<u>Recall</u> Let $V \subseteq \bar{K}^n$ be an irr δ-Variety over K with a generic point $\xi = (\xi_1, \dots, \xi_n)$.

- $\delta\text{-dim}(V) = \delta\text{-tr.deg}\, K\langle\xi\rangle / K$;
- $\omega_V(t) = \text{tr.deg}\, K(\xi_1^{[t]}, \dots, \xi_n^{[t]})/K$ for $t >> 0$. ($\xi_i^{[t]} = \{\xi_i, \xi_i', \dots, \xi_i^{(t)}\}$)

  $= \delta\text{-dim}(V) \cdot (t+1) + \text{ord}(V)$;
- $\text{ord}(V) = \text{ord}(\mathcal{A})$ for each characteristic set $\mathcal{A}$ of I under an arbitrary orderly ranking ;

Assume $U = \{y_{i_1}, \dots, y_{i_d}\}$ is a parametric set of $\mathbb{I}(V)$.

- Relative order of V w.r.t. U:

  $\text{ord}_U(V) = \text{tr.deg}\, K\langle\xi\rangle / K\langle\xi_{i_1}, \dots, \xi_{i_d}\rangle$.

If $\mathcal{B}$ is a char set of $\mathbb{I}(V)$ w.r.t. some elimination ranking and U is the set of non-leading variables of $\mathcal{B}$,

then $\delta\text{-dim}(V) = n - \text{Card}(\mathcal{B})$ & $\text{ord}_U(V) = \text{ord}(\mathcal{B}) = \sum_{B \in \mathcal{B}} \text{ord}(\text{ld}(B))$.

- Differential Resolvent theory: Sps K contains a nonconstant elt.

  Let $P \subseteq K\{u_1, \dots, u_d, y_1, \dots, y_{n-d}\}$ be a prime δ-ideal with $\{u_1, \dots, u_d\}$ a parametric set of P. Introduce a new δ-indeterminate w over $K\{u_1, \dots, y_{n-d}\}$ and consider an elimination ranking R:

  $u_1 < \dots < u_d < w < y_1 < \dots < y_{n-d}$. Then $\exists\, a_1, \dots, a_{n-d} \in K$

s.t. $[P, w - a_1 y_1 - \dots - a_{n-d} y_{n-d}]$ has a char set of the form

$R(u_1,\dots,u_d,w)$: δ-Resolvent of P or $W(P)$

$I_1(u_1,\dots,u_d,w)Y_1 + T_1(u_1,\dots,u_d,w)$

$\vdots$

$I_{n-d}(u_1,\dots,u_d,w)Y_{n-d} + T_{n-d}(u_1,\dots,u_d,w)$.

Geometric Form

(K,δ): non-constant δ-field and V: an irr δ-variety

Then V is δ-birationally equivalent to an irr δ-variety of codim 1. (i.e., the general component of some irr δ-poly.)

Chapter 5 Algorithms and constructive methods for algebraic differential equations

§5.1 Well-ordering principle for diff polys

Let (K,δ) be a diff field of char 0 and consider the δ-poly ring $K\{Y\} = K\{Y_1,\dots,Y_n\}$.

In this section, we introduce the notion of characteristic sets for finite sets of nonzero diff polys following Wu's characteristic methods.

Lemma 5.1.1 Let $\Sigma \subseteq K\{Y\}$ be a finite set of nonzero δ-polys. We can find an autoreduced set $\mathcal{A} \subseteq \Sigma$ which is of lowest rank among all autoreduced sets contained in Σ with a mechanical method. Such an autoreduced set is called a ==basic set== of Σ.

Proof. First select $A_1 \in \Sigma$ which has the lowest rank in Σ. Let $\Sigma_1 = \{f \in \Sigma \mid f \text{ is reduced w.r.t. } A_1\}$.

If $\Sigma_1 = \phi$, then A_1 satisfies the condition of the lemma. Otherwise, choose $A_2 \in \Sigma_1$ of lowest rank.

Let $\Sigma_2 = \{f \in \Sigma_1 \mid f \text{ is reduced w.r.t. } A_1, A_2\}$.

