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Equivalence and reduction of bivariate polynomial matrices to their Smith forms ☆

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ABSTRACT

This paper is concerned with Smith forms of bivariate polynomial matrices. For a bivariate polynomial square matrix with the determinant being the product of two distinct and irreducible univariate polynomials, we prove that it is equivalent to its Smith form. We design an algorithm to reduce this class of bivariate polynomial matrices to their Smith forms, and an example is given to illustrate the algorithm. Furthermore, we extend the above class of matrices to a more general case, and derive a necessary and sufficient condition for the equivalence of a matrix and one of its all possible existing Smith forms.

1. Introduction

It is well known that many multidimensional systems such as iterative learning control systems and multidimensional finite-impulse response (FIR) filter banks may be represented by multivariate polynomial matrices (see Bose (1982); Bose et al. (2003) and the references therein). We can obtain many important properties such as controllability, stability and solvability of a multidimen-

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sional system by studying the corresponding multivariate polynomial matrix. It is often necessary to transform a given matrix into an equivalent but simpler form to reduce the number of equations and unknowns. Consequently, the equivalence and reduction of multivariate polynomial matrices are important research problems in the multidimensional systems theory. Symbolic computation provides many effective tools such as Euclidean division (Cox et al., 2005, 2007) and Gröbner basis (Buchberger, 1965, 2001) for related research.

For a univariate polynomial matrix, it is always equivalent to its Smith form since the univariate polynomial ring has the Euclidean division property (Rosenbrock, 1970; Kailath, 1980), and many algorithms for reducing matrices to their Smith forms have already been implemented in computer algebra systems such as *Maple*, *Singular*, *Magma* and so on. Meanwhile, many researchers studied the equivalence and reduction of bivariate polynomial matrices to their Smith forms, and obtained some judgment conditions (see, e.g., Morf et al. (1977); Frost and Storey (1978); Lee and Zak (1983); Frost and Boudellioua (1986)). Due to the lack of mature multivariate (more than one variable) polynomial matrix theory, there are still challenging open problems on the equivalence and reduction of multivariate polynomial matrices.

Recently, the equivalence and reduction for different types of multivariate polynomial matrices have been widely investigated. For instance, Lin et al. (2006) showed that a square matrix F with $\det(F) = x_1 - f(x_2, \ldots, x_n)$ is equivalent to its Smith form by using the Quillen-Suslin theorem (Quillen, 1976; Suslin, 1976), where $x_1, x_2, \ldots, x_n$ are variables, $f \in K[x_2, \ldots, x_n]$ and K is a field. After that, Li et al. (2017, 2022) and Lu et al. (2020) generalized the above result to the case of $\det(F) = (x_1 - f(x_2, \ldots, x_n))^k$, where k is a positive integer. In addition, Boudellioua (2012, 2013) and Cluzeau and Quadrat (2013, 2015) used computer algebra and homological algebra to study the equivalence of other classes of multivariate polynomial matrices, and designed several algorithms for reducing matrices to their Smith forms.

In this paper, we focus on the equivalence and reduction of a class of bivariate polynomial matrices. Li et al. (2019) proved that a square matrix $F \in K[x, y]^{l\times l}$ with $\det(F)$ being an irreducible polynomial in $K[x]$ and its Smith form are equivalent, where x, y are variables. The main idea is to establish a homomorphic mapping from $K[x, y]$ to $K[x]/(\det(F))[y]$. Since $\det(F)$ is an irreducible polynomial in $K[x]$, $K[x]/(\det(F))$ is a field. This implies that $K[x]/(\det(F))[y]$ is a principal ideal domain and has the Euclidean division property. Then they used elementary transformations over the Euclidean ring $K[x]/(\det(F))[y]$ to reduce F and obtained the above result. Inspired by their work, we first consider whether a square matrix $F \in K[x, y]^{l\times l}$ with $\det(F) = pq$ is equivalent to its Smith form, where p, q are two distinct and irreducible polynomials in $K[x]$. Noting that $K[x]/(pq)$ is not a field, $K[x]/(pq)[y]$ is just a quotient ring. It follows that we need further processing to get the desired result. The next thing is how to reduce F to its Smith form. Furthermore, we will extend the above class of matrices to the more general case of $F \in K[x, y]^{l\times m}$ with $d_l(F) = p^{r_1}q^{r_2}$, where $d_l(F)$ is the greatest common divisor of all the $l \times l$ minors of F, r_1, r_2 are two positive integers and $l \leq m$.

The rest of the paper is organized as follows. In Section 2, we introduce some basic concepts and present three major problems that we shall consider. The main aim of Section 3 is to present the proof for the equivalence of a square matrix $F \in K[x, y]^{l\times l}$ with $\det(F) = pq$ and its Smith form. In Section 4, we construct an algorithm to reduce F to its Smith form and use an example to illustrate the effectiveness of the algorithm. In Section 5, we propose a necessary and sufficient condition for the equivalence of $F \in K[x, y]^{l\times m}$ with $d_l(F) = p^{r_1}q^{r_2}$ and one of its all possible existing Smith forms. Some concluding remarks are provided in Section 6.

2. Preliminaries and problems

Let K be a field, $K[x, y]$ be the polynomial ring in variables x, y over K, and $K[x, y]^{l\times m}$ be the set of $l \times m$ matrices with entries in $K[x, y]$. Without loss of generality, we assume that $l \leq m$. Let $h \in K[x, y]$, the leading coefficient of h with respect to y is denoted by $\mathrm{lc}_y(h)$. Moreover, we use $\deg_x(h)$ and $\deg_y(h)$ to denote the degree of h with respect to x and y, respectively. Let $F \in K[x, y]^{l\times l}$, we use $\det(F)$ to denote the determinant of F. Let $f_1, \ldots, f_l \in K[x, y]$, we use $\mathrm{diag}\{f_1, \ldots, f_l\}$ to denote the $l \times l$ diagonal matrix whose diagonal elements are $f_1, \ldots, f_l$. Throughout this paper, we write $\mathrm{diag}\{\underbrace{1, \ldots, 1}_{l-1}, g\}$ as $\mathrm{diag}\{1, \ldots, 1, g\}$ unless otherwise specified, where $g \in K[x, y]$.

2.1. Basic notions

We first introduce a concept which plays an important role in the equivalence of multivariate polynomial matrices.

Definition 1. Let $U \in K[x, y]^{l\times l}$, then U is said to be unimodular if $\det(U)$ is a nonzero constant in K.

Now, we present the concepts of equivalence and Smith forms for bivariate polynomial matrices.

Definition 2. Let $F, Q \in K[x, y]^{l\times m}$, then F is said to be equivalent to Q if there are two unimodular matrices $U \in K[x, y]^{l\times l}$ and $V \in K[x, y]^{m\times m}$ such that $F = UQV$.

For convenience, F being equivalent to Q is denoted by $F \sim Q$.

