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Article *in* Journal of Geometry · July 1995

DOI: 10.1007/BF01224042

2 authors:

Xiao-Shan Gao
Chinese Academy of Sciences

Dingkang Wang
Chinese Academy of Sciences

ON THE AUTOMATIC DERIVATION OF A SET OF GEOMETRIC FORMULAE

Xiao-Shan Gao and Ding-Kang Wang
Institute of Systems Science, Academia Sinica, Beijing 100800 China

Abstract

Let a, b, and c be the three sides of a triangle ABC, a_i, b_i, c_i and a_e, b_e, c_e be the lengths of the three internal and external bisectors of the three angles A, B, and C respectively. It is easy to express the bisectors as formulae of the sides. In this paper, we solve a problem proposed by H. Zassenhaus: for any three different bisectors in $\{a_i, b_i, c_i, a_e, b_e, c_e\}$, finding the relations between each side of the triangle and the three chosen bisectors. We also prove that given any general values for three different bisectors (internal or external) of a triangle, we can not draw the triangle using a ruler and a pair of compasses alone. The formulae mentioned above are derived automatically using a general method of mechanical formula derivation.

1 INTRODUCTION

Let a, b, and c be the three sides of a triangle ABC, a_i, b_i, and c_i be the lengths of the internal bisectors of the three angles A, B, and C respectively. It is well known that the bisectors can be expressed as formulae of a, b, c:

$$a_i^2 = \frac{cb(c+b-a)(c+b+a)}{(b+c)^2}$$
$$b_i^2 = \frac{ac(c-b+a)(c+b+a)}{(c+a)^2} \quad (1.1)$$
$$c_i^2 = \frac{ab(a+b-c)(a+b+c)}{(a+b)^2}$$

However the inverse problem, i.e., to express the three sides as formulae of the three bisectors, is not an easy problem.

The problem of determining a triangle from its three angular bisectors was proposed by Heymann [3]. In [7], a polynomial relation between the radius r of the incircle of the triangle and a_i, b_i, c_i which is of degree 10 in r, was derived. As a consequence, given three bisectors of a triangle, we cannot draw the triangle using a ruler and a pair of compasses alone [7,4]. But Heymann's problem is still not solved completely. Zassenhaus proposed to determine the three sides of the triangle as formulae of its three angular bisectors [5].

In this paper, with the aid of a computer we obtain a relation among a^2, a_i, b_i, and c_i which is a polynomial of degree 10 in a^2 and has 331 terms. We also consider more general problems.

Let a_e, b_e, and c_e be the lengths of the external bisectors of the three angles A, B, and C respectively. Then we have

$$\begin{aligned} a_e^2 &= \frac{cb(a+b-c)(c-b+a)}{(c-b)^2} \\ b_e^2 &= \frac{ac(a+b-c)(c+b-a)}{(c-a)^2} \\ c_e^2 &= \frac{ab(c-b+a)(c+b-a)}{(a-b)^2} \end{aligned} \quad (1.2)$$

The problem is *to express the three sides of the triangle as formulae of any three different (internal or external) bisectors.* To choose three different bisectors from six ones, there are $C_6^3 = 20$ cases, among which four cases, $\{a_i, b_i, c_i\}$, $\{a_e, b_e, c_i\}$, $\{a_i, b_e, c_e\}$, and $\{a_e, b_i, c_e\}$, were studied in [7]. We solve this problem completely and our results show that given general values for any three different bisectors (internal or external) of a triangle, we cannot draw the triangle using a ruler and a pair of compasses alone.

We obtain the formulae using a *general method of mechanical formula derivation* presented in [1,9,10]. The system used is implemented by Chou and the first author using Common Lisp on a SUN4/470 workstation [1]. This system has been used to derive more than 100 geometry formulae *totally automatically* [2]. The total computation time for the five formulae in this paper is about 32 hours[1] on a SUN4/470. The results reported here show the ability of solving challenge mathematical problems via symbolic computation.

This paper is organized as follows. In Section 2, we consider the classical case, i.e. expressing the sides of a triangle as formulae of the three internal angular bisectors of the triangle. In Section 3, we consider the general case. In the Appendix, we give the explicit expressions for the formulae.

2 EXPRESSING a, b, AND c AS FORMULAE OF a_i, b_i, AND c_i

First we use a simple example to illustrate the method of mechanical formula derivation. For details of the method, see [1,9,10].

EXAMPLE 2.1. Find the formula for the area of a triangle ABC in terms of its three sides.

Let a, b, and c be the three sides of the triangle, $B = (0,0)$, $C = (a, 0)$, and $A = (x_1, x_2)$. Then the geometry conditions can be expressed by the following set of polynomial equations:

$$\begin{aligned} &h_1 = x_2^2 + x_1^2 - 2ax_1 - b^2 + a^2 = 0, && b = AC \\ &h_2 = x_2^2 + x_1^2 - c^2 = 0, && c = AB \\ &h_3 = ax_2 - 2k = 0, && k = \text{the area of } ABC. \end{aligned}$$

Our task is to express the area k in terms of the parameters a, b, and c. To do so, let $h_4 = prem(h_2, h_3, x_2)$ be the *pseudo-remainder* of h_2 with respect to h_1 for variable x_2 (see [8]). Then $h_4 = a^2x_1^2 + 4k^2 - a^2c^2$. Similarly, let $h_5 = prem(h_1, h_2, x_2) = 2ax_1 - c^2 + b^2 - a^2$;

[1] We recently recompute the problem using Maple on a Spark-10. The computation costs less than three hours. The reason that our program based on Lisp is very slow is that we use a bad polynomial factorization program.

$h_6 = prem(h_4, h_5, x_1) = 16k^2 + c^4 + (-2b^2 - 2a^2)c^2 + b^4 - 2a^2b^2 + a^4$. Then $h_6 = 0$ is Herron's formula

$$k = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = (a+b+c)/2$. The complete method is more complicated, see [1].
For Zassenhaus's problem, rewriting (1.1) as polynomial equations, we have

$$\begin{aligned}
f_1 &= (c^2 + 2bc + b^2)a_i^2 - bc^3 - 2b^2c^2 + (-b^3 + a^2b)c = 0 \\
f_2 &= (c^2 + 2ac + a^2)b_i^2 - ac^3 - 2a^2c^2 + (ab^2 - a^3)c = 0 \\
f_3 &= (b^2 + 2ab + a^2)c_i^2 + abc^2 - ab^3 - 2a^2b^2 - a^3b = 0
\end{aligned}$$

Let $f_4 = prem(f_1, f_3, c)$ be the pseudo-remainder of f_1 with respect to f_3 for variable c [8]. Let $f_5 = prem(f_2, f_3, c)$. Then f_4 and f_5 are polynomials of degree one in c and have 18 terms.

