Introduction

What is differential algebra? Roughly speaking, it is the subject studying algebraic differential equations from the algebraic standpoint.

Examples of algebraic differential equations:

(1) $\frac{\mathrm{d}y(x)}{\mathrm{d}x} - \frac{1}{2x}y(x) = 0$ (linear ordinary differential equation).

(2) $(\frac{\mathrm{d}y}{\mathrm{d}t})^2 - 4y = 0$ (nonlinear ordinary differential equation).

(3) Heat Equation: $\frac{\partial u}{\partial t} = \gamma(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$ (linear partial differential equation).

(4) KDV Equation: $\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0$ (nonlinear partial differential equation).

In differential algebra, we are not interested in "solving". In fact, it is very hard to solve differential equations in closed form solutions and in general impossible. Our perspective is rather to study the solutions and their properties from an abstract, purely algebraic point of view. This subject enjoys many analogies with commutative algebra and algebraic geometry. Since polynomial equations are algebraic differential equations of order 0, differential algebra could be regarded as a generalization of classical algebraic geometry.

The main focus of this course is to study the set of solutions of a general system of differential polynomials in finitely many differential variables over a differential field. These solution sets are called *differential varieties*.

We address questions like:

(1) Can we replace an infinite system of algebraic differential equations by a finite system without changing the solutions? (**Ritt-Raudenbush basis theorem**)

(2) Give a criterion to test whether a system of differential equations have a solution or not? (**Differential Hilbert's Nullstellensatz**)

(3) Develop constructive methods in the elimination theory of algebraic differential equations; Decompose a system of algebraic differential equations into finitely many "irreducible" system. (**Wu-Ritt characteristic set methods**)

(4) Provide coherence conditions or integrability conditions for algebraic partial differential equations. (**Rosenfeld Lemma**)

References:

(1) An Introduction to Differential Algebra by I. Kaplansky, 1957.

(2) Differential Algebra by J. F. Ritt, 1950.

(3) Differential Algebra and Algebraic Groups by E. R. Kolchin, 1973.

Lecture notes will be uploaded to the website of the course every Tuesday after class:
`http://mmrc.iss.ac.cn/~weili/DA2022.html`

Course Contents:

- **Basic notions of differential algebra**
  differential rings, differential ideals, decomposition of radical differential ideals
- **Differential polynomial rings and the basis theorem**

  The ring of differential polynomials,

  Theory of Differential characteristic sets,

  The Ritt-Raudenbush basis theorem
- **The Differential Algebra-Geometry Dictionary**

  Ideal-Variety correspondence in differential algebra;

  Differential Nullstellensatz;

  Irreducible decomposition of differential varieties
- **Extensions of differential fields**

  Differential primitive theorem;

  Differential transcendence bases;

  Applications to differential varieties
- **Algorithms and constructive methods for algebraic differential equations**

  Well-ordering principle for differential polynomials;

  Differential Decomposition Algorithms
- **Systems of algebraic partial differential equations**

  Rosenfeld Lemma

Chapter 1

Basic Notions of Differential Algebra

In this chapter, we introduce the very basic definitions and constructions of differential algebra and establish some first theorems concerning differential ideals.

1.1 Differential rings

All rings in this course are assumed to be commutative rings with unity 1.

Definition 1.1.1. *A* **derivation** *on a ring R is a map $\delta : R \to R$ s.t. for $\forall\, a, b \in R$,*

1) $\delta(a+b) = \delta(a) + \delta(b)$*;*

2) (Leibniz rule) $\delta(ab) = \delta(a)b + a\delta(b)$.

Example 1. Let $R = \mathbb{Z}$. What are the possible derivations on R?
Note that (1) $\delta(0) = \delta(0+0) = 2\delta(0) \Rightarrow \delta(0) = 0$;
(2) $\delta(1) = \delta(1^2) = 2\delta(1) \Rightarrow \delta(1) = 0 \Rightarrow \forall n \in \mathbb{Z}, \delta(n) = 0$;
(3) $\delta(0) = \delta(1 + (-1)) = \delta(1) + \delta(-1) \Rightarrow \delta(-1) = 0 \Rightarrow \forall n \in \mathbb{Z}_{<0}, \delta(-n) = n\delta(-1) = 0$.
Thus, the only possible derivation on $\mathbb{Z}$ is the zero derivation (i.e., $\forall n \in \mathbb{Z}, \delta(n) = 0$.)

