Recall $V \subseteq \bar{K}^n$: an irr δ-variety over K ($\bar{K}$: a δ-closed field $\supseteq K$)

- Diff coordinate ring of V: $K\{V\} \triangleq K\{y_1,\dots,y_n\}/\mathbb{I}(V) \left(= K\{\bar{y}_1,\dots,\bar{y}_n\}\right)$

Each elt of $K\{V\}$ is also called a diff poly function on V.

$\left(\bar{f} \in K\{V\} \quad \bar{f}: \underset{a}{V} \longrightarrow \underset{f(a)}{\bar{K}} \ \&\ \bar{y}_i: \text{a diff coordinate function}\right)$

- The field of diff rational functions on V: $K\langle V\rangle = K\langle \bar{y}_1,\dots,\bar{y}_n\rangle$

$(\bar{y}_1,\dots,\bar{y}_n) \in K\langle V\rangle^n$ is a generic point of V.

- The diff dimension of V:

$\delta\text{-}\dim(V) = \delta\text{-tr.deg}\, K\langle \bar{y}_1,\dots,\bar{y}_n\rangle/K \quad \left(= \#(\text{parametric set of } \mathbb{I}(V))\right)$

For $W = \bigcup_i V_i$ with V_i irr component of W,

$$\delta\text{-}\dim(W) = \max_i \delta\text{-}\dim(V_i).$$

- Diff dimension poly of V:

$$\omega_V(t) = \text{tr.deg}\, K\langle \bar{y}_i^{(k)} : i=1,\dots,n;\ k \le t\rangle/K \quad (t \gg 0)$$
$$= \delta\text{-}\dim(V)\cdot(t+1) + \operatorname{ord}(V)$$

The Computation of $\omega_V(t)$:

First compute a char set $\mathcal{A}$ of $\mathbb{I}(V)$ under some orderly ranking,

then $\omega_V(t) = (n - \operatorname{card}(\mathcal{A}))(t+1) + \operatorname{ord}(\mathcal{A})$,

that is, $\delta\text{-}\dim(V) = n - \operatorname{card}(\mathcal{A})$

$\operatorname{ord}(V) = \operatorname{ord}(\mathcal{A}) = \sum_{A \in \mathcal{A}} \operatorname{ord}(A)$.

- Given two irr δ-varieties $W \subsetneq V$, it may happen that $\delta\text{-dim}(W) = \delta\text{-dim}(V)$, but $\underline{\omega_W(t) < \omega_V(t)}$ always hold.

  $\left(\begin{array}{l} \text{i.e., } \delta\text{-dim}(W) < \delta\text{-dim}(V) \\ \quad \text{or } \delta\text{-dim}(W) = \delta\text{-dim}(V) \text{ but } \mathrm{ord}(W) < \mathrm{ord}(V). \end{array} \right)$

If $W = \bigcup_i V_i$ with V_i irr components of W, then define $\omega_W(t) = \max_i \omega_{V_i}(t)$.

- Sps $U = \{u_1, \dots, u_d\}$ is a parametric set of $\mathbb{I}(V) \subseteq K\{Y\}$ and $(\bar{u}_1, \dots, \bar{u}_d, \xi_1, \dots, \xi_{n-d})$ is a generic point of $\mathbb{I}(V)$. Then

  $$\underline{\mathrm{ord}_U(V)} = \mathrm{tr.deg}\, K\langle \bar{u}_1, \dots, \bar{u}_d, \xi_1, \dots, \xi_{n-d} \rangle / K\langle \bar{u}_1, \dots, \bar{u}_d \rangle.$$

  relative order of V ($\mathbb{I}(V)$) relative to U.

<u>Prop:</u> Let $\mathcal{A}$ be a char set of $\mathbb{I}(V)$ under some elimination ranking. Sps $\mathrm{ld}(\mathcal{A}) = \{ y_{i_1}^{(o_1)}, \dots$

$Y_{i_2}^{(O_2)}, \dots, Y_{i_{n-d}}^{(O_{n-d})}\}$. Then $U = Y \setminus \{Y_{i_1}, \dots, Y_{i_{n-d}}\}$ is a parametric set of $\mathbb{I}(V)$ and

$$\mathrm{ord}_U(V) = \sum_{k=1}^{n-d} O_k .$$

proof. Let $\xi = (\xi_1, \dots, \xi_n)$ be a generic point of $\mathbb{I}(V)$.