$f_4 = -ab((2ab^4 + (2a^2 - c_i^2)b^3 + ((-2c_i^2 - 2a_i^2)a)b^2 - c_i^2a^2b)c + 2ab^5 + (4a^2 - 2c_i^2)b^4 + (2a^3 + (-4c_i^2 - 2a_i^2)a)b^3 + ((-2c_i^2 - 2a_i^2)a^2 + a_i^2c_i^2)b^2 + (-a_i^2a^3 + 2a_i^2c_i^2a)b + a_i^2c_i^2a^2)$
$f_5 = -ab(((2a^3 - c_i^2a)b^2 + (2a^4 + (-2c_i^2 - 2b_i^2)a^2)b - c_i^2a^3)c + (2a^3 - b_i^2a)b^3 + (4a^4 + (-2c_i^2 - 2b_i^2)a^2 + b_i^2c_i^2)b^2 + (2a^5 + (-4c_i^2 - 2b_i^2)a^3 + 2b_i^2c_i^2a)b - 2c_i^2a^4 + b_i^2c_i^2a^2)$
Let $f_6 = prem(f_5, f_4, c)$, $f_7 = prem(f_3, f_4, c)$. Then f_6 and f_7 are polynomials of degree 7 in b and of terms 42 and 37 respectively after removing some extraneous factors from them. Now we have eliminated variable c. The next step is to eliminate variable b. Let f_8 be the resultant of f_6 and f_7 with respect to variable b. Then f_8 is a polynomial of degree 10 in a^2 and of terms 331 after removing some extraneous factors from it. In the computation of f_8, there occurs a polynomial f_9 which is of degree one in b and has 2422 terms. Clearly,

$$ASC = \{f_8, f_9, f_4\}$$

is a triangular set with leading variables a, b, and c. All the extraneous factors occurring in the computation can be finally reduced to polynomials of a_i, b_i, and c_i alone. That is, the vanishing of the extraneous factors represents some special triangles. By Theorem 4.5 in [1], f_8 is the relation between a and a_i, b_i, c_i in the general case.
The above procedure has been carried out automatically on a computer. The details of the algorithm can be found in [1]. The total computation costs about 7 hours of CPU time on a SUN4/470.
THEOREM 2.2. Using the notations introduced in Section 1, we have

$$P_1(a, a_i, b_i, c_i) = 0, P_1(b, b_i, a_i, c_i) = 0, P_1(c, c_i, a_i, b_i) = 0$$

where $P_1(a, a_i, b_i, c_i)$ is an irreducible polynomial of degree ten in a^2 and of terms 331. The explicit expression for P_1 can be found in the Appendix.
As a corollary, we obtain the same result as [7].
COROLLARY 2.3. Given *general* values for the three internal angular bisectors of a triangle, we cannot draw the triangle using a ruler and a pair of compasses alone.
Proof. By a basic algebraic result, a necessary condition such that a quantity a can be constructed from a set of other quantities is that a and the given quantities satisfy an

algebraic relation whose degree in a is 2^m for a positive integer m. By Theorem 2.2, a, a_i, b_i, and c_i satisfy an irreducible polynomial relation P_1 of degree 20 in a. Then P_1 will still be irreducible and of degree 20 in a when replacing a_i, b_i, c_i in P_1 by a set of general values for them. Therefore, there exists no polynomial relations of degree 2^m in a between a and a set of general values for a_i, b_i, and c_i.

Remark. Corollary 2.3 is only true for general values of three bisectors. Because for some special cases, say $a_i = b_i = c_i$, we can clearly draw the triangle whose bisectors are a_i, b_i, and c_i. But such special values consist of a set which is of dimension lower than three, because to make $P_1 = 0$ become degree 2^m for an $m > 0$, the three bisectors have to satisfy some algebraic restrictions. It is an interesting problem to determine the conditions for a_i, b_i, and c_i under which the triangle can be drawn from a_i, b_i, c_i.

3 THE GENERAL CASE

To choose three different bisectors from the six bisectors $\{a_i, b_i, c_i, a_e, b_e, c_e\}$, there are $C_6^3 = 20$ possible cases. Actually, we only need to consider three cases.

LEMMA 3.1. Of the 20 cases mentioned above, we only need to consider three cases:

$$\{a_i, b_i, c_i\}, \{a_e, b_e, c_e\}, \{a_e, b_e, a_i\}. \tag{3.1.1}$$

That is, if we can obtain relations among any side of the triangle, say a, and each set of the three bisectors in (3.1.1) then we can obtain the relation among a and each of the 20 sets of bisectors.

Proof. By clear symmetry, the 20 cases may be reduced to six cases:

$$\{a_i, b_i, c_i\}, \{a_i, b_i, c_e\}, \{a_i, b_i, a_e\}, \{a_e, b_e, c_i\}, \{a_e, b_e, a_i\}, \{a_e, b_e, c_e\}.$$

(For example, two cases $\{a_i, c_i, b_e\}$ and $\{b_i, c_i, a_e\}$ can be reduced to $\{a_i, b_i, c_e\}$.) Note that when replacing c by $-c$ and a_i by a_e, the first formula in (1.1) will change to the first formula in (1.2). In other words, when changing c to $-c$, the relation among a,b, c, and a_i will change to the relation among a,b, c, and a_e. The second and the third formulae in (1.1) and (1.2) have similar properties. Therefore, when replacing c by $-c$, $\{a_e, b_e, c_i, \}$, $\{a_i, b_i, c_e\}$ and $\{a_i, b_i, a_e\}$ will reduce to $\{a_i, b_i, c_i\}$, $\{a_e, b_e, c_e\}$ and $\{a_e, b_e, a_i\}$ respectively. The Lemma is proved.

The first case in (3.1.1) has been solved in Section 2. Other cases have been solved similarly.

THEOREM 3.2. Using the notations introduced in Section 1, we have

$$P_2(a, a_e, b_e, c_e) = 0, P_2(b, b_e, a_e, c_e) = 0, P_2(c, c_e, a_e, b_e) = 0$$

where $P_2(a, a_e, b_e, c_e)$ is a polynomial of degree six in a^2 and of terms 139. We also have $P_2(a, a_e, b_e, c_e) = P_{21}P_{22}$ where P_{21} and P_{22} are irreducible polynomials of degree six in a. The explicit expression for P_2 can be found in the Appendix.

THEOREM 3.3. Using the notations introduced in Section 1, we have

$$P_3(a, a_e, b_e, a_i) = 0, P_4(b, a_e, b_e, a_i) = 0, P_5(c, a_e, b_e, a_i) = 0$$

where P_3, P_4, and P_5 are irreducible polynomials of degree ten in a^2, b^2, c^2 and of terms 330, 284, and 197 respectively. The explicit expressions for P_3, P_4, and P_5 can be found in the Appendix.

As a consequence, we have

COROLLARY 3.4. Using the notations introduced in Section 1, given *general* values for three different bisectors chosen from $\{a_i, b_i, c_i, a_e, b_e, c_e\}$, we cannot draw the triangle using a ruler and a pair of compasses alone.

ACKNOWLWGEMENT. We want to thank Dr. Dongming Wang for providing useful information on this problem. We also thank Prof. Shi He for valuable discussions.