Example 2. Let $R = \mathbb{Q}$. What are the possible derivations on R?
$\forall b \in \mathbb{Z}\backslash\{0\}$, $\delta(1) = \delta(b \cdot \frac{1}{b}) = b\delta(\frac{1}{b}) + \frac{1}{b}\delta(b) = 0 \Rightarrow \delta(\frac{1}{b}) = -\frac{\delta(b)}{b^2} = 0$.
For each $\frac{a}{b} \in \mathbb{Q}$, $\delta(\frac{a}{b}) = \delta(a \cdot \frac{1}{b}) = 0$, i.e., the only possible derivation on $\mathbb{Q}$ is the zero derivation.

More generally, we have the following result:

Lemma 1.1.2. *Let R be an integral domain and δ a derivation on R. Then δ has a unique extension to the quotient field* $\mathrm{Frac}(R)$.

Proof. To show Existence. Define for each $\frac{a}{b} \in \mathrm{Frac}(R)$, $\delta(\frac{a}{b}) = \frac{\delta(a)b - a\delta(b)}{b^2}$ and show $\delta : \mathrm{Frac}(R) \to \mathrm{Frac}(R)$ is ① well-defined and ② it is a derivation.

① Suppose $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$ and $\delta(a)d + a\delta(d) = \delta(b)c + b\delta(c)$. Show $\delta(\frac{a}{b}) = \frac{\delta(a)b - a\delta(b)}{b^2} = \delta(\frac{c}{d}) = \frac{\delta(c)d - c\delta(d)}{d^2}$.

② Show $\delta(\frac{a}{b} + \frac{c}{d}) = \delta(\frac{a}{b}) + \delta(\frac{c}{d})$ and $\delta(\frac{a}{b} \cdot \frac{c}{d}) = \delta(\frac{a}{b})\frac{c}{d} + \frac{a}{b}\delta(\frac{c}{d})$.
Uniqueness. $\forall \frac{a}{b} \in \mathrm{Frac}(R), \delta(a) = \delta(\frac{a}{b} \cdot b) = \delta(\frac{a}{b})b + \frac{a}{b}\delta(b) \Rightarrow \delta(\frac{a}{b}) = \frac{b\delta(a) - a\delta(b)}{b^2}$.

□

Suppose δ is a derivation on R. For $a \in R$, the element $\delta(a)$ is called the *derivative* of a, and we denote $\delta(a), \delta^2(a), \dots, \delta^n(a)$ for the successive derivatives.

Exercse 1. By induction on n, we can prove the following:

1) For all $a \in R$ and $n \geq 1$, $\delta(a^n) = na^{n-1}\delta(a)$.

2) For all $a, b \in R$ and $n \geq 1$, $\delta^n(ab) = \sum_{i=0}^{n} \binom{n}{i} \delta^{n-i}(a)\delta^i(b)$.

Definition 1.1.3. *A* **differential ring** *is a commutative ring R with unity 1 together with a finite set $\Delta = \{\delta_1, \dots, \delta_m\}$ of mutually commuting derivation operators$\big($i.e., $\forall a \in R, \delta_i(\delta_j(a)) = \delta_j(\delta_i(a))\big)$, denoted by (R, Δ).*

- *If card$(\Delta) = 1$(i.e., $\Delta = \{\delta\}$), (R, δ) is called an* **ordinary differential ring**.
- *If card$(\Delta) > 1$, (R, Δ) is called a* **partial differential ring**.

If R is also a field, (R, Δ) is called a differential field.

Example 3.

1) Let R be a commutative ring with unity. Define $\delta : R \to R$ by $\delta(a) = 0$ for $\forall a \in R$. Then (R, δ) is a differential ring. The rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ have no other derivation operators than the zero derivation.