Clearly, $\bar{U} = \{\xi_k : k \in \{1, \dots, n\} \setminus \{i_1, \dots, i_{n-d}\}\}$ is δ-alg indep over K. Let $A = A_1, \dots, A_{n-d}$ with $\mathrm{ld}(A_k) = Y_{i_k}^{(O_k)}$. Then $A_1(\xi) = 0 \Rightarrow \xi_{i_1}$ is δ-alg over $K\langle \bar{U} \rangle$. $A_2(\xi) = 0 \Rightarrow \xi_{i_2}$ is δ-alg over $K\langle \bar{U}, \xi_{i_1}$
$\Rightarrow \xi_{i_2}$ is δ-alg over $K\langle \bar{U} \rangle$.

Similarly in this way, you can show each ξ_{i_k} $(k=1, \dots, n-d)$ is δ-alg over $K\langle \bar{U} \rangle$.

$\Rightarrow \bar{U}$ is a δ-transcendence basis of $K\langle \xi \rangle$ over K.

$= U$ is a parametric set of $\mathbb{I}(V)$.

To show $\mathrm{ord}_U(V) = \sum_{k=1}^{n-d} O_k$, it suffices to show

$(*)\left\{\begin{array}{l}\tilde{\xi} \triangleq \{\underbrace{\xi_{i_1}, \xi_{i_1}', \dots, \xi_{i_1}^{(o_1-1)}}_{\xi_{i_1}^{[o_1-1]}}; \dots; \underbrace{\xi_{i_{n-d}}, \xi_{i_{n-d}}', \dots, \xi_{i_{n-d}}^{(o_{n-d}-1)}}_{\xi_{i_{n-d}}^{[o_{n-d}-1]}}\} \text{ is} \\ \text{a transcendence basis of } K\langle\xi\rangle \text{ over } K\langle\bar{U}\rangle.\end{array}\right.$

① $\tilde{\xi}$ is alg indep over $K\langle\bar{U}\rangle$, for a nonzero elt in $K\{U\}[Y_{i_1}, \dots, Y_{i_1}^{(o_1-1)}, \dots, Y_{i_{n-d}}, \dots, Y_{i_{n-d}}^{(o_{n-d}-1)}]$ is reduced w.r.t. $\mathcal{A}$;

② $A_1(\xi)=0 \Rightarrow \xi_{i_1}^{(o_1)}$ is alg over $K\langle\bar{U}\rangle(\xi_{i_1}, \dots, \xi_{i_1}^{(o_1-1)})$

& $\xi^{(o_1+k)} \in K\langle\bar{U}\rangle(\xi_{i_1}, \dots, \xi_{i_1}^{(o_1+k-1)})$ $(k \geq 1)$

$\Rightarrow \xi^{(o_1+k)}$ is alg over $K\langle\bar{U}\rangle(\xi_{i_1}, \dots, \xi_{i_1}^{(o_1-1)})$.

$A_2(\xi)=0 \Rightarrow \xi^{(o_2)}$ is alg over $K\langle\bar{U}\rangle(\xi_{i_1}, \dots, \xi_{i_1}^{(o_1)}, \xi_{i_2}, \dots, \xi_{i_2}^{(o_2-1)})$

$\xi_{i_2}^{(o_2+k)} \in K\langle\bar{U}\rangle(\xi_{i_1}^{[o_1+k]}, \xi_{i_2}^{[o_2+k-1]})$ $\Downarrow$

$\Rightarrow \xi^{(o_2+k)}$ is alg over $K\langle\bar{U}\rangle(\xi_{i_1}^{[o_1-1]}, \xi_{i_2}^{[o_2-1]})$

Similarly, we can show each $\xi_{i_k}^{(o_k+j)}$ for $j \geq 0$ is alg over $K\langle\bar{U}\rangle(\xi_{i_1}^{[o_1-1]}, \dots, \xi_{i_{n-d}}^{[o_{n-d}-1]})$.

Thus, $(*)$ holds & $\operatorname{ord}_U(V) = \sum_{k=1}^{n-d} o_k$. ☒.

Theorem 4.4.8 Suppose (K, δ) contains a nonconstant element. Let $p \subseteq K\{u_1, \dots, u_d, y_1, \dots, y_{n-d}\}$ be a prime δ-ideal with a parametric set $\{u_1, \dots, u_d\}$.