APPENDIX. THE EXPLICIT EXPRESSIONS OF THE FIVE FORMULAE

The relation among a, a_i, b_i, and c_i.

$$P_1(a, a_i, b_i, c_i) = \sum_{i=0}^{10} D_{2i} a^{2i} = 0 \qquad \text{where}$$

$D_{20} = 256c_i^2b_i^2(a_i^2 - c_i^2)^2(a_i^2 - b_i^2)^2((c_i + b_i)^2a_i^2 - b_i^2c_i^2)((c_i - b_i)^2a_i^2 - b_i^2c_i^2)$;

$D_{18} = 256((c_i^4 - b_i^4)^2a_i^{14} + (-2c_i^{10} - 9b_i^2c_i^8 + 3b_i^4c_i^6 + 3b_i^6c_i^4 - 9b_i^8c_i^2 - 2b_i^{10})a_i^{12} + (c_i^{12} + 11b_i^2c_i^{10} + 40b_i^4c_i^8 - b_i^6c_i^6 + 40b_i^8c_i^4 + 11b_i^{10}c_i^2 + b_i^{12})a_i^{10} + (-6b_i^2c_i^{12} - 31b_i^4c_i^{10} - 66b_i^6c_i^8 - 66b_i^8c_i^6 - 31b_i^{10}c_i^4 - 6b_i^{12}c_i^2)a_i^8 + (14b_i^4c_i^{12} + 44b_i^6c_i^{10} + 72b_i^8c_i^8 + 44b_i^{10}c_i^6 + 14b_i^{12}c_i^4)a_i^6 + (-16b_i^6c_i^{12} - 31b_i^8c_i^{10} - 31b_i^{10}c_i^8 - 16b_i^{12}c_i^6)a_i^4 + (9b_i^8c_i^{12} + 11b_i^{10}c_i^{10} + 9b_i^{12}c_i^8)a_i^2 - 2(b_i^2 + c_i^2)b_i^{10}c_i^{10})$;

$D_{16} = -32((40c_i^{10} + 40b_i^2c_i^8 - 80b_i^4c_i^6 - 80b_i^6c_i^4 + 40b_i^8c_i^2 + 40b_i^{10})a_i^{14} + (-24c_i^{12} - 299b_i^2c_i^{10} - 320b_i^4c_i^8 + 166b_i^6c_i^6 - 320b_i^8c_i^4 - 299b_i^{10}c_i^2 - 24b_i^{12})a_i^{12} + (16c_i^{14} + 137b_i^2c_i^{12} + 795b_i^4c_i^{10} + 971b_i^6c_i^8 + 971b_i^8c_i^6 + 795b_i^{10}c_i^4 + 137b_i^{12}c_i^2 + 16b_i^{14})a_i^{10} + (-72b_i^2c_i^{14} - 351b_i^4c_i^{12} - 1097b_i^6c_i^{10} - 1155b_i^8c_i^8 - 1097b_i^{10}c_i^6 - 351b_i^{12}c_i^4 - 72b_i^{14}c_i^2)a_i^8 + (128b_i^4c_i^{14} + 390b_i^6c_i^{12} + 762b_i^8c_i^{10} + 762b_i^{10}c_i^8 + 390b_i^{12}c_i^6 + 128b_i^{14}c_i^4)a_i^6 + (-112b_i^6c_i^{14} - 178b_i^8c_i^{12} - 274b_i^{10}c_i^{10} - 178b_i^{12}c_i^8 - 112b_i^{14}c_i^6)a_i^4 + (48b_i^8c_i^{14} + 49b_i^{10}c_i^{12} + 49b_i^{12}c_i^{10} + 48b_i^{14}c_i^8)a_i^2 - 8b_i^{10}c_i^{14} - 23b_i^{12}c_i^{12} - 8b_i^{14}c_i^{10})$;

$D_{14} = 16((118c_i^{12} + 428b_i^2c_i^{10} - 118b_i^4c_i^8 - 856b_i^6c_i^6 - 118b_i^8c_i^4 + 428b_i^{10}c_i^2 + 118b_i^{12})a_i^{14} + (-2c_i^{14} - 807b_i^2c_i^{12} - 2001b_i^4c_i^{10} - 1022b_i^6c_i^8 - 1022b_i^8c_i^6 - 2001b_i^{10}c_i^4 - 807b_i^{12}c_i^2 - 2b_i^{14})a_i^{12} + (16c_i^{16} + 172b_i^2c_i^{14} + 1542b_i^4c_i^{12} + 3120b_i^6c_i^{10} + 3533b_i^8c_i^8 + 3120b_i^{10}c_i^6 + 1542b_i^{12}c_i^4 + 172b_i^{14}c_i^2 + 16b_i^{16})a_i^{10} + (-64b_i^2c_i^{16} - 398b_i^4c_i^{14} - 1772b_i^6c_i^{12} - 2347b_i^8c_i^{10} - 2347b_i^{10}c_i^8 - 1772b_i^{12}c_i^6 - 398b_i^{14}c_i^4 - 64b_i^{16}c_i^2)a_i^8 + (96b_i^4c_i^{16} + 280b_i^6c_i^{14} + 817b_i^8c_i^{12} + 1040b_i^{10}c_i^{10} + 817b_i^{12}c_i^8 + 280b_i^{14}c_i^6 + 96b_i^{16}c_i^4)a_i^6 + (-64b_i^6c_i^{16} - 62b_i^8c_i^{14} - 365b_i^{10}c_i^{12} - 365b_i^{12}c_i^{10} - 62b_i^{14}c_i^8 - 64b_i^{16}c_i^6)a_i^4 + (16b_i^8c_i^{16} + 28b_i^{10}c_i^{14} - 87b_i^{12}c_i^{12} + 28b_i^{14}c_i^{10} + 16b_i^{16}c_i^8)a_i^2 - 18b_i^{12}c_i^{14} - 18b_i^{14}c_i^{12})$;

$D_{12} = -((720c_i^{14} + 10608b_i^2c_i^{12} + 11216b_i^4c_i^{10} - 22544b_i^6c_i^8 - 22544b_i^8c_i^6 + 11216b_i^{10}c_i^4 + 10608b_i^{12}c_i^2 + 720b_i^{14})a_i^{14} + (288c_i^{16} - 3217b_i^2c_i^{14} - 35778b_i^4c_i^{12} - 43775b_i^6c_i^{10} - 37340b_i^8c_i^8 - 43775b_i^{10}c_i^6 - 35778b_i^{12}c_i^4 - 3217b_i^{14}c_i^2 + 288b_i^{16})a_i^{12} + (288b_i^2c_i^{16} + 8666b_i^4c_i^{14} + 18872b_i^6c_i^{12} + 34894b_i^8c_i^{10} + 34894b_i^{10}c_i^8 + 18872b_i^{12}c_i^6 + 8666b_i^{14}c_i^4 + 288b_i^{16}c_i^2)a_i^{10} + (-576b_i^4c_i^{16} - 5967b_i^6c_i^{14} - 20854b_i^8c_i^{12} - 19615b_i^{10}c_i^{10} - 20854b_i^{12}c_i^8 - 5967b_i^{14}c_i^6 - 576b_i^{16}c_i^4)a_i^8 + (-576b_i^6c_i^{16} + 2140b_i^8c_i^{14} - 3050b_i^{10}c_i^{12} - 3050b_i^{12}c_i^{10} + 2140b_i^{14}c_i^8 - 576b_i^{16}c_i^6)a_i^6 + (288b_i^8c_i^{16} - 143b_i^{10}c_i^{14} - 6480b_i^{12}c_i^{12} - 143b_i^{14}c_i^{10} + 288b_i^{16}c_i^8)a_i^4 + (288b_i^{10}c_i^{16} - 966b_i^{12}c_i^{14} - 966b_i^{14}c_i^{12} + 288b_i^{16}c_i^{10})a_i^2 - 81b_i^{14}c_i^{14})$;