Definition 3. Let $F \in K[x, y]^{l\times m}$ with rank r, and Φ_i be a polynomial defined as follows:

$$\Phi_i = \begin{cases} \frac{d_i(F)}{d_{i-1}(F)}, & 1 \leq i \leq r; \\ 0, & r < i \leq l, \end{cases}$$

where $d_i(F)$ is the greatest common divisor of all the $i \times i$ minors of F. Here, we make the convention that $d_0(F) \equiv 1$ and $d_i(F) \equiv 0$ for $i > r$. Moreover, Φ_i satisfies the divisibility property

$$\Phi_1 \mid \Phi_2 \mid \cdots \mid \Phi_r.$$

Then the Smith form of F is given by

$$S = \begin{pmatrix} \mathrm{diag}\{\Phi_1, \ldots, \Phi_r\} & 0_{r\times(m-r)} \\ 0_{(l-r)\times r} & 0_{(l-r)\times(m-r)} \end{pmatrix}.$$

2.2. Problems

We construct a subset of bivariate polynomial matrices as follows:

$$\mathcal{F} := \{F \in K[x, y]^{l\times l} \mid \det(F) = pq \text{ with } p, q \in K[x] \text{ being distinct and irreducible}\}.$$

In the above set, p, q are irreducible polynomials over K. This paper first focus on the equivalence of polynomial matrices in $\mathcal{F}$ and their Smith forms.

Let $F \in \mathcal{F}$. It follows from $d_i(F) \mid \det(F)$ that $d_i(F)$ is equal to 1, or p, or q, or pq, where $i = 1, \ldots, l$. As $d_{i-1}(F) \mid d_i(F)$ and $\Phi_{i-1} \mid \Phi_i$ by Definition 3, it is easy to verify that the Smith form of F is

$$S = \mathrm{diag}\{1, \ldots, 1, pq\}.$$

Then, we address the following two specific problems.

Problem 4. Let $F \in \mathcal{F}$. Is F equivalent to its Smith form S?

Problem 5. If $F \sim S$, how to reduce F to S? That is to design an algorithm to construct two unimodular matrices $U, V \in K[x, y]^{l\times l}$ such that $S = UFV$.

When p or q is a nonzero constant, the above two problems have been solved by Li et al. (2019). In the following, we assume that p, q are two non-trivial polynomials in $K[x]$.

We now construct another subset of bivariate polynomial matrices:

$$\mathcal{F}^* := \{F \in K[x,y]^{l\times m} \mid d_l(F) = p^{r_1}q^{r_2} \text{ with } r_1, r_2 \text{ being two positive integers}\}.$$

It is obvious that $\mathcal{F} \subset \mathcal{F}^*$. Let $F \in \mathcal{F}^*$. According to Definition 3, the following matrix

$$S^* = \begin{pmatrix} \mathrm{diag}\{1,\ldots,1,p^{r_1}q^{r_2}\} & 0_{l\times(m-l)} \end{pmatrix}$$

is one of all the possible existing Smith forms of F. Then we consider the following problem.

Problem 6. Let $F \in \mathcal{F}^*$. What is the necessary and sufficient condition for the equivalence of F and S^*?

3. Matrix equivalence theorem

In this section, we will solve Problem 4. First, we give the main result in the paper.

Theorem 7. *Let $F \in \mathcal{F}$, then F is equivalent to its Smith form S.*

Although the description of Theorem 7 is brief, the proof process is very complicated. Before giving a detailed proof, we show the difficulty for proving Theorem 7.

Let $h \in K[x]$ be a nonzero polynomial and $R_h = K[x]/(h)$. If h is an irreducible polynomial, then R_h is a field; otherwise, it is a quotient ring. We consider the following homomorphism

$$\begin{array}{cccc} \phi_h: & K[x,y] & \rightarrow & R_h[y] \\ & \sum_{i=0}^n c_i y^i & \mapsto & \sum_{i=0}^n \overline{c_i} y^i, \end{array}$$

where $c_0,\ldots,c_n \in K[x]$ and $\overline{c_0},\ldots,\overline{c_n} \in R_h$. This homomorphism can extend canonically to the homomorphism $\phi_h: K[x,y]^{l\times l} \rightarrow R_h[y]^{l\times l}$ by applying ϕ_h entry-wise.

Lemma 8. *Let $F \in K[x,y]^{l\times l}$ and $h \in K[x]$ be an irreducible polynomial. If $h \mid \det(F)$, then there is a unimodular matrix $U \in K[x,y]^{l\times l}$ such that $UF = \mathrm{diag}\{1,\ldots,1,h\}\cdot G$, where $G \in K[x,y]^{l\times l}$.*

Proof. Since R_h is a field, $R_h[y]$ is an Euclidean ring. Let $\overline{F}$ be the polynomial matrix $\phi_h(F)$ in $R_h[y]^{l\times l}$. As $h \mid \det(F)$, the rank of $\overline{F}$ is less than l. Then, we can transform $\overline{F}$ into the following matrix

$$\overline{F_1} = \begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \\ \overline{0} & \overline{0} & \cdots & \overline{0} \end{pmatrix}$$

only by the first kind (interchanging two rows) and third kind (adding multiple of one row to another) of elementary transformations in $R_h[y]$. This implies that there exist a finite number of the first and third kinds of elementary matrices $\overline{U_1},\ldots,\overline{U_s} \in R_h[y]^{l\times l}$ such that

$$\overline{U_s}\cdots\overline{U_1}\cdot\overline{F} = \overline{F_1}. \tag{1}$$

For each entry of $\overline{U_i}$ with $1 \le i \le s$, we take the representation element whose degree with respect to x is less than $\deg_x(h)$. Then, we have unimodular matrices $U_1,\ldots,U_s \in K[x,y]^{l\times l}$ which satisfy

Equation (1). Let $U = U_s \cdots U_1$, then $\overline{U} \cdot \overline{F} = \overline{F_1}$. Note that the last row of $\overline{F_1}$ is zero vector. It follows that all elements of the last row of UF are divisible by h. Consequently,

$$UF = \mathrm{diag}\{1, \ldots, 1, h\} \cdot G$$

with $G \in K[x, y]^{l\times l}$ and the proof is completed. $\square$

Let $F \in \mathcal{F}$. Using Lemma 8 twice, we have

$$F \sim \mathrm{diag}\{1, \ldots, 1, p\} \cdot U' \cdot \mathrm{diag}\{1, \ldots, 1, q\}, \tag{2}$$

where $U' \in K[x, y]^{l\times l}$ is a unimodular matrix. It follows from Equation (2) that the difficulty for proving Theorem 7 is to prove

$$\mathrm{diag}\{1, \ldots, 1, p\} \cdot U' \cdot \mathrm{diag}\{1, \ldots, 1, q\} \sim \mathrm{diag}\{1, \ldots, 1, pq\}. \tag{3}$$

By mathematical induction, we should prove that when Equation (3) for any $l \le n-1$ is correct, then it is correct for $l = n$, where n is a positive integer greater than 2. But in order to enable everyone to understand the proof process more intuitively, we will prove the correctness of Equation (3) when $l = 2$ and $l = 3$, and then the detailed reduction process for $l = 2$ and $l = 3$ can be easily generalized to the case of $l \ge 4$.

Lemma 9. *Let $A = \mathrm{diag}\{1, p\} \cdot H \cdot \mathrm{diag}\{1, q\}$, where $p, q \in K[x] \setminus K$ are two distinct and irreducible polynomials, and $H \in K[x, y]^{2\times 2}$ (not necessarily unimodular). Then there exists $G \in K[x, y]^{2\times 2}$ such that*

$$A \sim \mathrm{diag}\{1, pq\} \cdot G.$$

The main idea of the proof of Lemma 9 is to apply Euclidean division over the quotient ring $R_{pq}[y]$, where $R_{pq} = K[x]/(pq)$. Since R_{pq} is not a field, the reductions of polynomials over $R_{pq}[y]$ are more complicated. To achieve reductions, we will frequently use the following operations in $R_{pq}[y]$:

1. construct some special polynomials (monic, or leading coefficient divided by p or q with respect to y) by using the irreducibility of p and q;
2. reduce the degree of a polynomial with respect to y by using special polynomials;
3. perform elementary row and column transformations simultaneously to reduce the entry at the upper left corner of a matrix each time.