2) Let $R = \mathbb{Q}[x], \delta(x) = 1$. For any $a_0, a_1, \dots, a_n \in \mathbb{Q}$, $\delta(a_0 + a_1x + \cdots + a_nx^n) = \delta(a_0) + \delta(a_1x) + \cdots + \delta(a_nx^n) = a_1 + 2a_2x + \cdots + na_nx^{n-1}$. (R, δ) is a differential ring.

3) Let F be a field of meromorphic functions of n complex variables $x_1, \dots, x_n$ in a region of $\mathbb{C}^n$. Then $(F, \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\})$ is a differential field.

4) If (S, δ) is an ordinary differential ring and $R = S[x]$,then for any arbitrary $f \in R$, $\delta(x) = f$ turns R into a differential ring.
But this notion of arbitrarily defining derivation doesn't work for the partial case.
Non-Exampe: $R = \mathbb{Q}[x]$. *Let $\delta_1(x) = 1, \delta_2(x) = x$. Since $\delta_1(\delta_2(x)) = 1 \neq \delta_2(\delta_1(x)) = 0$, $(R, \{\delta_1, \delta_2\})$ is not a differential ring.*

For $\Delta = \{\delta_1, \dots, \delta_m\}$, we denote $\Theta = \{\delta_1^{i_1} \cdots \delta_m^{i_m} \,|\, i_j \in \mathbb{N}\}$.

Definition 1.1.4. *Let (R, Δ) be a differential ring and $R_0 \subseteq R$ be a subring of R. If $\delta_i(R_0) \subseteq R_0$ for each $\delta_i \in \Delta$, then $(R_0, \Delta|_{R_0})$ is a differential ring. In this case, we say R_0 a differential subring of R and say R a differential overring of R_0.*

If $S \subseteq R$, there exists a smallest differential subring of R containing all the elements of R_0 and S, denoted by $R_0\{S\}$, and S is said to be a set of generators of the differential ring $R_0\{S\}$ over R_0. $R_0\{S\}$ coincides, as a ring, with the ring $R_0[(\theta(s))_{s \in S, \theta \in \Theta}]$. A differential overring of a differential ring R_0 is said to be finitely generated over R_0 if it has a finite set of generators over R_0.

If both R_0 and R are differential fields, R_0 is said to be a differential subfield of R and R is said to be a differential field extension of R_0.

Let L be a differential field extension of K and $S \subseteq L$. Denote by $K[S]$, $K\{S\}$, $K(S)$ and $K\langle S\rangle$ the smallest ring, the smallest differential ring, the smallest field, the smallest differential field containing K and S. Let $\Theta(S) = \{\theta(s) \,|\, s \in S, \theta \in \Theta\}$. Then $K\{S\} = K[\Theta(S)]$, $K\langle S\rangle = K(\Theta(S))$. L is said to be finitely generated if $\exists$ a finite subset $\{a_1, \dots, a_n\} \subseteq L$ s.t. $L = K\langle a_1, \dots, a_n\rangle$.

1.2 Differential ideals

Definition 1.2.1. *Let $(R, \Delta(= \{\delta_1, \ldots, \delta_m\}))$ be a differential ring. An ideal $I \lhd R$ is a* **differential ideal** *if $\delta_i(I) \subseteq I$ holds for each i.*

Example: Both $I = (0)$ and $I = R$ are differential ideals of R.

Proposition 1.2.2. *Let $I = (f_1, \ldots, f_s) \subseteq (R, \Delta)$ be the ideal in (R, Δ) generated by $f_1, \ldots, f_s$. Then I is a differential ideal $\Longleftrightarrow \forall i = 1, \ldots, m, j = 1, \ldots, s$, $\delta_i(f_j) \in I$.*

Proof. "$\Rightarrow$" Trivial by definition.
"$\Leftarrow$" For each $f \in I$, $\exists\, g_1, \ldots, g_s \in R$ s.t. $f = g_1 f_1 + \cdots + g_s f_s$. So $\delta_i(f) = \sum_{j=1}^{s} \delta(g_j) f_j + \sum_{j=1}^{s} \delta(f_j) g_j \in I$, for $\delta(f_j) \in I$ by hypothesis. Thus, $\delta_i(I) \subseteq I$ for each i. □

Notation: Let $S \subseteq (R, \Delta)$. We use $[S]$ to denote the smallest differential ideal of R generated by S. Clearly, $[S] = (\Theta(S)) = (\theta(s) : s \in S, \theta \in \Theta)$.