$D_{10} = a_i^2((81c_i^{16}+3972b_i^2c_i^{14}+22468b_i^4c_i^{12}-3972b_i^6c_i^{10}-45098b_i^8c_i^8-3972b_i^{10}c_i^6+22468b_i^{12}c_i^4+3972b_i^{14}c_i^2+81b_i^{16})a_i^{12}+(1254b_i^2c_i^{16}-8012b_i^4c_i^{14}-34520b_i^6c_i^{12}-35778b_i^8c_i^{10}-35778b_i^{10}c_i^8-34520b_i^{12}c_i^6-8012b_i^{14}c_i^4+1254b_i^{16}c_i^2)a_i^{10}+(1215b_i^4c_i^{16}+1672b_i^6c_i^{14}-11013b_i^8c_i^{12}-4764b_i^{10}c_i^{10}-11013b_i^{12}c_i^8+1672b_i^{14}c_i^6+1215b_i^{16}c_i^4)a_i^8+(84b_i^6c_i^{16}-2152b_i^8c_i^{14}-10400b_i^{10}c_i^{12}-10400b_i^{12}c_i^{10}-2152b_i^{14}c_i^8+84b_i^{16}c_i^6)a_i^6+(1215b_i^8c_i^{16}-860b_i^{10}c_i^{14}-12772b_i^{12}c_i^{12}-860b_i^{14}c_i^{10}+1215b_i^{16}c_i^8)a_i^4+(1254b_i^{10}c_i^{16}-1868b_i^{12}c_i^{14}-1868b_i^{14}c_i^{12}+1254b_i^{16}c_i^{10})a_i^2+81b_i^{12}c_i^{16}-432b_i^{14}c_i^{14}+81b_i^{16}c_i^{12});$

$D_8 = -(c_i^2b_i^2a_i^4((432c_i^{14}+8336b_i^2c_i^{12}+19760b_i^4c_i^{10}-28528b_i^6c_i^8-28528b_i^8c_i^6+19760b_i^{10}c_i^4+8336b_i^{12}c_i^2+432b_i^{14})a_i^{10}+(2672b_i^2c_i^{14}-4049b_i^4c_i^{12}-11252b_i^6c_i^{10}-4966b_i^8c_i^8-11252b_i^{10}c_i^6-4049b_i^{12}c_i^4+2672b_i^{14}c_i^2)a_i^8+(3808b_i^4c_i^{14}-3748b_i^6c_i^{12}-17532b_i^8c_i^{10}-17532b_i^{10}c_i^8-3748b_i^{12}c_i^6+3808b_i^{14}c_i^4)a_i^6+(3808b_i^6c_i^{14}+442b_i^8c_i^{12}-9980b_i^{10}c_i^{10}+442b_i^{12}c_i^8+3808b_i^{14}c_i^6)a_i^4+(2672b_i^8c_i^{14}-2740b_i^{10}c_i^{12}-2740b_i^{12}c_i^{10}+2672b_i^{14}c_i^8)a_i^2+432b_i^{10}c_i^{14}-945b_i^{12}c_i^{12}+432b_i^{14}c_i^{10});$

$D_6 = 16c_i^4b_i^4a_i^6((54c_i^{12}+492b_i^2c_i^{10}+202b_i^4c_i^8-1496b_i^6c_i^6+202b_i^8c_i^4+492b_i^{10}c_i^2+54b_i^{12})a_i^8+(216b_i^2c_i^{12}+121b_i^4c_i^{10}+235b_i^6c_i^8+235b_i^8c_i^6+121b_i^{10}c_i^4+216b_i^{12}c_i^2)a_i^6+(324b_i^4c_i^{12}+3b_i^6c_i^{10}-368b_i^8c_i^8+3b_i^{10}c_i^6+324b_i^{12}c_i^4)a_i^4+(216b_i^6c_i^{12}-149b_i^8c_i^{10}-149b_i^{10}c_i^8+216b_i^{12}c_i^6)a_i^2+54b_i^8c_i^{12}-75b_i^{10}c_i^{10}+54b_i^{12}c_i^8);$

$D_4 = -32c_i^6b_i^6a_i^8((24c_i^{10}+88b_i^2c_i^8-112b_i^4c_i^6-112b_i^6c_i^4+88b_i^8c_i^2+24b_i^{10})a_i^6+(72b_i^2c_i^{10}+29b_i^4c_i^8+10b_i^6c_i^6+29b_i^8c_i^4+72b_i^{10}c_i^2)a_i^4+(72b_i^4c_i^{10}-46b_i^6c_i^8-46b_i^8c_i^6+72b_i^{10}c_i^4)a_i^2+24b_i^6c_i^{10}-35b_i^8c_i^8+24b_i^{10}c_i^6);$

$D_2 = 256c_i^8b_i^8a_i^{10}((c_i^4-b_i^4)^2a_i^4+(2b_i^2c_i^8-3b_i^4c_i^6-3b_i^6c_i^4+2b_i^8c_i^2)a_i^2+b_i^4c_i^8-3b_i^6c_i^6+b_i^8c_i^4);$

$D_0 = 256c_i^{14}b_i^{14}a_i^{12}.$

P_1 is of degree 20 in a and has 331 terms. The computation of P_1 costs about 7 hours of CPU time on a SUN4/470.