Proof. For convenience, we use $\overline{A}$ to denote the polynomial matrix $\phi_{pq}(A)$ in $R_{pq}[y]^{2\times 2}$. In addition, we denote the i-th row and j-th column element of $\overline{A}$ by $\overline{A}[i, j]$, where $1 \le i, j \le 2$. We assume that there exist $v_{11}, v_{21}, v_{12} \in K[x, y]$ such that

$$\overline{A} = \begin{pmatrix} \overline{v_{11}} & \overline{v_{12}q} \\ \overline{v_{21}p} & \overline{0} \end{pmatrix}$$

with $\deg_x(v_{11}) < \deg_x(pq)$, $\deg_x(v_{21}) < \deg_x(q)$ and $\deg_x(v_{12}) < \deg_x(p)$.

Without loss of generality, we assume that $\overline{A}[1, 1] \ne \overline{0}$, $\overline{A}[2, 1] \ne \overline{0}$ and $\overline{A}[1, 2] \ne \overline{0}$. Now, we divide our proof into three steps.

Step 1. We first need to verify that there exists a polynomial matrix $A' \in K[x, y]^{2\times 2}$ such that

$$A \sim A' \text{ and } \overline{A'} = \begin{pmatrix} \overline{v'_{11}} & \overline{v'_{12}q} \\ \overline{v'_{21}p} & \overline{0} \end{pmatrix},$$

where $\deg_y(\overline{A'}[1, 1]) < \deg_y(\overline{A}[1, 1])$, or $\overline{A'}[2, 1] = \overline{0}$, or $\overline{A'}[1, 2] = \overline{0}$.

Proof of Step 1. Let $d = \deg_y(\overline{A}[1, 1])$, $d_1 = \deg_y(\overline{A}[2, 1])$, $d_2 = \deg_y(\overline{A}[1, 2])$ and $d_0 = \max\{d_1, d_2\}$. There are four cases.

(1.1) If $d \geq d_0$, then

$$\begin{aligned}\overline{A}[1,1] &= \overline{a_d}y^d + \cdots + \overline{a_{d_0}}y^{d_0} + \cdots + \overline{a_1}y + \overline{a_0} \\ &= (\overline{a_d}y^{d-d_0} + \cdots + \overline{a_{d_0}})y^{d_0} + \cdots + \overline{a_1}y + \overline{a_0} \\ &= \overline{f}y^{d_0} + \cdots + \overline{a_1}y + \overline{a_0}.\end{aligned}$$

As $\deg_x(v_{21}) < \deg_x(q)$ and $\deg_x(v_{12}) < \deg_x(p)$, we have

$$\gcd(\mathrm{lc}_y(v_{21}), q) = 1 \text{ and } \gcd(\mathrm{lc}_y(v_{12}), p) = 1.$$

It follows that

$$\begin{aligned}\gcd(\mathrm{lc}_y(\overline{A}[2,1]), \mathrm{lc}_y(\overline{A}[1,2])) &= \gcd(\mathrm{lc}_y(v_{21})p, \mathrm{lc}_y(v_{12})q) \\ &= \gcd(\mathrm{lc}_y(v_{21}), \mathrm{lc}_y(v_{12})).\end{aligned}$$

In addition,

$$\deg_x\left(\gcd(\mathrm{lc}_y(v_{21}), \mathrm{lc}_y(v_{12}))\right) < \min\{\deg_x(p), \deg_x(q)\}.$$

This implies that $\gcd(\mathrm{lc}_y(v_{12}), \mathrm{lc}_y(v_{21}))$ and pq are relatively prime. Then there exist polynomials $s, u, v \in K[x]$ such that

$$s \cdot (u \cdot \mathrm{lc}_y(\overline{A}[2,1]) + v \cdot \mathrm{lc}_y(\overline{A}[1,2])) = s \cdot \gcd(\mathrm{lc}_y(v_{21}), \mathrm{lc}_y(v_{12})) \equiv 1 \mod pq.$$

Therefore, $\overline{s} \cdot (\overline{u} \cdot \overline{A}[2,1]y^{d_0-d_1} + \overline{v} \cdot \overline{A}[1,2]y^{d_0-d_2})$ is monic in $R_{pq}[y]$. Let

$$U = \begin{pmatrix} 1 & -fsuy^{d_0-d_1} \\ 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & 0 \\ -fsvy^{d_0-d_2} & 1 \end{pmatrix}.$$

Obviously, U and V are two unimodular matrices. Let $A' = UAV$, then

$$\deg_y(\overline{A'}[1,1]) < d, \quad \overline{A'}[2,1] = \overline{v_{21}p}, \quad \overline{A'}[1,2] = \overline{v_{12}q}, \quad \overline{A'}[2,2] = \overline{0}.$$

(1.2) If $d < d_0$, $p \nmid \mathrm{lc}_y(\overline{A}[1,1])$ and $q \nmid \mathrm{lc}_y(\overline{A}[1,1])$, then $\mathrm{lc}_y(\overline{A}[1,1])$ is reversible in R_{pq}. There is a polynomial $s' \in K[x]$ such that $s' \cdot \mathrm{lc}_y(\overline{A}[1,1]) \equiv 1 \mod pq$. Note that $d_0 = \max\{d_1, d_2\} > d$. Without loss of generality, we assume that $d_1 > d$. Constructing the following unimodular matrix

$$U_1 = \begin{pmatrix} 1 & 0 \\ -s' \cdot \mathrm{lc}_y(\overline{A}[2,1]) \cdot y^{d_1-d} & 1 \end{pmatrix}.$$

Let $A_2 = U_1A$, then $\deg_y(\overline{A_2}[2,1]) < d_1$. It is obvious that $\overline{A_2}[1,1] = \overline{A}[1,1]$ and $\overline{A_2}[1,2] = \overline{v_{12}q}$. Since $p \mid \mathrm{lc}_y(\overline{A}[2,1])$, we have $p \mid \overline{A_2}[2,1]$ and $\overline{A_2}[2,2] = \overline{0}$. Repeat this process and there is a positive integer n_1 such that

$$\begin{aligned}&\overline{A_{n_1}}[1,1] = \overline{A}[1,1], \quad \deg_y(\overline{A_{n_1}}[2,1]) \leq d \text{ and } p \mid \overline{A_{n_1}}[2,1], \\ &\overline{A_{n_1}}[1,2] = \overline{v_{12}q}, \quad \overline{A_{n_1}}[2,2] = \overline{0}.\end{aligned}$$

If $d_2 > d$ at this moment, let

$$V_1 = \begin{pmatrix} 1 & -s' \cdot \mathrm{lc}_y(\overline{A_{n_1}}[1,2]) \cdot y^{d_2-d} \\ 0 & 1 \end{pmatrix} \text{ and } A_{n_1+1} = A_{n_1}V_1,$$

then $\deg_y(\overline{A_{n_1+1}}[1,2]) < d_2$. It is obvious that $\overline{A_{n_1+1}}[1,1] = \overline{A_{n_1}}[1,1]$ and $\overline{A_{n_1+1}}[2,1] = \overline{A_{n_1}}[2,1]$. Since $q \mid \mathrm{lc}_y(\overline{A_{n_1}}[1,2])$, we have $q \mid \overline{A_{n_1+1}}[1,2]$ and $\overline{A_{n_1+1}}[2,2] = \overline{0}$. Repeat this process and there is a positive integer n_2 such that

$$\overline{A_{n_2}}[1,1]=\overline{A_{n_1}}[1,1],\ \ \overline{A_{n_2}}[2,1]=\overline{A_{n_1}}[2,1],$$
$$\deg_y(\overline{A_{n_2}}[1,2])\le d \text{ and } q \mid \overline{A_{n_2}}[1,2],\ \ \overline{A_{n_2}}[2,2]=\overline{0}.$$