If $\Sigma_2 = \phi$, then $\mathcal{A} = A_1, A_2$ is a basic set of Σ.

Otherwise, let $A_3 \in \Sigma_2$ be of lowest rank.

Repeat the above process, at last we get an autoreduced set $\mathcal{A} = A_1, \cdots, A_k$ of the desired property. ∎.

<u>Lemma 5.1.2</u> If a nonzero δ-poly p is reduced w.r.t. a basic set of Σ, then a basic set of $\Sigma \cup \{p\}$ is of lower rank than a basic set of Σ.

proof. Let $\mathcal{A} = A_1, \dots, A_k$ be a basic set of Σ.

If $rk(p) < rk(A_1)$, then p is an autoreduced set of lower rank than $\mathcal{A}$. Otherwise, there exists an index i with $rk(A_i) < rk(p) < rk(A_{i+1})$. Since p is reduced w.r.t. $\mathcal{A}$, $A_1, \dots, A_i, p$ is an autoreduced set of lower rank than $\mathcal{A}$. So a basic set of $\Sigma \cup \{p\}$ will be of lower rank than $\mathcal{A}$. $\square$

Let $\Sigma \subseteq K\{Y\}$ be a finite set of nonzero δ-polys. If $\mathbb{V}(\Sigma) = \phi$, we call Σ a contradictory system. Given another diff poly set G, we denote

$$\mathbb{V}(\Sigma/G) \triangleq \{ \xi \in \bar{K}^n \mid \xi \in \mathbb{V}(\Sigma) \ \& \ G(\xi) \neq 0 \}$$ and

call $\mathbb{V}(\Sigma/G)$ a quasi-δ variety.

We now proceed to study the structure of $\mathbb{V}(\Sigma)$.

First set $\Sigma_0 = \Sigma$ and take a basic set $\mathcal{A}_0$ of Σ.

Let $R_0 = \{ \delta\text{-rem}(f, \mathcal{A}_0) \mid f \in \Sigma \setminus \mathcal{A}_0 \} \setminus \{0\}$.

If $R_0 \neq \phi$, set $\Sigma_1 = \Sigma \cup R_0$ and take a basic set $\mathcal{A}_1$ of Σ_1. Let $R_1 = \{ \delta\text{-rem}(f, \mathcal{A}_1) \mid f \in \Sigma_1 \setminus \mathcal{A}_1 \} \setminus \{0\}$.

If $R_1 \neq \phi$, set $\Sigma_2 = \Sigma_1 \cup R_1$ and construct a basic set $\mathcal{A}_2$ of Σ_2 and R_2 accordingly. In this way, we shall get a strictly increasing sequence of δ-poly sets

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$$

with a strictly decreasing sequence of autoreduced sets

$$\mathcal{A}_0 > \mathcal{A}_1 > \mathcal{A}_2 > \cdots.$$

This decreasing sequence can have only finite terms. Thus, $\exists\, q \geq 1$ s.t. $R_q = \phi$. The above gives an algorithm:

$$\boxed{\begin{array}{ccccccc} \Sigma_0 = \Sigma & \subseteq & \Sigma_1 = \Sigma_0 \cup R_1 & \subseteq & \cdots & \subseteq & \Sigma_q = \Sigma_{q-1} \cup R_{q-1} \\ \mathcal{A}_0 & > & \mathcal{A}_1 & > & \cdots & > & \mathcal{A}_q \\ R_0 \neq \phi & & R_1 \neq \phi & & \cdots & & R_q = \phi \end{array}} \quad (*)$$

where $\mathcal{A}_i$ = a basic set of Σ_i

$$R_i = \delta\text{-rem}(\Sigma_i \setminus \mathcal{A}_i, \mathcal{A}_i) \setminus \{0\}$$

$$\Sigma_i = \Sigma_{i-1} \cup R_{i-1}$$

Definition 5.13

The above obtained $\mathcal{A}_e$ is called a characteristic set of the finite set $\Sigma \subseteq K\{Y\}$.