The relation among a, a_e, b_e, and c_e.

$$P_2(a,a_e,b_e,c_e) = P_{21}(a,a_e,b_e,c_e)P_{22}(a,a_e,b_e,c_e) \qquad \text{where}$$

$$\begin{aligned}
&P_{21}(a,a_e,b_e,c_e) = \\
&16a_e^4b_e^3c_e^3a^6 \\
&+32a_e^5b_e^2c_e^2(b_e^2-c_e^2)a^5 \\
&+4(5a_e^6b_e^5c_e-(10a_e^6b_e^3-a_e^4b_e^5)c_e^3+(5a_e^6b_e+a_e^4b_e^3-2a_e^2b_e^5)c_e^5)a^4 \\
&+4(a_e^7b_e^6-(3a_e^7b_e^4-a_e^5b_e^6)c_e^2+(3a_e^7b_e^2-2a_e^3b_e^6)c_e^4+(-a_e^4-a_e^2b_e^2+2b_e^4)a_e^3c_e^6)a^3 \\
&+(-a_e^6b_e^7c_e+(a_e^6b_e^5+3a_e^4b_e^7)c_e^3+(a_e^4+11a_e^2b_e^2-3b_e^4)a_e^2b_e^3c_e^5+(b_e^2-a_e^2)^3b_ec_e^7)a^2 \\
&+(-a_e^7b_e^8+(2a_e^7b_e^6+3a_e^5b_e^8)c_e^2-(a_e^5b_e^6+3a_e^3b_e^8)c_e^4 \\
&\quad +(-2a_e^6+a_e^4b_e^2+b_e^6)a_eb_e^2c_e^6+(a_e^2-b_e^2)^3a_ec_e^8)a \\
&-a_e^6b_e^7c_e^3+2(a_e^6b_e^5+a_e^4b_e^7)c_e^5-(a_e^2-b_e^2)^2a_e^2b_e^3c_e^7; \\
&P_{22}(a,a_e,b_e,c_e) = P_{21}(-a,a_e,b_e,c_e).
\end{aligned}$$

P_2 is of degree 12 in a and has 139 terms. The computation of P_2 costs about two hours of CPU time on a SUN4/470.

The relation among a, a_e, a_i, and b_e.

$$P_3(a,a_e,a_i,b_e) = \sum_{i=0}^{10} E_{2i}a^{2i} = 0 \qquad \text{where}$$

$E_{20} = 4096a_i^4a_e^4(a_i^2 + a_e^2)^2(b_e^2 - a_i^2)^2(b_e^2 - a_e^2)^2;$

$E_{18} = -2048a_i^2a_e^2(a_i^2 + a_e^2)((a_i^2 + a_e^2)^3b_e^{10} + (-2a_i^8 - 7a_e^2a_i^6 - 18a_e^4a_i^4 - 7a_e^6a_i^2 - 2a_e^8)b_e^8 + (a_i^{10} + 15a_e^2a_i^8 + 40a_e^4a_i^6 + 40a_e^6a_i^4 + 15a_e^8a_i^2 + a_e^{10})b_e^6 + (-3a_e^2a_i^{10} - 37a_e^4a_i^8 - 84a_e^6a_i^6 - 37a_e^8a_i^4 - 3a_e^{10}a_i^2)b_e^4 + (4a_e^4a_i^{10} + 44a_e^6a_i^8 + 44a_e^8a_i^6 + 4a_e^{10}a_i^4)b_e^2 - 16a_e^8a_i^8);$

$E_{16} = 256(((a_i^2+a_e^2)^6 + 8a_i^2a_e^2(a_i^4-a_e^4)^2)b_e^{12} + (-2a_i^{14} - 28a_e^2a_i^{12} - 80a_e^4a_i^{10} - 114a_e^6a_i^8 - 114a_e^8a_i^6 - 80a_e^{10}a_i^4 - 28a_e^{12}a_i^2 - 2a_e^{14})b_e^{10} + (a_i^{16} + 54a_e^2a_i^{14} + 281a_e^4a_i^{12} + 558a_e^6a_i^{10} + 660a_e^8a_i^8 + 558a_e^{10}a_i^6 + 281a_e^{12}a_i^4 + 54a_e^{14}a_i^2 + a_e^{16})b_e^8 + (-8a_e^2a_i^{16} - 196a_e^4a_i^{14} - 996a_e^6a_i^{12} - 1552a_e^8a_i^{10} - 1552a_e^{10}a_i^8 - 996a_e^{12}a_i^6 - 196a_e^{14}a_i^4 - 8a_e^{16}a_i^2)b_e^6 + (16a_e^4a_i^{16} + 328a_e^6a_i^{14} + 1872a_e^8a_i^{12} + 3120a_e^{10}a_i^{10} + 1872a_e^{12}a_i^8 + 328a_e^{14}a_i^6 + 16a_e^{16}a_i^4)b_e^4 + (-256a_e^8a_i^{14} - 1280a_e^{10}a_i^{12} - 1280a_e^{12}a_i^{10} - 256a_e^{14}a_i^8)b_e^2 + 256a_e^{12}a_i^{12});$

$E_{14} = -128b_e^2(a_i^2 + a_e^2)((4a_i^{10} + 4a_e^2a_i^8 - 8a_e^4a_i^6 - 8a_e^6a_i^4 + 4a_e^8a_i^2 + 4a_e^{10})b_e^{12} + (-6a_i^{12} - 9a_e^2a_i^{10} - 62a_e^4a_i^8 - 118a_e^6a_i^6 - 62a_e^8a_i^4 - 9a_e^{10}a_i^2 - 6a_e^{12})b_e^{10} + (10a_i^{14} + 125a_e^2a_i^{12} + 149a_e^4a_i^{10} + 404a_e^6a_i^8 + 404a_e^8a_i^6 + 149a_e^{10}a_i^4 + 125a_e^{12}a_i^2 + 10a_e^{14})b_e^8 + (-76a_e^2a_i^{14} - 739a_e^4a_i^{12} - 1272a_e^6a_i^{10} - 1218a_e^8a_i^8 - 1272a_e^{10}a_i^6 - 739a_e^{12}a_i^4 - 76a_e^{14}a_i^2)b_e^6 + (208a_e^4a_i^{14} + 2032a_e^6a_i^{12} + 3520a_e^8a_i^{10} + 3520a_e^{10}a_i^8 + 2032a_e^{12}a_i^6 + 208a_e^{14}a_i^4)b_e^4 + (-256a_e^6a_i^{14} - 2432a_e^8a_i^{12} - 5632a_e^{10}a_i^{10} - 2432a_e^{12}a_i^8 - 256a_e^{14}a_i^6)b_e^2 + 1024a_e^{10}a_i^{12} + 1024a_e^{12}a_i^{10});$

$E_{12} = 32b_e^4(a_i^2 + a_e^2)^2((8a_i^8 - 32a_e^2a_i^6 + 48a_e^4a_i^4 - 32a_e^6a_i^2 + 8a_e^8)b_e^{12} + (-a_i^{10} + 91a_e^2a_i^8 - 234a_e^4a_i^6 - 234a_e^6a_i^4 + 91a_e^8a_i^2 - a_e^{10})b_e^{10} + (59a_i^{12} - 157a_e^2a_i^{10} - 7a_e^4a_i^8 + 674a_e^6a_i^6 - 7a_e^8a_i^4 - 157a_e^{10}a_i^2 + 59a_e^{12})b_e^8 + (-332a_e^2a_i^{12} - 844a_e^4a_i^{10} - 2152a_e^6a_i^8 - 2152a_e^8a_i^6 - 844a_e^{10}a_i^4 - 332a_e^{12}a_i^2)b_e^6 + (1352a_e^4a_i^{12} + 4496a_e^6a_i^{10} + 3600a_e^8a_i^8 + 4496a_e^{10}a_i^6 + 1352a_e^{12}a_i^4)b_e^4$
$+(-3584a_e^6a_i^{12} - 13312a_e^8a_i^{10} - 13312a_e^{10}a_i^8 - 3584a_e^{12}a_i^6)b_e^2 + 2048a_e^8a_i^{12} + 9216a_e^{10}a_i^{10} + 2048a_e^{12}a_i^8);$