If $\overline{A_{n_2}}[2,1]\neq\overline{0}$ or $\overline{A_{n_2}}[1,2]\neq\overline{0}$, then we perform the calculation process of the case (1.1) and obtain a polynomial matrix $A'\in K[x,y]^{2\times 2}$ such that $A'\sim A_{n_2}$ with

$$\deg_y(\overline{A'}[1,1])<\deg_y(\overline{A_{n_2}}[1,1])=d,\ \ \overline{A'}[2,1]=\overline{A_{n_2}}[2,1],$$
$$\overline{A'}[1,2]=\overline{A_{n_2}}[1,2],\ \ \overline{A'}[2,2]=\overline{0},$$

where $p\mid\overline{A'}[2,1]$ and $q\mid\overline{A'}[1,2]$.

(1.3) If $d<d_0$ and $p\mid \mathrm{lc}_y(\overline{A}[1,1])$. It follows from $q\nmid \mathrm{lc}_y(\overline{A}[1,1])$ that

$$\gcd\big(p\cdot \mathrm{lc}_y(\overline{A}[1,1]),pq\big)=p.$$

Hence, there exists a polynomial $s'\in K[x]$ such that $s'\cdot p\cdot \mathrm{lc}_y(\overline{A}[1,1])\equiv p \mod pq$. If $d_1>d$, then let

$$U_1=\begin{pmatrix}1 & 0\\ -s'\cdot p\cdot \frac{\mathrm{lc}_y(\overline{A}[2,1])}{p}\cdot y^{d_1-d} & 1\end{pmatrix} \text{ and } A_2=U_1A.$$

In this case, we have $\deg_y(\overline{A_2}[2,1])<d_1$. Obviously, $\overline{A_2}[1,1]=\overline{A}[1,1]$ and $\overline{A_2}[1,2]=\overline{v_{12}q}$. Since $p\mid U_1[2,1]$, we have $p\mid\overline{A_2}[2,1]$ and $\overline{A_2}[2,2]=\overline{0}$. Repeat this process and there is a positive integer n_1 such that

$$\overline{A_{n_1}}[1,1]=\overline{A}[1,1],\ \ \deg_y(\overline{A_{n_1}}[2,1])\le d \text{ and } p\mid\overline{A_{n_1}}[2,1],$$
$$\overline{A_{n_1}}[1,2]=\overline{v_{12}q},\ \ \overline{A_{n_1}}[2,2]=\overline{0}.$$

If $A_{n_1}[2,1]\neq\overline{0}$ at this moment, then let

$$d_1'=\deg_y(\overline{A_{n_1}}[2,1]),\ \ U_1=\begin{pmatrix}1 & -s''\cdot\frac{\mathrm{lc}_y(\overline{A_{n_1}}[1,1])}{p}\cdot y^{d-d_1'}\\ 0 & 1\end{pmatrix} \text{ and } A'=U_1A_{n_1},$$

where $s''\in K[x]$ satisfies $s''\cdot \mathrm{lc}_y(\overline{A_{n_1}}[2,1])\equiv p \mod pq$. Then

$$\deg_y(\overline{A'}[1,1])<\deg_y(\overline{A_{n_1}}[1,1])=d,\ \ \overline{A'}[2,1]=\overline{A_{n_1}}[2,1],$$
$$\overline{A'}[1,2]=\overline{A_{n_1}}[1,2],\ \ \overline{A'}[2,2]=\overline{0},$$

where $p\mid\overline{A'}[2,1]$ and $q\mid\overline{A'}[1,2]$.

(1.4) If $d<d_0$ and $q\mid \mathrm{lc}_y(\overline{A}[1,1])$, then we use the calculation process similar to that of the case (1.3) to reduce A. The details are omitted here.

Step 2. The next thing to do is to prove that there exists a polynomial matrix $A''\in K[x,y]^{2\times 2}$ such that $A\sim A''$ and

$$\overline{A''}=\begin{pmatrix}\overline{v''_{11}} & \overline{v''_{12}q}\\ \overline{0} & \overline{0}\end{pmatrix},\ \text{ or } \overline{A''}=\begin{pmatrix}\overline{0} & \overline{v''_{12}q}\\ \overline{v''_{21}p} & \overline{0}\end{pmatrix},\ \text{ or } \overline{A''}=\begin{pmatrix}\overline{v''_{11}} & \overline{0}\\ \overline{v''_{21}p} & \overline{0}\end{pmatrix}.$$

Proof of Step 2. Repeat *Step 1* and we can get the result. This process will stop in finite steps since the degrees of entries in A are strictly decreasing in the whole reduction process.

Step 3. Finally, we have to show that there exists a polynomial matrix $G\in K[x,y]^{2\times 2}$ such that $A\sim \mathrm{diag}\{1,pq\}\cdot G$.

Proof of Step 3. According to *Step 2*, there are three cases.

(3.1) If $\overline{A''} = \begin{pmatrix} \overline{v''_{11}} & \overline{v''_{12}q} \\ \bar{0} & \bar{0} \end{pmatrix}$, then there exists a polynomial matrix $G \in K[x, y]^{2\times 2}$ such that

$$A \sim A'' = \mathrm{diag}\{1, pq\} \cdot G.$$

(3.2) If $\overline{A''} = \begin{pmatrix} \bar{0} & \overline{v''_{12}q} \\ \overline{v''_{21}p} & \bar{0} \end{pmatrix}$, then let $A''' = \begin{pmatrix} 1 & 0 \\ -s''' \cdot p & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot A''$, where $s''' \in K[x]$ satisfies $s''' \cdot p^2 \equiv p \mod pq$. It is easy to compute that $\overline{A'''} = \begin{pmatrix} \overline{v''_{21}p} & \overline{v''_{12}q} \\ \bar{0} & \bar{0} \end{pmatrix}$. Then there exists a polynomial matrix $G \in K[x, y]^{2\times 2}$ such that

$$A \sim A''' = \mathrm{diag}\{1, pq\} \cdot G.$$

(3.3) If $\overline{A''} = \begin{pmatrix} \overline{v''_{11}} & \bar{0} \\ \overline{v''_{21}p} & \bar{0} \end{pmatrix}$, then we discuss it in the following three situations.

(a) If $p \mid \overline{v''_{11}}$, then we repeat the calculation process of the four cases in *Step 1* to reduce $\overline{A''}$ until one of the elements in the first column is $\bar{0}$. Thus, we can obtain a polynomial matrix $G \in K[x, y]^{2\times 2}$ such that $A \sim \mathrm{diag}\{1, pq\} \cdot G$.

(b) If $q \mid \overline{v''_{11}}$, then let $A''' = A'' \cdot \begin{pmatrix} 1 & s''' \cdot q \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, where $s''' \in K[x]$ satisfies $s''' \cdot q^2 \equiv q \mod pq$. It is easy to see that $\overline{A'''} = \begin{pmatrix} \bar{0} & \overline{v''_{11}s'''q} \\ \overline{v''_{21}p} & \bar{0} \end{pmatrix}$. Based on the case (3.2), we have $A \sim A''' \sim \mathrm{diag}\{1, pq\} \cdot G$, where $G \in K[x, y]^{2\times 2}$.