Theorem 5.14 (Ritt-Wu's Well-Ordering principle)

Let $\Sigma \subseteq K\{Y\}$ be a finite nonempty δ-poly set. There is an algorithm to construct an autoreduced set $\mathcal{A}$, which is a characteristic set of Σ, such that either

(1) $\mathcal{A}$ consists of a nonzero elt in K; in this case Σ is a contradictory system;

or

(2) $\mathcal{A} = A_1, \dots, A_p$ with $A_1 \in K\{Y\} \setminus K$ such that

$A_i \in [\Sigma]$ and $\forall f \in \Sigma$, $\delta\text{-rem}(f, \mathcal{A}) = 0$;

in this case, we have

$$\mathbb{W}(\mathcal{A}/H_{\mathcal{A}}) \subseteq \mathbb{W}(\Sigma) \subseteq \mathbb{W}(\mathcal{A}) \text{ and}$$

$$\underline{\mathbb{W}(\Sigma) = \mathbb{W}(\mathcal{A}/H_{\mathcal{A}}) \cup \bigcup_{A \in \mathcal{A}} \left(\mathbb{W}(\Sigma, I_A) \cup \mathbb{W}(\Sigma, S_A)\right).} \quad (\text{Wu1})$$

the formula for the structure of $\mathbb{W}(\Sigma)$

<u>Proof</u>. The first assertion has been shown above the scheme (*). That is, $\exists\, q \in \mathbb{N}$ s.t. $R_q = \phi$ and $\mathcal{A} := \mathcal{A}_q$ is a characteristic set of Σ.

Note that $\mathcal{A} = \mathcal{A}_q \subseteq \Sigma_q$ and $\delta\text{-rem}(\Sigma_q, \mathcal{A}) = \{0\}$. So $\mathbb{W}(\Sigma_q) \subseteq \mathbb{W}(\mathcal{A})$ and $\delta\text{-rem}(\Sigma, \mathcal{A}) = \{0\}$ for $\Sigma \subseteq \Sigma_m$. For each i, $R_i = \delta\text{-rem}(\Sigma_i \setminus \mathcal{A}_i, \mathcal{A}_i) \setminus \{0\}$, we have $R_i \subseteq [\Sigma_i]$ and $\Sigma_{i+1} = \Sigma_i \cup R_i \subseteq [\Sigma_i]$. So $\mathbb{W}(\Sigma_i) = \mathbb{W}(\Sigma_{i+1})$

Thus, $\mathbb{W}(\Sigma) = \mathbb{W}(\Sigma_1) = \cdots = \mathbb{W}(\Sigma_q) \subseteq \mathbb{W}(\mathcal{A})$ and $\mathcal{A} = \mathcal{A}_q \subseteq \Sigma_q \subseteq [\Sigma]$.

If $\mathcal{A}$ consists a nonzero elt in K, $\mathbb{W}(\Sigma) \subseteq \mathbb{W}(\mathcal{A}) = \phi$, so in this case, $\mathbb{W}(\Sigma) = \phi$.

Otherwise, $\mathcal{A} = A_1, \ldots, A_p$ for some p and $A_1 \in K\{Y\} \setminus K$.

Since $\delta\text{-rem}(\Sigma, \mathcal{A}) = \{0\}$, $\forall f \in \Sigma$, $\exists H_f \in H_{\mathcal{A}}^{\infty}$ s.t. $H_f \cdot f \in [\mathcal{A}]$. So $\mathbb{V}(\mathcal{A}/H_{\mathcal{A}}) \subseteq \mathbb{V}(\Sigma)$.

Thus, $\mathbb{V}(\mathcal{A}/H_{\mathcal{A}}) \subseteq \mathbb{V}(\Sigma) \subseteq \mathbb{V}(\mathcal{A})$ and $\mathbb{V}(\mathcal{A}/H_{\mathcal{A}}) = \mathbb{V}(\Sigma/H_{\mathcal{A}})$.

Note that a δ-zero y of Σ which annul $H_{\mathcal{A}} = \prod_{A \in \mathcal{A}} I_A S_A$ is necessarily a zero of some I_A or S_A.