$E_{10} = -16b_e^6(a_i^2 + a_e^2)((18a_i^{12} + 36a_e^2a_i^{10} - 18a_e^4a_i^8 - 72a_e^6a_i^6 - 18a_e^8a_i^4 + 36a_e^{10}a_i^2 + 18a_e^{12})b_e^{10} + (45a_i^{14} - 261a_e^2a_i^{12} + 113a_e^4a_i^{10} - 89a_e^6a_i^8 - 89a_e^8a_i^6 + 113a_e^{10}a_i^4 - 261a_e^{12}a_i^2 + 45a_e^{14})b_e^8 + (206a_e^2a_i^{14} - 548a_e^4a_i^{12} - 2126a_e^6a_i^{10} - 2744a_e^8a_i^8 2126a_e^{10}a_i^6 - 548a_e^{12}a_i^4 + 206a_e^{14}a_i^2)b_e^6 + (800a_e^4a_i^{14} + 2592a_e^6a_i^{12}$
$+5824a_e^8a_i^{10} + 5824a_e^{10}a_i^8 + 2592a_e^{12}a_i^6 + 800a_e^{14}a_i^4)b_e^4 + (-4736a_e^6a_i^{14} - 22656a_e^8a_i^{12} - 36864a_e^{10}a_i^{10} - 22656a_e^{12}a_i^8 - 4736a_e^{14}a_i^6)b_e^2 + 10240a_e^8a_i^{14} + 34816a_e^{10}a_i^{12} + 34816a_e^{12}a_i^{10} + 10240a_e^{14}a_i^8);$

$E_8 = b_e^8(a_i^2 + a_e^2)^2((81a_i^{12} + 1254a_e^2a_i^{10} + 1215a_e^4a_i^8 + 84a_e^6a_i^6 + 1215a_e^8a_i^4 + 1254a_e^{10}a_i^2 + 81a_e^{12})b_e^8 + (3072a_e^2a_i^{12} - 8192a_e^4a_i^{10} + 9216a_e^6a_i^8 + 9216a_e^8a_i^6 - 8192a_e^{10}a_i^4 + 3072a_e^{12}a_i^2)b_e^6 + (13696a_e^4a_i^{12} + 3072a_e^6a_i^{10} + 27904a_e^8a_i^8 + 3072a_e^{10}a_i^6 + 13696a_e^{12}a_i^4)b_e^4 + (14336a_e^6a_i^{12} - 79872a_e^8a_i^{10}$
$-79872a_e^{10}a_i^8 + 14336a_e^{12}a_i^6)b_e^2 + 167936a_e^8a_i^{12} + 344064a_e^{10}a_i^{10} + 167936a_e^{12}a_i^8);$

$E_6 = -16b_e^{10}a_i^2a_e^2(a_i^2 + a_e^2)^3((27a_i^8 + 140a_e^2a_i^6 + 98a_e^4a_i^4 + 140a_e^6a_i^2 + 27a_e^8)b_e^6 + (416a_e^2a_i^8 + 224a_e^4a_i^6 + 224a_e^6a_i^4 + 416a_e^8a_i^2)b_e^4 + (1696a_e^4a_i^8 - 320a_e^6a_i^6 + 1696a_e^8a_i^4)b_e^2 + 5632a_e^6a_i^8 + 5632a_e^8a_i^6);$

$E_4 = 32b_e^{12}a_i^4a_e^4(a_i^2 + a_e^2)^3((27a_i^6 + 81a_e^2a_i^4 + 81a_e^4a_i^2 + 27a_e^6)b_e^4 + (256a_e^2a_i^6 + 256a_e^4a_i^4 + 256a_e^6a_i^2)b_e^2 + 832a_e^4a_i^6 + 832a_e^6a_i^4);$

$E_2 = -256b_e^{14}a_i^6a_e^6(a_i^2 + a_e^2)^4((3a_i^2 + 3a_e^2)b_e^2 + 16a_e^2a_i^2);$

$E_0 = 256b_e^{16}a_i^8a_e^8(a_i^2 + a_e^2)^4.$

P_3 is of degree 20 in a and has 330 terms. The computation of P_3 costs about 19 hours of CPU time on a SUN4/470.

The relation among b, a_e, a_i, and b_e.

$$P_4(b, b_e, a_i, a_e) = \sum_{i=0}^{10} F_{2i}a^{2i} = 0 \qquad \text{where}$$

$F_{20} = 4096(a_i^2 + a_e^2)^2(b_e^2 - a_i^2)^2(b_e^2 - a_e^2)^2;$

$F_{18} = -2048(a_i^2 + a_e^2)(11(a_i^2 + a_e^2)^2 b_e^8 + (-8a_i^6 - 56a_e^2a_i^4 - 56a_e^4a_i^2 - 8a_e^6)b_e^6 + (5a_i^8 + 37a_e^2a_i^6 + 72a_e^4a_i^4 + 37a_e^6a_i^2 + 5a_e^8)b_e^4 + (-11a_e^2a_i^8 - 41a_e^4a_i^6 - 41a_e^6a_i^4 - 11a_e^8a_i^2)b_e^2 + 8a_e^4a_i^8 + 16a_e^6a_i^6 + 8a_e^8a_i^4);$

$F_{16} = 256(a_i^2 + a_e^2)((159a_i^6 + 669a_e^2a_i^4 + 669a_e^4a_i^2 + 159a_e^6)b_e^8 + (-28a_i^8 - 784a_e^2a_i^6 - 1768a_e^4a_i^4 - 784a_e^6a_i^2 - 28a_e^8)b_e^6 + (33a_i^{10} + 403a_e^2a_i^8 + 1460a_e^4a_i^6 + 1460a_e^6a_i^4 + 403a_e^8a_i^2 + 33a_e^{10})b_e^4 + (-88a_e^2a_i^{10} - 576a_e^4a_i^8 - 976a_e^6a_i^6 - 576a_e^8a_i^4 - 88a_e^{10}a_i^2)b_e^2 + 96a_e^4a_i^{10} + 352a_e^6a_i^8 + 352a_e^8a_i^6 + 96a_e^{10}a_i^4);$

$F_{14} = -128(a_i^2 + a_e^2)((218a_i^8 + 2120a_e^2a_i^6 + 4060a_e^4a_i^4 + 2120a_e^6a_i^2 + 218a_e^8)b_e^8 + (28a_i^{10} - 916a_e^2a_i^8 - 5192a_e^4a_i^6 - 5192a_e^6a_i^4 - 916a_e^8a_i^2 + 28a_e^{10})b_e^6 + (20a_i^{12} + 455a_e^2a_i^{10} + 3408a_e^4a_i^8 + 6458a_e^6a_i^6 + 3408a_e^8a_i^4 + 455a_e^{10}a_i^2 + 20a_e^{12})b_e^4 + (-80a_e^2a_i^{12} - 896a_e^4a_i^{10} - 2544a_e^6a_i^8 - 2544a_e^8a_i^6 - 896a_e^{10}a_i^4 - 80a_e^{12}a_i^2)b_e^2 + 128a_e^4a_i^{12} + 896a_e^6a_i^{10} + 1536a_e^8a_i^8 + 896a_e^{10}a_i^6 + 128a_e^{12}a_i^4);$