(c) If $p \nmid \overline{v''_{11}}$ and $q \nmid \overline{v''_{11}}$, then let $B = A'' \cdot \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$. We get $\overline{B}[1, 1] = \overline{A''}[1, 1]$ and $\deg_y(\overline{B}[1, 1]) = \deg_y(\overline{B}[1, 2])$. We repeat the calculation process of *Step 2* and *Step 3* alternately to reduce B until case (3.1) or case (3.2) or case (3.3.a) or case (3.3.b) occurs. This process will stop in finite steps since the degrees of the first column entries in B are strictly decreasing.

The proof is completed. $\square$

It follows from Lemma 9 that Equation (3) is correct for the case of $l = 2$. Next, we will focus on the case of $l \geq 3$.

Lemma 10. *Let* $A = \mathrm{diag}\{1, \ldots, 1, p\} \cdot H \cdot \mathrm{diag}\{1, \ldots, 1, q\}$, *where* $p, q \in K[x] \setminus K$ *are two distinct and irreducible polynomials,* $H \in K[x, y]^{l\times l}$ *(not necessarily unimodular) and* $l \geq 3$. *Then there exists* $G \in K[x, y]^{l\times l}$ *such that*

$$A \sim \mathrm{diag}\{1, \ldots, 1, pq\} \cdot G.$$

Proof. We first consider the case of $l = 3$. We assume that there exist $v_{ij} \in K[x, y]$ with $1 \leq i, j \leq 3$ such that

$$\overline{A} = \begin{pmatrix} \overline{v_{11}} & \overline{v_{12}} & \overline{v_{13}q} \\ \overline{v_{21}} & \overline{v_{22}} & \overline{v_{23}q} \\ \overline{v_{31}p} & \overline{v_{32}p} & \bar{0} \end{pmatrix},$$

where $\deg_x(v_{ij}) < \deg_x(pq)$ with $i = 1, 2$ and $j = 1, 2$, $\deg_x(v_{3j}) < \deg_x(q)$ with $j = 1, 2$, and $\deg_x(v_{i3}) < \deg_x(p)$ with $i = 1, 2$.

Without loss of generality, we assume that $\overline{A}[3,1] \neq \overline{0}$ and $\overline{A}[3,2] \neq \overline{0}$. Let

$$\overline{A_r} = \begin{pmatrix} \overline{v_{22}} & \overline{v_{23}q} \\ \overline{v_{32}p} & \overline{0} \end{pmatrix},$$

then we use the method in Lemma 9 to reduce $\overline{A_r}$. In the following, we will explain an invariant property in the whole reduction process.

First, we use $\overline{A_r}[2,1]$ and $\overline{A_r}[1,2]$ to reduce $\overline{A_r}[1,1]$. Let

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & g \\ 0 & 0 & 1 \end{pmatrix}, \; V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & h & 1 \end{pmatrix} \text{ and } A_1 = UAV,$$

where $g, h \in K[x, y]$ satisfy some conditions, then

$$\overline{A_1} = \begin{pmatrix} \overline{v_{11}} & \overline{v_{12}^*} & \overline{v_{13}q} \\ \overline{v_{21}^*} & \overline{v_{22}^*} & \overline{v_{23}q} \\ \overline{v_{31}p} & \overline{v_{32}p} & \overline{0} \end{pmatrix}.$$

It is obvious that $p \mid \overline{A_1}[3, j]$ for $j = 1, 2, 3$ and $q \mid \overline{A_1}[i, 3]$ for $i = 1, 2, 3$.

Second, we use $\overline{A_1}[2,2]$ to reduce $\overline{A_1}[3,2]$ and $\overline{A_1}[2,3]$. According to the three cases of Lemma 9, we add $f' \cdot p$ multiple of the second row of $\overline{A_1}$ to the third row of $\overline{A_1}$, and add $f'' \cdot q$ multiple of the second column of $\overline{A_1}$ to the third column of $\overline{A_1}$, where $f', f'' \in K[x, y]$ satisfy some conditions. Let

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & f' \cdot p & 1 \end{pmatrix}, \; V_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f'' \cdot q \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A_2 = U_1 A_1 V_1,$$

then

$$\overline{A_2} = \begin{pmatrix} \overline{v_{11}} & \overline{v_{12}^*} & \overline{(v_{13} + f'' v_{12}^*)q} \\ \overline{v_{21}^*} & \overline{v_{22}^*} & \overline{(v_{23} + f'' v_{22}^*)q} \\ \overline{(v_{31} + f' v_{21}^*)p} & \overline{(v_{32} + f' v_{22}^*)p} & \overline{0} \end{pmatrix}.$$

Obviously, $p \mid \overline{A_2}[3, j]$ for $j = 1, 2, 3$ and $q \mid \overline{A_2}[i, 3]$ for $i = 1, 2, 3$.

Repeat the above process and there is a positive integer n_k such that

$$\overline{A} \sim \overline{A_{n_k}} = \begin{pmatrix} \overline{h_{11}} & \overline{h_{12}} & \overline{h_{13}q} \\ \overline{h_{21}} & \overline{h_{22}} & \overline{h_{23}q} \\ \overline{h_{31}p} & \overline{0} & \overline{0} \end{pmatrix}.$$

If $q \mid h_{22}$, then we use Lemma 9 again to reduce $\overline{A_{n_k}}$ and obtain the result, that is, there exists $G \in K[x, y]^{3\times 3}$ such that

$$A \sim \mathrm{diag}\{1, 1, pq\} \cdot G.$$

Otherwise, let $B = \begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix}$. Then $\phi_q(B)$ is a polynomial matrix in $R_q[y]^{2\times 1}$. It is easy to see that $\mathrm{rank}(\phi_q(B)) \leq 1$. This implies that there is a unimodular matrix $C \in K[x, y]^{2\times 2}$ such that

$$C \cdot B = \mathrm{diag}\{1, q\} \cdot \begin{pmatrix} h_{12}^* \\ h_{22}^* \end{pmatrix}.$$

Let

$$U_2 = \mathrm{diag}\{C, 1\} \text{ and } A_3 = U_2 A_{n_k},$$

then

$$\overline{A_3} = \begin{pmatrix} \overline{h_{11}^*} & \overline{h_{12}^*} & \overline{h_{13}^* q} \\ \overline{h_{21}^*} & \overline{h_{22}^* q} & \overline{h_{23}^* q} \\ \overline{h_{31}^* p} & \overline{0} & \overline{0} \end{pmatrix}.$$

We use the method in Lemma 9 again to reduce $\overline{A_3}$. Similarly, in finite steps we get the result

$$A \sim \mathrm{diag}\{1, 1, pq\} \cdot G,$$

where $G \in K[x, y]^{3\times 3}$.

Using the same procedure as above, we can easily carry out the proof for the case of $l \geq 4$. Consequently, we derive that $A \sim \mathrm{diag}\{1, \ldots, 1, pq\} \cdot G$ for $l \geq 3$. □

Based on Lemma 10, Equation (3) is correct for the case of $l \geq 3$. Thus, we give a complete proof of Theorem 7 and solve Problem 4.

Before proceeding further, let us remark on Theorem 7. We first introduce a lemma, which can be easily proved by Binet-Cauchy formula. Thus, the proof is omitted here.