Introduce a new δ-indeterminate z and let R be the elimination ranking $u_1 < \cdots < u_d < z < y_1 < \cdots < y_{n-d}$. Then $\exists\ a_1, \ldots, a_{n-d} \in K$ s.t. $[p, z - a_1 y_1 - \cdots - a_{n-d} y_{n-d}] \subseteq K\{u_1, \ldots, u_d, y_1, \ldots, y_{n-d}, z\}$ has a characteristic set of the form

$$\begin{aligned}
&X(u_1, \ldots, u_d, z) \\
&I_1(u_1, \ldots, u_d, z) y_1 - T_1(u_1, \ldots, u_d, z) \\
&\qquad \vdots \\
&I_{n-d}(u_1, \ldots, u_d, z) y_{n-d} - T_{n-d}(u_1, \ldots, u_d, z)
\end{aligned}$$

Moreover, $\mathrm{ord}(X, z) = \mathrm{ord}_{u_1, \ldots, u_d}(p)$.

<u>Proof.</u> Let $\eta = (\bar{u}_1, \ldots, \bar{u}_d, \bar{y}_1, \ldots, \bar{y}_{n-d})$ be a generic point of p. Introduce $n-d$ new δ-indeterminates $\lambda_1, \ldots, \lambda_{n-d}$ over $K\langle\eta\rangle$. Let $J = [p, z - \lambda_1 y_1 - \cdots - \lambda_{n-d} y_{n-d}]$

$$\subseteq K\{u_1, \ldots, u_d, y_1, \ldots, y_{n-d}, \lambda_1, \ldots, \lambda_{n-d}, z\}.$$

Then J is a prime δ-ideal with a generic point

$\xi = (\eta, \lambda_1, \dots, \lambda_{n-d}, \sum_{i=1}^{n-d} \lambda_i \bar{y}_i)$.

Since $\delta\text{-}\dim(\mathfrak{p}) = d$, $\delta\text{-tr.deg}\, K\langle \eta\rangle / K = d$ and

$\delta\text{-tr.deg}\, K\langle\xi\rangle/K = \delta\text{-tr.deg}\, K\langle\eta\rangle/K + \delta\text{-tr.deg}\, K\langle\xi\rangle/K\langle\eta\rangle$

$= d + (n-d) = n$.

So $J_\lambda = J \cap K\{u_1, \dots, u_d, \lambda_1, \dots, \lambda_{n-d}, z\} \neq \{0\}$ &

$\{u_1, \dots, u_d, \lambda_1, \dots, \lambda_{n-d}\}$ is a parametric set of J_λ.

Consider the elimination ranking R_λ: $u_1 < \dots < u_d < \lambda_1 < \dots < \lambda_{n-d} < z$,

then $\exists$ an irr δ-poly $R(u_1, \dots, u_d, \lambda_1, \dots, \lambda_{n-d}, z)$ s.t.

$\{R\}$ is a δ-char set of J_λ under R_λ.

Sps $\mathrm{ord}(R, z) = s$. Since $R(\bar{u}_1, \dots, \bar{u}_d, \lambda_1, \dots, \lambda_{n-d}, \sum_{i=1}^{n-d} \lambda_i \bar{y}_i) = 0$,

for $j = 1, \dots, n-d$, taking the partial derivative of

this identity w.r.t. $\lambda_j^{(s)}$, then we have

$$(**) \quad \overline{\frac{\partial R}{\partial \lambda_j^{(s)}}} + \overline{\frac{\partial R}{\partial z^{(s)}}} \cdot \bar{y}_j = 0,$$

where $\overline{\frac{\partial R}{\partial \lambda_j^{(s)}}}$ & $\overline{\frac{\partial R}{\partial z^{(s)}}}$ are obtained by substituting

$u_i = \bar{u}_i$ & $z = \sum \lambda_i \bar{y}_i$ in $\frac{\partial R}{\partial \lambda_j^{(s)}}$ & $\frac{\partial R}{\partial z^{(s)}}$.

Since $\frac{\partial R}{\partial z^{(s)}} \notin J_\lambda$, $\overline{\frac{\partial R}{\partial z^{(s)}}} \neq 0$, $\bar{y}_j = -\frac{\overline{\frac{\partial R}{\partial \lambda_j^{(s)}}}}{\overline{\frac{\partial R}{\partial z^{(s)}}}}$ & $\frac{\partial R}{\partial \lambda_j^{(s)}} + \frac{\partial R}{\partial z^{(s)}} y_j \in J_\lambda$.