Example: Consider $(\mathbb{Q}[x], \delta)$ with $\delta(x) = 1$. Then $[0]$ and $\mathbb{Q}[x]$ are the only differential ideals in $\mathbb{Q}[x]$. (Indeed, let $[0] \neq I \lhd \mathbb{Q}[x]$ be a differential ideal. Then $\exists\, 0 \neq f \in \mathbb{Q}[x]$ s.t. $I = (f)$. Since I is a differential ideal, $\delta(f) = \frac{\partial f}{\partial x} \in (f)$. If $f \notin \mathbb{Q}, f \nmid \frac{\partial f}{\partial x}$. So, $f \in \mathbb{Q} \backslash \{0\}$ and $I = \mathbb{Q}[x]$ follows.)

An ideal $I \lhd (R, \Delta)$ is called a **radical** (resp. **prime**) **differential ideal** if

1) $\delta_i(I) \subseteq I$ for each $\delta_i \in \Delta$, and

2) I is a radical ideal (resp. prime ideal).

Notation: Given $I \lhd R$, let $\sqrt{I} = \{f \in R \mid \exists\, n \in \mathbb{N} \text{ s.t. } f^n \in I\}$.

Given $S \subseteq (R, \Delta)$, let $\{S\}$[1] be the smallest radical differential ideal containing S, and say $\{S\}$ is a radical differential ideal generated by S.

Now we turn to the construction of radical differnetial ideals. Normally, one may intuitively start with S, consider $[S]$ and then take its radical $\sqrt{[S]}$. However, this might not be sufficient.

Example: Let (R, δ) with $R = \mathbb{Z}_2[x, y], \delta(x) = y$ and $\delta(y) = 0$. Consider $I = [x^2]$. Since $\delta(x^2) = 0, I = (x^2)$. So $\sqrt{I} = (x)$. But $\sqrt{I}$ is not a differential ideal for $\delta(x) = y \notin \sqrt{I}$. So $\{x^2\} \neq \sqrt{[x^2]}$.

Exercise: Construct an example of an ideal $I \subseteq (R, \delta)$ s.t. $[\sqrt{I}]$ is not radical.
(Let $R = \mathbb{C}[x, y]$, $\delta(x) = y$ and $\delta(y) = 0$. Let $I = (xy)$. $\sqrt{I} = (xy), [\sqrt{I}] = [xy] = (xy, y^2)$. $J := [\sqrt{I}]$ is not radical for $y^2 \in J$ but $y \notin J$.)

Example: A maximal differential ideal (i.e., a maximal element in the set of all proper differential ideals) is not necessarily prime. For example, let $R = \mathbb{Z}_2[x]$ with $\delta(x) = 1$. Let $J = [x^2] = (x^2)$. Clearly, J is not prime but J is a maximal differential ideal. Indeed, if $\exists\, I \lhd (R, \delta)$ with $J \subsetneqq I \subseteq R$, then $\exists\, x + b \in I$. But $\delta(x + b) = 1 \in I$, so $I = R$.

However, if the ring R contains the rational field $\mathbb{Q}$, then the radical of a differential ideal is a radical differential ideal (i.e., $\{S\} = \sqrt{[S]}$).

Theorem 1.2.3. *Let (R, δ) be a differential ring, $\mathbb{Q} \subseteq R$ and let $I \subseteq (R, \delta)$ be a differential ideal. Then, $\sqrt{I}$ is a radical differential ideal.*

The proof will be given next class (March 8).

[1] It will be clear in which context $\{\cdot\}$ denotes a radical differential ideal or a set.