So $\mathbb{V}(\Sigma) = \mathbb{V}(\mathcal{A}/H_{\mathcal{A}}) \cup \bigcup_{A \in \mathcal{A}} \mathbb{V}(\Sigma, I_A) \cup \mathbb{V}(\Sigma, S_A)$. $\boxed{\Sigma}$.

<u>Example</u>: Let $f = Y_1' + 1$, $g = Y_1 + Y_2'$ in $\mathbb{Q}(t)\{Y_1, Y_2\}$.

(1) Consider the elimination ranking $\mathcal{R}_1$ with $Y_1 > Y_2$. We compute a char set of the set $\Sigma = \{f, g\}$ w.r.t. $\mathcal{R}_1$ following the scheme $(*)$.

Let $\Sigma_0 := \Sigma$. $\mathcal{A}_0 := g$ is a basic set of Σ_0. Compute $r_1 := \delta\text{-rem}(f, g) = 1 - Y_2''$ and $R_0 := \{r_1\}$.

Let $\Sigma_1 := \Sigma_0 \cup R_0 = \{f, g, r_1\}$. $\mathcal{A}_1 := r_1, g$ is a basic set of Σ_1. Compute $r_2 \triangleq \delta\text{-rem}(f, \mathcal{A}_1) = 0$. So $R_1 = \phi$.

Thus, $\mathcal{A}_1 = r_1, g$ is a characteristic set of Σ.

(2) Consider the orderly ranking R_2 with $y_1 > y_2$.
Let $\Sigma_0 := \Sigma$. $A_0 := g, f$ is a basic set of Σ.
$R_0 = \phi$. So $A_0 = g, f$ is a char set w.r.t. R_2.

Remark To simplify the algorithm, we can replace $\Sigma_i = \Sigma_{i-1} \cup R_{i-1}$ by $\Sigma_i := \Sigma_0 \cup B_{i-1} \cup R_{i-1}$.
(Excerise: show $\mathbb{W}(\Sigma_i) = \mathbb{W}(\Sigma)$ for $\forall i$).

Theorem 5.1.5 (Zero Decomposition Theorem: Weak Form)
There is a mechanical procedure which permits to compute in a finite number of steps for a given finite system Σ of nonzero diff polys, a finite number of autoreduced sets $CS_1, \dots, CS_m$ such that $\mathbb{W}(\Sigma) = \bigcup_j \mathbb{W}(CS_j / H_{CS_j})$

and $\delta\text{-rem}(\Sigma / CS_j) = \{0\}$.
Here, H_{CS_j} is the product of initials and separants of

δ-polys in CS_j. And each CS_j is of rank less than or equal to that of a basic set of Σ.

proof. By the well-ordering principle,

$$\mathbb{V}(\Sigma) = \mathbb{V}(CS/H_{CS}) \cup \bigcup_{A \in CS} \left(\mathbb{V}(\Sigma, I_A) \cup \mathbb{V}(\Sigma, S_A)\right),$$

where CS is a char set of Σ, H_{CS} is the product of initials and separants of CS.

If CS consisting of a nonzero elt in K, $\mathbb{V}(\Sigma) = \mathbb{V}(CS/H_{CS}) = \phi$.

Otherwise, let $\Sigma_{i_1} = \Sigma \cup \{I_A\} \cup CS$ (resp., $\Sigma \cup \{S_A\} \cup CS$).

Since $\mathbb{V}(\Sigma) \subseteq \mathbb{V}(CS)$, we have

$$\mathbb{V}(\Sigma, I_A) = \mathbb{V}(\Sigma_{i_1}) \ (\text{resp.}, \mathbb{V}(\Sigma, S_A) = \mathbb{V}(\Sigma_{i_1})).$$

So $\mathbb{V}(\Sigma) = \mathbb{V}(CS/H_{CS}) \cup \bigcup_{i_1} \mathbb{V}(\Sigma_{i_1})$.