Note that $\overline{\frac{\partial R}{\partial z^{(s)}}} \in K\{\eta\}\{\lambda_1, \ldots, \lambda_{n-d}\} \setminus \{0\}$, so by the nonvanishing theorem for diff polynomials,

$$\exists\, a_1, \ldots, a_{n-d} \text{ s.t. } \overline{\frac{\partial R}{\partial z^{(s)}}}\Big|_{\lambda_i = a_i,\, i=1,\ldots,n-d} \neq 0.$$

$$\Big\| $$

$$\frac{\partial R}{\partial z^{(s)}}\left(\bar{u}_1, \ldots, \bar{u}_d, a_1, \ldots, a_{n-d}, \sum_i a_i \bar{y}_i\right)$$

Let $I(u_1, \ldots, u_d, z) = \frac{\partial R}{\partial z^{(s)}}\Big|_{\lambda_i = a_i,\, i=1,\ldots,n-d} \in K\{u_1, \ldots, u_d, z\}$.

Then $I(\bar{u}_1, \ldots, \bar{u}_d, \sum_i a_i \bar{y}_i) \neq 0$. Let

$$J_a = [P, z - a_1 y_1 - \cdots - a_{n-d} y_{n-d}] \subseteq K\{u_1, \ldots, u_d, y_1, \ldots, y_{n-d}, z\}$$

Then J_a is a prime δ-ideal with a generic point $\xi_a \triangleq (\eta, a_1 \bar{y}_1 + \cdots + a_{n-d} \bar{y}_{n-d})$. Clearly, $I(u_1, \ldots, u_d, z) \notin J_a$

Set $T_j = \frac{\partial R}{\partial \lambda_j^{(s)}}\Big|_{\lambda_i = a_i,\, i=1,\ldots,n-d} \in K\{u_1, \ldots, u_d, z\}$ for $j = 1, \ldots, n-d$.

Then $I(u_1, \ldots, u_d, z) y_j + T_j(u_1, \ldots, u_d, z) \in J_a$.

Since δ-tr.deg $K\langle \xi_a \rangle / K = d$,

$\{u_1, \dots, u_d\}$ is a parametric set of J_a

& $\underbrace{J_a \cap K\{u_1, \dots, u_d, z\} \neq [0]}$. codimension$=1$

So $\exists$ an irr δ-poly $X(u_1, \dots, u_d, z)$ s.t.

$\{X\}$ is a δ-char set of $J_a \cap K\{u_1, \dots, u_d, z\}$

under the elimination ranking $u_1 < \dots < u_d < z$.

For each j, take the δ-remainder of $IY_j + T_j$

w.r.t. X, then we get

$$\underset{\delta\text{rem}(IY_j + T_j)}{\underset{\shortparallel}{I_j Y_j + G_j}} \quad \text{for some } I_j,\ G_j \in K\{u_1, \dots, u_d, z\} \setminus \{0\} \quad (\text{for } I \notin J_a)$$

<u>Claim</u> $X(u_1, \dots, u_d, z)$, $I_1 Y_1 + G_1, \dots, I_{n-d} Y_{n-d} + G_{n-d}$

is a δ-char set of J_a w.r.t. the elimination

ranking $u_1 < \dots < u_d < z < Y_1 < \dots < Y_{n-d}$.

<u>proof of the claim</u> Let $f \in J_a \setminus \{0\}$. and

$f_1 = \delta\text{-rem}(f, \{I_1 y_1 + G_1, \ldots, I_{n-d} y_{n-d} + G_{n-d}\})$.

Then $f_1 \in J_a \cap K\{u_1, \ldots, u_d, z\} = \mathrm{Sat}(X)$.

So $\delta\text{-rem}(f_1, X) = 0$.

It remains to show $\mathrm{ord}(X, z) = \mathrm{ord}_{u_1,\ldots,u_d}(p)$.

