polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{2\times 3}$, where $\mathbb{C}$ is the complex field.

By calculation, we have $d_2(\mathbf{F}) = z_2^3 z_3$ and $S^* = \{z_2, z_2^2, z_2^3, z_3, z_2z_3, z_2^2z_3, z_2^3z_3\}$. Set $W = \emptyset$, $S = S^*$ and $f_1 = z_2$, $f_2 = z_2^2, f_3 = z_2^3, f_4 = z_3, f_5 = z_2z_3, f_6 = z_2^2z_3, f_7 = z_2^3z_3$.

(1) Choose f_1 from S and $S := S\setminus\{f_1\}$. By checking, $\rho(\mathbf{F}) : f_1$ is free. However, f_1 is not regular w.r.t. $\mathbf{F}$ and there is no $f' \in S^*$ such that f' is weakly regular w.r.t. $\mathbf{F}$ and f_1. Therefore, we need to determine what is f'' equal to in Step 15. We first compute a free basis $\mathcal{G}_1$ of $\rho(\mathbf{F}) : f_1$, and then use all the elements in $\mathcal{G}_1$ to form the following matrix

$$\mathbf{F}_1 = \begin{bmatrix} 0 & z_2z_3 & z_3 \\ z_2(z_1-1) & z_2 & z_2(z_3-2) \end{bmatrix}.$$

By calculation, $d_2(\mathbf{F}_1) = z_2z_3$ and $f'' = \frac{d_2(\mathbf{F})}{d_2(\mathbf{F}_1)} = z_2^2 = f_2$. Then, $\mathbf{F}$ can be factorized as

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 \text{ with } \det(\mathbf{G}_1) = f_2 \text{ and } \rho(\mathbf{F}_1) = \rho(\mathbf{F}) : f_1.$$

Thus, $W := W \cup \{(\rho(\mathbf{F}) : f_1, f_2)\}$.

(2) Choose f_2 from S and $S := S\backslash\{f_2\}$. By checking, $\rho(\mathbf{F}) : f_2$ is free and f_2 is not regular w.r.t. $\mathbf{F}$. But f_3 is weakly regular w.r.t. $\mathbf{F}$ and f_2. Then, $W := W \cup \{(\rho(\mathbf{F}) : f_2, f_3)\}$. Through further verification, f_3 is regular w.r.t. $\mathbf{F}$. Thus, $S := S \setminus \{f_3\}$.

(3) Choose f_4 from S and $S := S\backslash\{f_4\}$. By checking, $\rho(\mathbf{F}) : f_4$ is free and f_4 is regular w.r.t. $\mathbf{F}$. So, $W := W \cup \{(\rho(\mathbf{F}) : f_4, f_4)\}$.

(4) Choose f_5 from S and $S := S\backslash\{f_5\}$. By checking, $\rho(\mathbf{F}) : f_5$ is free. However, f_5 is not regular w.r.t. $\mathbf{F}$ and there is no $f' \in S^*$ such that f' is weakly regular w.r.t. $\mathbf{F}$ and f_5. Similar to (1), we can obtain a general factorization $\mathbf{F} = \mathbf{G}_5\mathbf{F}_5$ of $\mathbf{F}$, where $\det(\mathbf{G}_5) = f_6$ and $\rho(\mathbf{F}_5) = \rho(\mathbf{F}) : f_5$. Thus, $W := W \cup \{(\rho(\mathbf{F}) : f_5, f_6)\}$.

(5) Choose f_6 from S and $S := S\backslash\{f_6\}$. By checking, $\rho(\mathbf{F}) : f_6$ is free and f_6 is not regular w.r.t. $\mathbf{F}$. But f_7 is weakly regular w.r.t. $\mathbf{F}$ and f_6. Then, $W := W \cup \{(\rho(\mathbf{F}) : f_6, f_7)\}$. Furthermore, f_7 is regular w.r.t. $\mathbf{F}$. Thus, $S := S \setminus \{f_7\}$.

As $S = \emptyset$, we end the algorithm and output W. Therefore, $\mathbf{F}$ has the following general factorizations:

$$(\rho(\mathbf{F}) : f_1, f_2), (\rho(\mathbf{F}) : f_2, f_3), (\rho(\mathbf{F}) : f_4, f_4), \\ (\rho(\mathbf{F}) : f_5, f_6), (\rho(\mathbf{F}) : f_6, f_7).$$

For Example 2, we can draw the following conclusions.

1) We cannot obtain general factorizations of $\mathbf{F}$ w.r.t. f_1 and f_5 by computing quotient modules of $\mathbf{F}$ w.r.t. some divisors of $d_2(\mathbf{F})$.
2) $\rho(\mathbf{F}) : f_2 = \rho(\mathbf{F}) : f_3$ and $\rho(\mathbf{F}) : f_6 = \rho(\mathbf{F}) : f_7$. The calculations for verifying that $\rho(\mathbf{F}) : f_3$ and $\rho(\mathbf{F}) : f_7$ are free can be avoided.

IV. Concluding Remarks

In this brief we obtained several results about two general factorization problems of n-D polynomial matrices. In particular, for any general factorization $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ of $\mathbf{F}$ w.r.t. f, knowing the form of $\rho(\mathbf{F}_1)$ can help us better understand general factorizations. Therefore, we established two relationships between $\rho(\mathbf{F}_1)$ and quotient modules of $\rho(\mathbf{F})$ w.r.t. some polynomials of $d_l(\mathbf{F})$. From the perspective of practical calculation, these relationships can be used to compute some possible existing general factorizations of n-D polynomial matrices.

It follows from the second general factorization of Example 1 that there is no divisor h' of $d_2(\mathbf{F})$ such that $\rho(\mathbf{F}'_1) = \rho(\mathbf{F}) : h'$. This indicates that the form of $\rho(\mathbf{F}'_1)$ needs further study. If this problem is solved, then we can solve general factorizations of n-D polynomial matrices.

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