Using the well-ordering principle and the above procedure for Σ_{i_1}, we have

$$\begin{cases} \mathbb{W}(\Sigma_{i_1}) = \mathbb{W}(CS_{i_1}/H_{CS_{i_1}}) \cup \bigcup_{i_2} \mathbb{W}(\Sigma_{i_1 i_2}) \\ \Sigma_{i_1 i_2} = \Sigma_{i_1} \cup \{I_A\} \cup CS_{i_1} \text{ or } \Sigma_{i_1} \cup \{S_A\} \cup CS_{i_1} \\ \text{a basic set of } \Sigma_{i_1 i_2} < CS_{i_1} \lesssim \text{a basic set of } \Sigma_{i_1}. \end{cases}$$

Here, CS_{i_1} is a char set of Σ_{i_1} and $A \in CS_{i_1}$.

Since $\Sigma \subseteq \Sigma_{i_1}$, $\text{f-rem}(\Sigma, CS_{i_1}) = \{0\}$.

If for each i_1, $CS_{i_1} := a \in K \setminus \{0\}$, $\mathbb{W}(\Sigma_{i_1}) = \phi$ and we have $\mathbb{W}(\Sigma) = \mathbb{W}(CS/H_{CS})$.

Otherwise, $\mathbb{W}(\Sigma) = \mathbb{W}(CS/H_{CS}) \cup \bigcup_{i_1} \mathbb{W}(CS_{i_1}/H_{CS_{i_1}}) \cup \bigcup_{i_1 i_2} \mathbb{W}(\Sigma_{i_1 i_2})$

Note that $CS_{i_1} < CS \lesssim$ a basic set of Σ.

Perform the above procedures for each $\Sigma_{i_1 i_2}$ and keep doing in this way, we have

$$\mathbb{W}(\Sigma) = \mathbb{W}(CS/H_{CS}) \cup \bigcup_{i_1} \mathbb{W}(CS_{i_1}/H_{CS_{i_1}}) \cup \bigcup_{i_1 i_2} \mathbb{W}(CS_{i_1 i_2}/H_{CS_{i_1 i_2}})$$
$$\cup \cdots \bigcup_{i_1 i_2 \cdots i_k} \mathbb{W}(\Sigma_{i_1 i_2 \cdots i_k}).$$

Now we have a strictly decreasing sequence of autoreduced sets $CS > CS_{i_1} > CS_{i_1 i_2} > \cdots > CS_{i_1 i_2 \cdots i_{k-1}} > \cdots$.

So there exist k s.t. all the $\mathbb{W}(\Sigma_{i_1 \cdots i_k}) = \phi$.

Thus, $\exists$ a finite number of autoreduced sets $\mathcal{A}_i$ s.t. $\mathbb{W}(\Sigma) = \bigcup_i \mathbb{W}(\mathcal{A}_i / H_{\mathcal{A}_i})$ and by induction $\delta\text{-rem}(\Sigma, \mathcal{A}_i) = \{0\}$ for each i.

And each $\mathcal{A}_i$ is of rank not higher than a basic set of Σ. $\boxed{\xi}$.

§5.2 Decomposition algorithms for differential Varieties

<u>Problem</u>: Given a finite set Σ of nonzero δ-polys, whether there exists a mechanical procedure to decompose $\mathbb{W}(\Sigma)$ into the irredundant union of irreducible components: $\mathbb{W}(\Sigma) = V_1 \cup \cdots \cup V_r$.

Or equivalently, decompose $\{\Sigma\}$ into an irredundant intersection of prime diff ideals:

$$\{\Sigma\} = P_1 \cap \cdots \cap P_r.$$

Since a prime δ-ideal P is completely determined by its characteristic set $\mathcal{A}$ (i.e., $P = \mathrm{Sat}(\mathcal{A})$), the above decomposition problem can be separated into the following two problems:

Problem 1: Given Σ, to find a finite set Λ of autoreduced sets of $K\{Y\}$, each of which is a characteristic set of a prime δ-ideal containing Σ, such that Λ contains a characteristic set of each component of $\{\Sigma\}$. That is, $\{\Sigma\} = \mathrm{Sat}(\mathcal{B}_1) \cap \cdots \cap \mathrm{Sat}(\mathcal{B}_e)$, where $\Lambda = \{\mathcal{B}_1, \cdots, \mathcal{B}_e\}$.