Lemma 11. *Suppose $F = F_1 F_2$, where $F, F_2 \in K[x, y]^{l\times m}$ and $F_1 \in K[x, y]^{l\times l}$. If all the $(l-1)\times(l-1)$ minors of F generate $K[x, y]$, then all the $(l-1)\times(l-1)$ minors of F_i generate $K[x, y]$ for $i = 1, 2$.*

Remark 12. If $l = m$, then the above lemma is the same as Lemma 3.4 in Li et al. (2022). In addition, the lemma still holds when $F_1 \in K[x, y]^{l\times m}$ and $F_2 \in K[x, y]^{m\times m}$.

Let $F \in \mathcal{F}$. According to Theorem 7, there exist two unimodular matrices $U, V \in K[x, y]^{l\times l}$ such that $S = UFV$. Since $S = \mathrm{diag}\{1, \ldots, 1, pq\}$, it is easy to see that all the $(l-1)\times(l-1)$ minors of S generate $K[x, y]$. By Lemma 11, we have that all the $(l-1)\times(l-1)$ minors of F generate $K[x, y]$. Therefore, Theorem 7 implies the following corollary.

Corollary 13. *Let $F \in \mathcal{F}$, then all the $(l-1)\times(l-1)$ minors of F generate $K[x, y]$.*

4. Algorithm and example

According to the proofs of Lemmas 8 and 10, it is easy to construct the following algorithm to solve Problem 5.

Algorithm 1: Smith Form.

Input : $F \in \mathcal{F}$.
Output: two unimodular matrices U and V such that UFV is the Smith form of F.

1 **begin**
2 compute the determinant pq of F;
3 based on Lemma 8, compute three unimodular matrices U_1, U_2, U_3 such that $U_1 F U_2 = \mathrm{diag}\{1, \ldots, 1, p\} \cdot U_3 \cdot \mathrm{diag}\{1, \ldots, 1, q\}$;
4 use Lemma 10 to compute two unimodular matrices U_4, U_5 such that $U_4 \cdot \mathrm{diag}\{1, \ldots, 1, p\} \cdot U_3 \cdot \mathrm{diag}\{1, \ldots, 1, q\} \cdot U_5 = \mathrm{diag}\{1, \ldots, 1, pq\}$;
5 set $U = U_4 U_1$ and $V = U_2 U_5$;
6 **return** U and V.

Theorem 14. *Algorithm 1 outputs as specified within a finite number of steps.*

Proof. The correctness and termination follow directly from Lemmas 8 and 10. □

We now use an example to illustrate the calculation process of Algorithm 1.

Example 15. Let

$$F=\begin{pmatrix} xy^4+y^5+1 & xy^3 & xy^2+y^3 \\ F[2,1] & x^4y^5+x^4y^4+1 & x^3y^3(y+1)(y+x) \\ F[3,1] & (y+x)xy^3 & x^2y^2+2xy^3+y^4+x^3-x^2+x-1 \end{pmatrix}$$

be a bivariate polynomial matrix in $\mathbb{Q}[x,y]^{3\times 3}$, where $F[2,1]=x^4y^6+x^3y^7+x^4y^5+x^3y^6+x^3y^2+x^3y$, $F[3,1]=x^2y^4+2xy^5+y^6+x^3y^2-x^2y^2+xy^2-y^2+x+y$, and $\mathbb{Q}$ is the rational number field.

It is easy to compute that $\det(F)=(x-1)(x^2+1)$. Based on Theorem 7, F is equivalent to its Smith form S, where

$$S=\mathrm{diag}\{1,1,(x-1)(x^2+1)\}.$$

Let $p=x-1$ and $q=x^2+1$. According to Lemma 8, there is a unimodular matrix

$$U_1=\begin{pmatrix} -y^2-y & 1 & 0 \\ y^5+y^4+1 & -y^3 & 0 \\ -y-1 & 0 & 1 \end{pmatrix}$$

such that

$$U_1F=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix}G_1,$$

where

$$G_1=\begin{pmatrix} G_1[1,1] & x^4y^5+x^4y^4-xy^5-xy^4+1 & G_1[1,3] \\ G_1[2,1] & -x^4y^8-x^4y^7+xy^8+xy^7+xy^3-y^3 & G_1[2,3] \\ G_1[3,1] & xy^3 & G_1[3,3] \end{pmatrix}$$

with $G_1[1,1]=x^4y^6+x^3y^7+x^4y^5+x^3y^6-xy^6-y^7-xy^5-y^6+x^3y^2+x^3y-y^2-y$, $G_1[1,3]=x^4y^4+x^3y^5+x^4y^3+x^3y^4-xy^4-y^5-xy^3-y^4$, $G_1[2,1]=-x^4y^9-x^3y^{10}-x^4y^8-x^3y^9+xy^9+y^{10}+xy^8+y^9-x^3y^5-x^3y^4+xy^4+2y^5+y^4+1$, $G_1[2,3]=-x^4y^7-x^3y^8-x^4y^6-x^3y^7+xy^7+y^8+xy^6+y^7+xy^2+y^3$, $G_1[3,1]=xy^4+y^5+x^2y^2+y^2+1$ and $G_1[3,3]=xy^2+y^3+x^2+1$.

Using Lemma 8 again, there exists a unimodular matrix

$$U_2=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & xy^2+xy+y^2+y \\ y^3 & 1 & -1 \end{pmatrix}$$

such that

$$U_2G_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix}G_2,$$

where

$$G_2 = \begin{pmatrix} xy^4 + y^5 + x^2y^2 + y^2 + 1 & xy^3 & xy^2 + y^3 + x^2 + 1 \\ G_2[2,1] & x^4y^5 + x^4y^4 + x^2y^5 + x^2y^4 + 1 & G_2[2,3] \\ -y^2 & 0 & -1 \end{pmatrix}$$

with $G_2[2,1] = x^4y^6 + x^3y^7 + x^4y^5 + x^3y^6 + x^2y^6 + xy^7 + x^3y^4 + x^2y^5 + xy^6 + x^3y^3 + x^2y^4 + x^3y^2 + x^2y^3 + xy^4 + x^3y + xy^3 + y^4 + xy^2 + y^3 + xy$ and $G_2[2,3] = x^4y^4 + x^3y^5 + x^4y^3 + x^3y^4 + x^2y^4 + xy^5 + x^3y^2 + x^2y^3 + xy^4 + x^3y + x^2y^2 + x^2y + xy^2 + xy + y^2 + y$.

It follows that

$$U_1F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix} U_2^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix} G_2.$$

Since U_1, U_2 are unimodular matrices and $\det(F) = pq$, G_2 is a unimodular matrix. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix} U_2^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix} = \begin{pmatrix} -xy^2 - xy - y^2 - y & 1 & 0 \\ xy^5 + xy^4 + y^5 + y^4 + 1 & -y^3 & q \\ p & 0 & 0 \end{pmatrix}.$$

Now, we use Lemma 10 to reduce A. The first thing is to reduce A to satisfy $q \mid A[2,2]$. Let

$$U_3 = \begin{pmatrix} 1 & 0 & 0 \\ y^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then U_3 is a unimodular matrix. Hence,

$$U_3A = \begin{pmatrix} -xy^2 - xy - y^2 - y & 1 & 0 \\ 1 & 0 & q \\ p & 0 & 0 \end{pmatrix}.$$

Let

$$U_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -p & 1 \end{pmatrix},$$

then U_4 is a unimodular matrix and

$$U_4U_3A = \begin{pmatrix} -xy^2 - xy - y^2 - y & 1 & 0 \\ 1 & 0 & q \\ 0 & 0 & -pq \end{pmatrix} = S \cdot G_3.$$

It is obvious that G_3 is a unimodular matrix. Let $U = U_4U_3U_1$ and $V = G_2^{-1}G_3^{-1}$, then $S = UFV$.