Since $\bar{y}_j = \dfrac{G_j(\bar{u}_1, \ldots, \bar{u}_d, \sum a_i \bar{y}_i)}{I_j(\bar{u}_1, \ldots, \bar{u}_d, \sum a_i \bar{y}_i)}$ for $j = 1, \ldots, n-d$,

$$K\langle y\rangle = K\langle \bar{u}_1, \ldots, \bar{u}_d, \textstyle\sum a_i \bar{y}_i\rangle.$$

Thus, $\mathrm{ord}_{u_1,\ldots,u_d}(p) = \mathrm{tr.deg}\, K\langle y\rangle / K\langle \bar{u}_1, \ldots, \bar{u}_d\rangle$

$= \mathrm{tr.deg}\, K\langle \bar{u}_1, \ldots, \bar{u}_d, \sum a_i \bar{y}_i\rangle / K\langle \bar{u}_1, \ldots, \bar{u}_d\rangle$

$= \mathrm{ord}_{u_1,\ldots,u_d}(J_a \cap K\{u_1, \ldots, u_d, z\})$

$= \mathrm{ord}(X, z)$. $\boxed{\xi}$.

<u>Remark</u>: ① The above irr δ-poly $X(u_1, \ldots, u_d, z)$ is called the differential resolvent of p or $\mathbb{V}(p)$.

② With the obtain $a_1, \ldots, a_{n-d}$, we have $K\langle \bar{u}_1, \dots, \bar{u}_d, \bar{y}_1, \dots, \bar{y}_{n-d} \rangle$

$$= K\langle \bar{u}_1, \dots, \bar{u}_d, a_1\bar{y}_1 + \cdots + a_{n-d}\bar{y}_{n-d} \rangle.$$

In the case $d=0$, $a_1\bar{y}_1 + \cdots + a_{n-d}\bar{y}_{n-d}$ is the primitive elt of $K\langle \bar{y}_1, \dots, \bar{y}_n \rangle$.

Consider the field of diff rational functions on V over K, $K\langle V \rangle$. Each elt of $K\langle V \rangle$ can be identified as a diff rational function on V. If $\eta \in V$, a diff rational function on V is defined at η if it can be represented as a quotient of diff poly functions whose denominator doesn't vanish at η.

Definition 4.4.9

Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be irreducible δ-varieties over K. A diff rational map $\varphi : V \dashrightarrow W$ is a family $(f_1, \dots, f_m) \in K\langle V \rangle^m$ s.t.

$$\varphi(\eta) = (f_1(\eta), \dots, f_m(\eta)) \in W$$

whenever the coordinate functions $f_1, \dots, f_m$ are defined at η. φ is called dominant if the Kolchin closure of $\varphi(V)$ is W (or equivalently, φ maps a generic point of V to a generic point of W).

And φ is called a diff birational map if φ is dominant and there exists a dominant diff rational map $\psi: W \to V$, called the generic inverse of φ s.t.

- if φ is defined at η & ψ is defined at $\varphi(\eta)$, then $\psi(\varphi(\eta)) = \eta$;
- if ψ is defined at ξ and φ is defined at $\psi(\xi)$, then $\varphi(\psi(\xi)) = \xi$.

In this case, V & W are called δ-birationally equivalent.

Corollary 4.4.10

Let (K, δ) contain a non-constant element. Let $V \subseteq \mathbb{A}^n$ be an irreducible δ-variety. Then V is δ-birationally equivalent to the general component of an irr δ-poly (i.e., an irreducible δ-variety of codimension 1).

proof. Sps $\delta\text{-dim}(V) = d$ and $\{u_1, \dots, u_d\}$ is a parametric set of $p = \mathbb{I}(V) \subseteq K\{u_1, \dots, u_d, y_1, \dots, y_{n-d}\}$.

By Theorem 4.4.8, $\exists\, a_1, \dots, a_{n-d} \in K$ s.t.

$$J_a = [p,\ z - a_1 y_1 - \cdots - a_{n-d} y_{n-d}] \subseteq K\{u_1, \dots, u_d, y_1, \dots, y_{n-d}, z\}$$

has a characteristic set of the form

$$\begin{gathered} X(u_1, \dots, u_d, z) \\ I_1(u_1, \dots, u_d, z) y_1 + T_1(u_1, \dots, u_d, z) \\ \vdots \\ I_{n-d}(u_1, \dots, u_d, z) y_{n-d} + T_{n-d}(u_1, \dots, u_d, z) \end{gathered}$$

under the elimination ranking $u_1 < \cdots < u_d < z < y_1 < \cdots < y_{n-d}$ where X is irr.