Problem 2 Given an autoreduced set $\mathcal{A}$ of

$K\{Y\}$, to determine whether $\mathcal{A}$ is a characteristic set of a prime component of $\{\Sigma\}$ or not.

<u>Problem 2'</u> Given that $\mathcal{A}$ and $\mathcal{B}$ are characteristic sets of prime δ-ideals P and Q respectively, to determine whether $P \subseteq Q$ or not.

$$\begin{array}{c} \text{Decomposition problem} \\ \| \\ \text{Problem 1} + \text{Problem 2} \\ \| \\ \text{Problem 1} + \text{Problem 2'}. \end{array}$$

Remark: ① Problem 1 has been solved (Wu-Ritt irr decomposition algorithm to be introduced in this section)

② Problem 2 is still not solved in the general case, and we have a complete answer for the special case that $\mathcal{Z}$ consists of a single δ-poly given by Ritt's Component theorem and the low power theorem.

③ Although it is trivial to decide whether $\mathfrak{p} = Q$, problem 2' is still open, even for the special case below:

<u>Ritt's problem</u> Given $A \in K\{Y\}$ irr with $A(0,\dots,0)=0$ to determine whether $(0,\dots,0)$ is a zero of $\mathrm{sat}(A)$. Or equivalently, whether $\mathrm{sat}(A) \subseteq [y_1,\dots,y_n]$.

In this section, we focus on a solution to Problem 1.

<u>Question 1</u>. Given an autoreduced set $\mathcal{A} \subseteq K\{Y\}$, give a necessary and sufficient condition for $\mathcal{A}$ to be a characteristic set of a prime δ-ideal $\mathfrak{p} \subseteq K\{Y\}$?

Part I. Rosenfeld's Lemma and the reduction of Question 1 to an algebraic problem.

<u>Lemma 5.2.1</u> (Rosenfeld's lemma in ordinary diff case)
Let $\mathcal{A} = A_1, \dots, A_p$ be an autoreduced set in $K\{Y\}$ w.r.t. a ranking and $f \in K\{Y\}$ be partially reduced w.r.t. $\mathcal{A}$. Then

$$f \in \operatorname{sat}(\mathcal{A}) = [\mathcal{A}] : H_{\mathcal{A}}^{\infty} \iff f \in (\mathcal{A}) : H_{\mathcal{A}}^{\infty}.$$

proof. "$\Leftarrow$" Trivial.

"$\Rightarrow$" Sps $f \in \operatorname{sat}(\mathcal{A})$. Then $\exists\, m \in \mathbb{N}$ and $g_{ij} \in K\{Y\}$ s.t.

$$H_{\mathcal{A}}^{m} \cdot f = \sum_{i=1}^{p} g_{i0} A_i + \sum_{i=1}^{p} \sum_{j=1}^{k_i} g_{ij} \cdot A_i^{(j)} \quad (*).$$

Note that for $j \geq 1$, $A_i^{(j)} = S_{A_i} \cdot \delta^j(\mathrm{ld}(A_i)) + T_{ij}$ for some T_{ij} free of $\delta^j(\mathrm{ld}(A_i))$.

Let $\Phi = \{\, \delta^j(\mathrm{ld}(A_i)) \mid g_{ij} \neq 0,\ j \geq 1,\ i = 1, \dots, p \,\}$.

If $\Phi \neq \phi$, take the greatest $v = \delta^j(\mathrm{ld}(A_i))$ in Φ, substitute $\delta^j(\mathrm{ld}(A_i)) = -\dfrac{T_{ij}}{S_{A_i}}$ at both sides of $(*)$, and set $\Phi \triangleq \Phi \setminus \{v\}$. Continuing this process and successively substitute $\delta^j(\mathrm{ld}(A_i)) = -\frac{T_{ij}}{S_{A_i}}$ into $(*)$ for all $\delta^j(\mathrm{ld}(A_i))$ in Φ.