5. Necessary and sufficient condition

In order to solve Problem 6, we first introduce some useful concepts and conclusions which will be used to prove our results.

Definition 16 *(Youla and Gnavi (1979)).* Let $F \in K[x,y]^{l\times m}$ be of full row rank. F is said to be a zero left prime (ZLP) matrix if the $l \times l$ minors of F generate $K[x,y]$.

Lemma 17 *(Quillen-Suslin theorem, Quillen (1976); Suslin (1976)). If $F \in K[x, y]^{l\times m}$ is a ZLP matrix, then there is a unimodular matrix $U \in K[x, y]^{m\times m}$ such that $FU = \begin{pmatrix} I_l & 0_{l\times(m-l)} \end{pmatrix}$, where I_l is the $l \times l$ identity matrix.*

The above Quillen-Suslin theorem has an important effect on the development of multidimensional systems. We now use it to prove the following case for the equivalence of bivariate polynomial matrices.

Lemma 18. *Let $U \in K[x, y]^{l\times l}$ be a unimodular matrix, $H_1 = \mathrm{diag}\{1, \ldots, 1, h_1\}$ and $H_2 = \mathrm{diag}\{1, \ldots, 1, h_2\}$, where $h_1, h_2 \in K[x, y]$ satisfy $h_2 \mid h_1$. Then $\mathrm{diag}\{1, \ldots, 1, h_1h_2\}$ is equivalent to H_1UH_2 if and only if all the $(l-1) \times (l-1)$ minors of H_1UH_2 generate $K[x, y]$.*

This lemma is a generalization of Lemma 2.3 in Li et al. (2019). Although the proof of the lemma is similar to that of Lemma 2.3, for the sake of the rigor of the argument and the ease of understanding we still give a detailed proof here.

Proof. *Necessity.* It is straightforward from Lemma 11 that all the $(l-1) \times (l-1)$ minors of H_1UH_2 generate $K[x, y]$.

Sufficiency. Suppose $h_1 = gh_2$, where $g \in K[x, y]$. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, where $U_{11} \in K[x, y]^{(l-1)\times(l-1)}$, $U_{12} \in K[x, y]^{(l-1)\times 1}$, $U_{21} \in K[x, y]^{1\times(l-1)}$ and $U_{22} \in K[x, y]^{1\times 1}$. Then

$$H_1UH_2 = \begin{pmatrix} U_{11} & h_2U_{12} \\ gh_2U_{21} & gh_2^2U_{22} \end{pmatrix}.$$

We next prove that $\begin{pmatrix} U_{11} & h_2U_{12} \end{pmatrix}$ is a ZLP matrix. Note that U is unimodular, $\begin{pmatrix} U_{11} & U_{12} \end{pmatrix}$ is a ZLP matrix. Let $c_1, c_2, \ldots, c_\gamma \in K[x, y]$ be all the $(l-1) \times (l-1)$ minors of $\begin{pmatrix} U_{11} & U_{12} \end{pmatrix}$, then the ideal generated by $c_1, c_2, \ldots, c_\gamma$ is $K[x, y]$. Without loss of generality, we assume that $\det(U_{11}) = c_1$. It is easy to see that $c_1, h_2c_2, \ldots, h_2c_\gamma$ are all the $(l-1) \times (l-1)$ minors of $\begin{pmatrix} U_{11} & h_2U_{12} \end{pmatrix}$. We assert that the ideal generated by $c_1, h_2c_2, \ldots, h_2c_\gamma$ is $K[x, y]$. If otherwise, $c_1, h_2c_2, \ldots, h_2c_\gamma$ have a common zero $(\alpha_1, \alpha_2) \in \widehat{K}^2$, where $\widehat{K}$ is the algebraic closure of K. This implies that c_1 and h_2 have the common zero (α_1, α_2). Since the polynomials in the last row of H_1UH_2 have the common factor h_2, all the $(l-1) \times (l-1)$ minors of H_1UH_2 have the common zero (α_1, α_2). This contradicts the fact that all the $(l-1) \times (l-1)$ minors of H_1UH_2 generate $K[x, y]$. Therefore, $\begin{pmatrix} U_{11} & h_2U_{12} \end{pmatrix}$ is a ZLP matrix. Based on the Quillen-Suslin theorem, there is a unimodular matrix $N \in K[x, y]^{l\times l}$ such that $\begin{pmatrix} U_{11} & h_2U_{12} \end{pmatrix} \cdot N = \begin{pmatrix} I_{l-1} & 0_{(l-1)\times 1} \end{pmatrix}$. It follows that

$$H_1UH_2 \cdot N = \begin{pmatrix} I_{l-1} & 0_{(l-1)\times 1} \\ R_1 & R_2 \end{pmatrix},$$

where $R_1 \in K[x, y]^{1\times(l-1)}$ and $R_2 \in K[x, y]^{1\times 1}$. Using elementary row transformations, we have

$$H_1UH_2 \cdot N \sim \begin{pmatrix} I_{l-1} & 0_{(l-1)\times 1} \\ 0_{1\times(l-1)} & R_2 \end{pmatrix}.$$

It is easy to check that $R_2 = \begin{pmatrix} h_1h_2 \end{pmatrix}$. Thus, $\mathrm{diag}\{1, \ldots, 1, h_1h_2\}$ is equivalent to H_1UH_2. □

Remark 19. When $h_1 \mid h_2$, the above lemma still holds.

Based on Lemma 18, we can solve the special case of Problem 6 where F is a square matrix.

Theorem 20. *Let $F \in K[x, y]^{l\times l}$ with $\det(F) = p^{r_1}q^{r_2}$, where r_1, r_2 are positive integers. F is equivalent to $\mathrm{diag}\{1, \ldots, 1, p^{r_1}q^{r_2}\}$ if and only if all the $(l-1) \times (l-1)$ minors of F generate $K[x, y]$.*

Proof. *Necessity.* If F is equivalent to $\mathrm{diag}\{1,\ldots,1,p^{r_1}q^{r_2}\}$, then there exist two unimodular matrices $U, V \in K[x,y]^{l\times l}$ such that $\mathrm{diag}\{1,\ldots,1,p^{r_1}q^{r_2}\} = UFV$. It follows from Lemma 11 that all the $(l-1)\times(l-1)$ minors of F generate $K[x,y]$.