$F_{12} = 16(a_i^2 + a_e^2)((559a_i^{10} + 10859a_e^2a_i^8 + 45654a_e^4a_i^6 + 45654a_e^6a_i^4 + 10859a_e^8a_i^2 + 559a_e^{10})b_e^8 + (142a_i^{12} - 556a_e^2a_i^{10} - 23982a_e^4a_i^8 - 54760a_e^6a_i^6 - 23982a_e^8a_i^4 - 556a_e^{10}a_i^2 + 142a_e^{12})b_e^6 + (16a_i^{14} + 672a_e^2a_i^{12} + 14144a_e^4a_i^{10} + 57104a_e^6a_i^8 + 57104a_e^8a_i^6 + 14144a_e^{10}a_i^4 + 672a_e^{12}a_i^2 + 16a_e^{14})b_e^4 + (-128a_e^2a_i^{14} - 2560a_e^4a_i^{12} - 12672a_e^6a_i^{10} - 20480a_e^8a_i^8 - 12672a_e^{10}a_i^6 - 2560a_e^{12}a_i^4 - 128a_e^{14}a_i^2)b_e^2 + 256a_e^4a_i^{14} + 4352a_e^6a_i^{12} + 13312a_e^8a_i^{10} + 13312a_e^{10}a_i^8 + 4352a_e^{12}a_i^6 + 256a_e^{14}a_i^4);$

$F_{10} = -8(a_i^2 + a_e^2)((171a_i^{12} + 6594a_e^2a_i^{10} + 54021a_e^4a_i^8 + 119772a_e^6a_i^6 + 54021a_e^8a_i^4 + 6594a_e^{10}a_i^2 + 171a_e^{12})b_e^8(36a_i^{14} + 828a_e^2a_i^{12} - 7884a_e^4a_i^{10} - 60308a_e^6a_i^8 - 60308a_e^8a_i^6 - 7884a_e^{10}a_i^4 + 828a_e^{12}a_i^2 + 36a_e^{14})b_e^6 + (-80a_e^2a_i^{14} + 4480a_e^4a_i^{12} + 58832a_e^6a_i^{10} + 122880a_e^8a_i^8 + 58832a_e^{10}a_i^6 + 4480a_e^{12}a_i^4 - 80a_e^{14}a_i^2)b_e^4 + (-768a_e^4a_i^{14} - 7168a_e^6a_i^{12} - 19712a_e^8a_i^{10} - 19712a_e^{10}a_i^8 - 7168a_e^{12}a_i^6 - 768a_e^{14}a_i^4)b_e^2 + 2048a_e^6a_i^{14} + 14336a_e^8a_i^{12} + 24576a_e^{10}a_i^{10} + 14336a_e^{12}a_i^8 + 2048a_e^{14}a_i^6);$

$F_8 = ((81a_i^{16} + 7752a_e^2a_i^{14} + 129884a_e^4a_i^{12} + 647672a_e^6a_i^{10} + 1116454a_e^8a_i^8 + 647672a_e^{10}a_i^6 + 129884a_e^{12}a_i^4 + 7752a_e^{14}a_i^2 + 81a_e^{16})b_e^8 + (1056a_e^2a_i^{16} + 5600a_e^4a_i^{14} - 85088a_e^6a_i^{12} - 341408a_e^8a_i^{10} - 341408a_e^{10}a_i^8 - 85088a_e^{12}a_i^6 + 5600a_e^{14}a_i^4 + 1056a_e^{16}a_i^2)b_e^6 + (-5120a_e^4a_i^{16} + 73216a_e^6a_i^{14} + 584704a_e^8a_i^{12} + 1012736a_e^{10}a_i^{10} + 584704a_e^{12}a_i^8 + 73216a_e^{14}a_i^6 - 5120a_e^{16}a_i^4)b_e^4 + (-6144a_e^6a_i^{16} - 38912a_e^8a_i^{14} - 86016a_e^{10}a_i^{12} - 86016a_e^{12}a_i^{10} - 38912a_e^{14}a_i^8 - 6144a_e^{16}a_i^6)b_e^2 + 24576a_e^8a_i^{16} + 114688a_e^{10}a_i^{14} + 180224a_e^{12}a_i^{12} + 114688a_e^{14}a_i^{10} + 24576a_e^{16}a_i^8);$

$F_6 = -16a_i^2a_e^2(a_i^2 + a_e^2)((27a_i^{12} + 1034a_e^2a_i^{10} + 8501a_e^4a_i^8 + 19084a_e^6a_i^6 + 8501a_e^8a_i^4 + 1034a_e^{10}a_i^2 + 27a_e^{12})b_e^8 + (80a_e^2a_i^{12} - 112a_e^4a_i^{10} - 2784a_e^6a_i^8 - 2784a_e^8a_i^6 - 112a_e^{10}a_i^4 + 80a_e^{12}a_i^2)b_e^6 + (-584a_e^4a_i^{12} + 5664a_e^6a_i^{10} + 16592a_e^8a_i^8 + 5664a_e^{10}a_i^6 - 584a_e^{12}a_i^4)b_e^4 + (-128a_e^6a_i^{12} - 384a_e^8a_i^{10} - 384a_e^{10}a_i^8 - 128a_e^{12}a_i^6)b_e^2 + 1024a_e^8a_i^{12} + 2048a_e^{10}a_i^{10} + 1024a_e^{12}a_i^8);$

$F_4 = 32a_i^4a_e^4(a_i^2 + a_e^2)^2((27a_i^8 + 492a_e^2a_i^6 + 1698a_e^4a_i^4 + 492a_e^6a_i^2 + 27a_e^8)b_e^8 + (16a_e^2a_i^8 - 80a_e^4a_i^6 - 80a_e^6a_i^4 + 16a_e^8a_i^2)b_e^6 + (-224a_e^4a_i^8 + 1856a_e^6a_i^6 - 224a_e^8a_i^4)b_e^4 + 128a_e^8a_i^8);$

$F_2 = -256a_i^6a_e^6b_e^4(a_i^2 + a_e^2)((3a_i^8 + 28a_e^2a_i^6 + 50a_e^4a_i^4 + 28a_e^6a_i^2 + 3a_e^8)b_e^4 - 8a_e^4a_i^8 + 48a_e^6a_i^6 - 8a_e^8a_i^4);$

$F_0 = 256a_i^8a_e^8b_e^8(a_i^2 + a_e^2)^4.$

P_4 is of degree 20 in a and has 284 terms. The computation of P_4 costs about half an hour of CPU time on a SUN4/470.