Let $W = \mathbb{V}(\mathrm{sat}(X)) \subseteq \mathbb{A}^{d+1}$ be the general component of X. Define

- $\varphi : V \dashrightarrow W$ by

$$\varphi(u_1, \ldots, u_d, y_1, \ldots, y_{n-d}) = (u_1, \ldots, u_d, a_1 y_1 + \cdots + a_{n-d} y_{n-d})$$

and

- $\psi : W \dashrightarrow V$ by

$$\psi(u_1, \ldots, u_d, z) = \left(u_1, \ldots, u_d, -\frac{T_1(u_1, \ldots, u_d, z)}{I_1(u_1, \ldots, u_d, z)}, \ldots, -\frac{T_{n-d}(u_1, \ldots, u_d, z)}{I_{n-d}(u_1, \ldots, u_d, z)}\right).$$

Let $\eta = (\bar{u}_1, \ldots, \bar{u}_d, \bar{y}_1, \ldots, \bar{y}_{n-d})$ be a generic point of V. By the proof of Theorem 4.4.8,

$\xi = (\bar{u}_1, \ldots, \bar{u}_d, a_1 \bar{y}_1 + \cdots + a_{n-d} \bar{y}_{n-d})$ is a generic point of W and $\bar{y}_i = -\dfrac{T_i(u_1, \ldots, u_d, \sum a_i \bar{y}_i)}{I_i(u_1, \ldots, u_d, \sum a_i \bar{y}_i)}$ for $i = 1, \ldots, n-d$.

So both φ and ψ are dominant and $(\varphi\circ\psi)|_W = id_W$ & $(\psi\circ\varphi)|_V = id_V$.

Thus, V and W are δ-birationally equivalent. $\boxtimes$

Example

Let $K = (\mathbb{Q}(t), \frac{d}{dt})$ and $V = \mathbb{V}(Y_1', Y_2') \subseteq \mathbb{A}^2(\bar{K})$.

Introduce new δ-indeterminates z, λ_1, λ_2 and consider $J = [Y_1', Y_2', z - \lambda_1 Y_1 - \lambda_2 Y_2] \subseteq K\{Y_1, Y_2, \lambda_1, \lambda_2, z\}$.

To eliminate Y_1, Y_2 in order to get $X(z) \in K\{z\}$, we have

$$R(\lambda_1, \lambda_2, z) = \begin{vmatrix} z & -\lambda_1 & -\lambda_2 \\ z' & -\lambda_1' & -\lambda_2' \\ z'' & -\lambda_1'' & -\lambda_2'' \end{vmatrix}$$

$$= (\lambda_1\lambda_2' - \lambda_1'\lambda_2)\, z'' - (\lambda_1\lambda_2'' - \lambda_1''\lambda_2) z' + (\lambda_1'\lambda_2'' - \lambda_1''\lambda_2') z.$$

$$S_R = \frac{\partial R}{\partial z''} = \lambda_1\lambda_2' - \lambda_1'\lambda_2 .$$

Select $\lambda_1 = 1$ and $\lambda_2 = t$, then $\overline{S_R} = 1 \neq 0$.

So $X(z) = z''$, $\overline{S_R} Y_1 + \overline{\frac{\partial R}{\partial \lambda_1''}} = Y_1 + (tz' - z)$

$$\overline{S_R} Y_2 + \overline{\frac{\partial R}{\partial \lambda_2''}} = Y_2 - z'$$

is a characteristic set of $[Y_1', Y_2', z - Y_1 - tY_2]$

w.r.t. the elimination ranking $z < Y_1 < Y_2$.

Let $W = \mathbb{V}(z'') \subseteq \mathbb{A}^1$. Then V and W are

δ-birationally equivalent.

Indeed, $\varphi : V \dashrightarrow W$ and $\psi : W \dashrightarrow V$

$(Y_1, Y_2) \quad Y_1 + tY_2 \qquad z \quad (z - tz', z')$

Then $\psi \circ \varphi(Y_1, Y_2) = (Y_1, Y_2)$

and $\varphi \circ \psi(z) = \varphi(z - tz', z') = z - tz' + tz' = z$.

So $X(z) = z''$ is a diff resolvent of V,

and if C_1, C_2 are alg indeterminates with $C_1' = C_2' = 0$,

then $\mathbb{Q}(t)\langle C_1, C_2\rangle = \mathbb{Q}(t)\langle C_1 + tC_2\rangle$.

↳ primitive element

Chapter 5 Algorithms and Constructive methods for algebraic differential equations

In section 2.1, we have introduced the (Ritt's) theory of diff characteristic sets for differential ideals
In this chapter, we study Wu's characteristic set methods for finite set of diff poly and in particular, introduce the Ritt-Wu's irreducible decomposition algorithm.