Sufficiency. Without loss of generality, we assume that $1 \le r_1 \le r_2$. Using Lemma 8 repeatedly we have

$$F \sim S_1V_1S_2U_1S_1V_2S_2U_2\cdots S_1V_{r_1}S_2U_{r_1}S_2U_{r_1+1}S_2U_{r_1+2}S_2\cdots U_{r_2-1}S_2,$$

where $S_1 = \mathrm{diag}\{1,\ldots,1,p\}$, $S_2 = \mathrm{diag}\{1,\ldots,1,q\}$, and $V_i, U_j \in K[x,y]^{l\times l}$ are unimodular matrices. According to Lemma 10, we get

$$F \sim S\widetilde{U_1}S\widetilde{U_2}\cdots S\widetilde{U_{r_1}}S_2U_{r_1+1}S_2U_{r_1+2}S_2\cdots U_{r_2-1}S_2,$$

where $S = \mathrm{diag}\{1,\ldots,1,pq\}$ and $\widetilde{U_j} \in K[x,y]^{l\times l}$ is a unimodular matrix, $j = 1,\ldots,r_1$. If all the $(l-1)\times(l-1)$ minors of F generate $K[x,y]$, then by Lemmas 11 and 18 repeatedly we obtain

$$F \sim \mathrm{diag}\{1,\ldots,1,p^{r_1}q^{r_2}\}.$$

The proof is completed. □

Definition 21 *(Lin (1988); Sule (1994)).* Let $F \in K[x,y]^{l\times m}$ be of full row rank, and $a_1,\ldots,a_\beta$ denote all the $l\times l$ minors of F, where $\beta = \binom{m}{l}$. Extracting $d_l(F)$ from $a_1,\ldots,a_\beta$ yields $a_i = d_l(F)\cdot b_i$, where $i = 1,\ldots,\beta$. Then $b_1,\ldots,b_\beta$ are called the reduced minors of F.

Lemma 22 *(Wang and Feng (2004)). Let $F \in K[x,y]^{l\times m}$ be of full row rank. If the reduced minors of F generate $K[x,y]$, then there exist a matrix $G_1 \in K[x,y]^{l\times l}$ and a ZLP matrix $F_1 \in K[x,y]^{l\times m}$ such that $F = G_1F_1$ and $\det(G_1) = d_l(F)$.*

With the help of the above conclusions, we can now give the necessary and sufficient condition for the equivalence of F and S^*, where $F \in \mathcal{F}^*$.

Theorem 23. *Let $F \in \mathcal{F}^*$, then F is equivalent to S^* if and only if both the ideals generated by the reduced minors and $(l-1)\times(l-1)$ minors of F respectively are $K[x,y]$.*

Proof. *Necessity.* If F is equivalent to S^*, then there are two unimodular matrices $U \in K[x,y]^{l\times l}$ and $V \in K[x,y]^{m\times m}$ such that $S^* = UFV$. Based on Lemma 11, all the $(l-1)\times(l-1)$ minors of F generate $K[x,y]$. Since V is a unimodular matrix, we have $UF = S^*V^{-1}$. Without loss of generality, we assume that $V_l^{-1} \in K[x,y]^{l\times m}$ is the matrix composed of the first l rows of V^{-1} and

$$V_l^{-1} = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{l1} & \cdots & v_{lm} \end{pmatrix},$$

where $v_{ij} \in K[x,y]$, $1 \le i \le l$ and $1 \le j \le m$. Obviously, V_l^{-1} is a ZLP matrix. Let $e_1,\ldots,e_\eta \in K[x,y]$ be all the $l\times l$ minors of V_l^{-1}, then the ideal generated by $e_1,\ldots,e_\eta$ is $K[x,y]$, where $\eta = \binom{m}{l}$. It follows from $UF = S^*V^{-1}$ that

$$UF = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{(l-1)1} & \cdots & v_{(l-1)m} \\ p^{r_1}q^{r_2}v_{l1} & \cdots & p^{r_1}q^{r_2}v_{lm} \end{pmatrix}.$$

Since $\det(U)$ is a nonzero constant in K, we can derive that $p^{r_1}q^{r_2}e_1, \ldots, p^{r_1}q^{r_2}e_\eta$ are all the $l \times l$ minors of F up to multiplication by a nonzero constant. Combining the fact that $e_1, \ldots, e_\eta$ generate $K[x, y]$, we get that $e_1, \ldots, e_\eta$ are the reduced minors of F. Therefore, the reduced minors of F generate $K[x, y]$.

Sufficiency. If the reduced minors of F generate $K[x, y]$, then by Lemma 22 there exist a matrix $G_1 \in K[x, y]^{l\times l}$ and a ZLP matrix $F_1 \in K[x, y]^{l\times m}$ such that $F = G_1F_1$ and $\det(G_1) = d_l(F) = p^{r_1}q^{r_2}$. Using the Quillen-Suslin theorem, there exists a unimodular matrix $V \in K[x, y]^{m\times m}$ such that $F_1V = \begin{pmatrix} I_l & 0_{l\times(m-l)} \end{pmatrix}$. Then $FV = \begin{pmatrix} G_1 & 0_{l\times(m-l)} \end{pmatrix}$. If all the $(l-1) \times (l-1)$ minors of F generate $K[x, y]$, then all the $(l-1) \times (l-1)$ minors of G_1 generate $K[x, y]$ by Lemma 11. By Theorem 20, there exist two unimodular matrices $U_1, V_1 \in K[x, y]^{l\times l}$ such that $\mathrm{diag}\{1, \ldots, 1, p^{r_1}q^{r_2}\} = U_1G_1V_1$. Combining the above two equations we get

$$U_1FV \begin{pmatrix} V_1 & 0_{l\times(m-l)} \\ 0_{(m-l)\times l} & I_{m-l} \end{pmatrix} = U_1 \begin{pmatrix} G_1 & 0_{l\times(m-l)} \end{pmatrix} \begin{pmatrix} V_1 & 0_{l\times(m-l)} \\ 0_{(m-l)\times l} & I_{m-l} \end{pmatrix} = S^*.$$

Therefore, F is equivalent to S^*. $\square$

Theorem 23 gives a positive answer to Problem 6. Let $F \in \mathcal{F}^*$. If F satisfies the sufficient condition of Theorem 23, then we can use Algorithm 1 to reduce F to its Smith form S^*. Of course, some simple modifications need to be made to Algorithm 1, and they are omitted here.

6. Concluding remarks

In general, a bivariate polynomial matrix is not equivalent to its Smith form. Naturally, there comes the problem that under what conditions is a bivariate polynomial matrix equivalent to its Smith form. Moreover, if a matrix and its Smith form are equivalent, how to reduce it to its Smith form? In this paper, we are devoted to studying the equivalence and reduction of a class of bivariate polynomial matrices to their Smith forms.

Let $F \in \mathcal{F}$, we prove that F and its Smith form $S = \mathrm{diag}\{1, \ldots, 1, pq\}$ are equivalent. The main idea is to extend the Euclidean division of a univariate polynomial ring to $R_{pq}[y]$, where $R_{pq} = K[x]/(pq)$. Although R_{pq} is just a quotient ring, we can use the irreducibility of p and q to construct some special polynomials in $R_{pq}[y]$. Based on the Euclidean division in $R_{pq}[y]$, we can reduce the degrees of entries in F with respect to y. Since the degrees are strictly descending, the reduction process stops in finite steps. Then, we solved the considering problem (Problem 4) and designed an algorithm (Algorithm 1) for reducing the matrix to its Smith form. Finally, we extend the main theorem (Theorem 7) to the case of $F \in \mathcal{F}^*$, and obtain a necessary and sufficient condition (Theorem 23) for the equivalence of F and S^*.

With the resolution of the problem (Problem 6), the following problems will naturally be considered. Let $F \in K[x, y]^{l\times m}$ be of full row rank and $d_l(F) = p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$, where $s \geq 3$, $p_1, \ldots, p_s$ are distinct and irreducible polynomials in $K[x]$ and $r_1, \ldots, r_s$ are positive integers. What is the necessary and sufficient condition for the equivalence of F and its Smith form? If $p_1, \ldots, p_s \in K[x, y]$, what will be the conclusion? How to reduce F to its Smith form? We hope the results provided in the paper will motivate further research in the area of the equivalence and reduction of bivariate polynomial matrices.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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