The relation among c, a_e, a_i, and b_e.

$$P_5(c, b_e, a_i, a_e) = \sum_{i=0}^{10} G_{2i} a^{2i} = 0 \qquad \text{where}$$

$G_{20} = 4096a_i^4a_e^4(a_i^2 + a_e^2);$

$G_{18} = -2048a_i^2a_e^2(a_i^2 + a_e^2)^2((a_i^2 + a_e^2)b_e^2 + 8a_e^2a_i^2);$

$G_{16} = 256(a_i^2 + a_e^2)((a_i^8 + 12a_e^2a_i^6 - 10a_e^4a_i^4 + 12a_e^6a_i^2 + a_e^8)b_e^4 + (8a_e^2a_i^8 + 88a_e^4a_i^6 + 88a_e^6a_i^4 + 8a_e^8a_i^2)b_e^2 + 96a_e^4a_i^8 + 256a_e^6a_i^6 + 96a_e^8a_i^4);$

$G_{14} = -128((4a_i^{10} + 4a_e^2a_i^8 - 8a_e^4a_i^6 - 8a_e^6a_i^4 + 4a_e^8a_i^2 + 4a_e^{10})b_e^6 + (4a_i^{12} + 15a_e^2a_i^{10} + 56a_e^4a_i^8 - 38a_e^6a_i^6 + 56a_e^8a_i^4 + 15a_e^{10}a_i^2 + 4a_e^{12})b_e^4 + (-16a_e^2a_i^{12} + 192a_e^4a_i^{10} + 784a_e^6a_i^8 + 784a_e^8a_i^6 + 192a_e^{10}a_i^4 - 16a_e^{12}a_i^2)b_e^2 + 128a_e^4a_i^{12} + 896a_e^6a_i^{10} + 1536a_e^8a_i^8 + 896a_e^{10}a_i^6 + 128a_e^{12}a_i^4);$

$G_{12} = 32(a_i^2 + a_e^2)((8a_i^8 - 32a_e^2a_i^6 + 48a_e^4a_i^4 - 32a_e^6a_i^2 + 8a_e^8)b_e^8 + (23a_i^{10} + 19a_e^2a_i^8 - 186a_e^4a_i^6 - 186a_e^6a_i^4 + 19a_e^8a_i^2 + 23a_e^{10})b_e^6 + (8a_i^{12} - 40a_e^2a_i^{10} - 200a_e^4a_i^8 - 304a_e^6a_i^6 - 200a_e^8a_i^4 - 40a_e^{10}a_i^2 + 8a_e^{12})b_e^4 + (-64a_e^2a_i^{12} + 64a_e^4a_i^{10} + 2560a_e^6a_i^8 + 2560a_e^8a_i^6 + 64a_e^{10}a_i^4 - 64a_e^{12}a_i^2)b_e^2 + 128a_e^4a_i^{12} + 2048a_e^6a_i^{10} + 4608a_e^8a_i^8 + 2048a_e^{10}a_i^6 + 128a_e^{12}a_i^4);$

$G_{10} = -32(a_i^2 + a_e^2)((9a_i^{10} + 9a_e^2a_i^8 - 18a_e^4a_i^6 - 18a_e^6a_i^4 + 9a_e^8a_i^2 + 9a_e^{10})b_e^8 + (9a_i^{12} + 54a_e^2a_i^{10} - 249a_e^4a_i^8 - 588a_e^6a_i^6 - 249a_e^8a_i^4 + 54a_e^{10}a_i^2 + 9a_e^{12})b_e^6 + (-20a_e^2a_i^{12} - 460a_e^4a_i^{10} - 960a_e^6a_i^8 - 960a_e^8a_i^6 - 460a_e^{10}a_i^4 - 20a_e^{12}a_i^2)b_e^4 + (-192a_e^4a_i^{12} + 704a_e^6a_i^{10} + 2816a_e^8a_i^8 + 704a_e^{10}a_i^6 - 192a_e^{12}a_i^4)b_e^2$
$+512a_e^6a_i^{12} + 3072a_e^8a_i^{10} + 3072a_e^{10}a_i^8 + 512a_e^{12}a_i^6);$

$G_8 = (a_i^2 + a_e^2)((81a_i^{12} + 1254a_e^2a_i^{10} + 1215a_e^4a_i^8 + 84a_e^6a_i^6 + 1215a_e^8a_i^4 + 1254a_e^{10}a_i^2 + 81a_e^{12})b_e^8 + (1056a_e^2a_i^{12} - 1632a_e^4a_i^{10} - 18368a_e^6a_i^8 - 18368a_e^8a_i^6 - 1632a_e^{10}a_i^4 + 1056a_e^{12}a_i^2)b_e^6 + (-5120a_e^4a_i^{12} - 32768a_e^6a_i^{10} - 47104a_e^8a_i^8 - 32768a_e^{10}a_i^6 - 5120a_e^{12}a_i^4)b_e^4 + (-6144a_e^6a_i^{12} + 30720a_e^8a_i^{10} + 30720a_e^{10}a_i^8$
$-6144a_e^{12}a_i^6)b_e^2 + 24576a_e^8a_i^{12} + 65536a_e^{10}a_i^{10} + 24576a_e^{12}a_i^8);$

$G_6 = 16a_i^2a_e^2(a_i^2 + a_e^2)((27a_i^{10} + 167a_e^2a_i^8 + 238a_e^4a_i^6 + 238a_e^6a_i^4 + 167a_e^8a_i^2 + 27a_e^{10})b_e^8 + (80a_e^2a_i^{10} - 320a_e^4a_i^8 - 800a_e^6a_i^6 - 320a_e^8a_i^4 + 80a_e^{10}a_i^2)b_e^6 + (-584a_e^4a_i^{10} - 2008a_e^6a_i^8 - 2008a_e^8a_i^6 - 584a_e^{10}a_i^4)b_e^4 + (-128a_e^6a_i^{10} + 768a_e^8a_i^8 - 128a_e^{10}a_i^6)b_e^2 + 1024a_e^8a_i^{10} + 1024a_e^{10}a_i^8);$

$G_4 = 32a_i^4a_e^4(a_i^2 + a_e^2)((27a_i^8 + 108a_e^2a_i^6 + 162a_e^4a_i^4 + 108a_e^6a_i^2 + 27a_e^8)b_e^8 + (16a_e^2a_i^8 - 80a_e^4a_i^6 - 80a_e^6a_i^4 + 16a_e^8a_i^2)b_e^6 + (-224a_e^4a_i^8 - 448a_e^6a_i^6 - 224a_e^8a_i^4)b_e^4 + 128a_e^8a_i^8);$

$G_2 = -256a_i^6a_e^6b_e^4(a_i^2 + a_e^2)^2((3a_i^4 + 6a_e^2a_i^2 + 3a_e^4)b_e^4 - 8a_e^4a_i^4);$

$G_0 = 256a_i^8a_e^8b_e^8(a_i^2 + a_e^2)^3.$

P_5 is of degree 20 in a and has 197 terms. The computation of P_4 costs about half an hour of CPU time on a SUN4/470.

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