Let (K, δ) be a δ-field of char 0 and consider the diff poly ring $K\{Y\} = K\{y_1, \ldots, y_n\}$.

§5.1 Well-ordering principle for diff poly

First, we recall basic notions and lemmas about characteristic sets.

- A ranking $\mathcal{R}$ on $K\{Y\}$ is a total ordering on $\Theta(Y) = \{y_i^{(k)} : k \in \mathbb{N}, i = 1, \dots, n\}$ satisfying

  1) $u < \delta u$ 2) $u < v \Rightarrow \delta(u) < \delta(v)$.

  Elimination ranking & orderly ranking

- Fix a ranking $\mathcal{R}$. Given $f \in K\{Y\} \setminus K$, $ld(f)$, I_f, S_f, $rk(f)$ are the leader initial, separant & rank of f under $\mathcal{R}$.

  $$\left(\begin{array}{l} f = I_f (ld(f))^d + I_{d-1}(ld(f))^{d-1} + \cdots + I_0 \\ rk(f) = (ld(f), d) \in \Theta(Y) \times \mathbb{N} \end{array}\right)$$

- Given $f, g \in K\{Y\}$ with $f \notin K$,

  g is $\left\{\begin{array}{l}\text{partially reduced} \\ \text{reduced}\end{array}\right.$ w.r.t. f if

  $\left\{\begin{array}{l} \Theta_{\geq 1}(ld(f)) \text{ doesn't appear in } g. \\ \text{partially reduced} + \deg(g, ld(f)) < \deg(f, ld(f)). \end{array}\right.$

- An autoreduced set $\mathcal{A} \subseteq K\{Y\}$. $(\Rightarrow |\mathcal{A}| < \infty)$

- Given $f, g \in K\{Y\}$, $f < g$ if $rk(f) <_{lex} rk(g)$.

Two autoreduced sets $\mathcal{A} = A_1 < A_2 < \cdots < A_p$,
$\mathcal{B} = B_1 < B_2 < \cdots < B_q$

$\mathcal{A} < \mathcal{B}$, if either i) $\exists\, i \le \min\{p, q\}$ s.t. $rk(A_k) = rk(B_k)$ for $i < k$ & $rk(A_i) < rk(B_i)$

or 2) $p > q$ & $\forall\, k \le q$, $rk(A_k) = rk(B_k)$.

$\mathcal{A} \sim \mathcal{B}$ if $p = q$ & $\forall\, k$, $rk(A_k) = rk(B_k)$.

$\mathcal{A}_1 \geqslant \mathcal{A}_2 \geqslant \cdots \geqslant \mathcal{A}_i \geqslant \cdots \Rightarrow \exists\, i_0$ s.t. $\forall\, i \geqslant i_0$ $\mathcal{A}_i \sim \mathcal{A}_{i_0}$.

Equivalently, any nonempty set of autoreduced sets contains an autoreduced set of lowest rank.

- Differential reduction

Given $f \in K\{Y\}$ & an autoreduced set $\mathcal{A}$,

$\exists\, r = \mathrm{prem}(f, \mathcal{A})$ reduced w.r.t. $\mathcal{A}$ & $i_A, s_A \in \mathbb{N}$ s.t.

(*) $\displaystyle\prod_{A \in \mathcal{A}} I_A^{i_A} S_A^{s_A} \cdot f \equiv r \mod [\mathcal{A}]$.

(*) is the diff reduction formula.

- $\mathcal{I} \subseteq K\{Y\}$ a proper ideal. A char set of $\mathcal{I}$ is an autoreduced set $\mathcal{A}$ contained in $\mathcal{I}$ of lowest rank.

$\Longleftrightarrow$ $\forall f \in \mathcal{I}$, $\operatorname{prem}(f, \mathcal{A}) = 0$.

In this section, we introduce the notion of characteristic sets for finite sets of δ-poly, instead of δ-ideals.

<u>Lemma 5.1.1</u> Let $\Sigma \subseteq K\{Y\}$ be a finite set of nonzero δ-polys. We can find an autoreduced set $\mathcal{A} \subseteq \Sigma$ which is of lowest rank among all autoreduced sets contained in Σ with a mechanical method. Such an autoreduced set is called a <u>basic set</u> of Σ.