Clearing denominators by multiplying a power product $S_{\mathcal{A}}^{l}$ of S_{A_i} at both sides of the obtained equality, we have $S_{\mathcal{A}}^{l} \cdot H_{\mathcal{A}}^{m} \cdot f = \sum_{i=1}^{p} \overline{g_{i0}} \cdot A_i$ for $\overline{g_{i0}} \in K\{Y\}$.

Thus, $f \in (\mathcal{A}) : H_{\mathcal{A}}^{\infty}$. $\boxed{\checkmark}$

In the following, we will use "δ-characteristic set" to distinguish with the algebraic case.

Lemma 5.2.2 Let $\mathcal{A}$ be an autoreduced set in $K\{Y\}$ w.r.t. a ranking $\mathcal{R}$. Then $\mathcal{A}$ is a δ-characteristic set of a prime δ-ideal $\Longleftrightarrow$ $(\mathcal{A}) : H_{\mathcal{A}}^{\infty}$ is a prime algebraic ideal in $K\{Y\}$ and $(\mathcal{A}) : H_{\mathcal{A}}^{\infty}$ contains no nonzero element reduced w.r.t. $\mathcal{A}$.

(i.e., $\mathcal{A}$ is a characteristic set of $(\mathcal{A}) : H_{\mathcal{A}}^{\infty}$.)

Proof. "$\Rightarrow$" Take a minimal subset $V \subseteq \Theta(Y)$ s.t. $\mathcal{A} \subseteq K[V]$.

Let $P_{\mathcal{A}} = \{ f \in K[V] \mid \exists\, m \in \mathbb{N} \text{ s.t. } H_{\mathcal{A}}^{m} \cdot f \in (\mathcal{A}) \}$.

Then we have $(\mathcal{A}) : H_{\mathcal{A}}^{\infty} = (P_{\mathcal{A}})_{K\{Y\}}$ and

$$P_{\mathcal{A}} = \left((\mathcal{A}) : H_{\mathcal{A}}^{\infty} \right) \cap K[V].$$

Indeed, $\forall f \in (\mathcal{A}) : H_{\mathcal{A}}^{\infty}$, $\exists\, m \in \mathbb{N}$ s.t. $H_{\mathcal{A}}^{m} \cdot f = \sum_{A \in \mathcal{A}} C_A \cdot A$, $C_A \in K\{Y\}$.

Rewrite f and C_A as δ-polys in $\Theta(Y) \setminus V$ with coefficients in $K[V]$, then $f = \sum_i f_i(V) \cdot M_i$ and $C_A = \sum_i C_{A,i}(V) \cdot M_i$ with M_i being distinct δ-monomials in $\Theta(Y) \setminus V$. Then we have

$$H_{\mathcal{A}}^{m} \cdot f_i = \sum_{A \in \mathcal{A}} C_{A,i} \cdot A, \text{ i.e., } f_i \in P_{\mathcal{A}}. \text{ So } f \in (P_{\mathcal{A}})$$

and $(\mathcal{A}) : H_{\mathcal{A}}^{\infty} = (P_{\mathcal{A}})_{K\{Y\}}$. Similarly, $P_{\mathcal{A}} = ((\mathcal{A}) : H_{\mathcal{A}}^{\infty}) \cap K[V]$ follows.

By Lemma 5.2.1, $\mathrm{Sat}(\mathcal{A}) \cap K[V] = (\mathcal{A}) : H_{\mathcal{A}}^{\infty} \cap K[V] = P_{\mathcal{A}}$.

Since $\mathrm{sat}(\mathcal{A})$ is a prime δ-ideal, $P_{\mathcal{A}}$ is a prime ideal and consequently, $(\mathcal{A}) : H_{\mathcal{A}}^{\infty} = (P_{\mathcal{A}})_{K\{Y\}}$ is prime too.

Since $\mathcal{A}$ is a δ-characteristic set of $\mathrm{Sat}(\mathcal{A})$, $(\mathcal{A}) : H_{\mathcal{A}}^{\infty}$ contains no nonzero polynomial reduced w.r.t. $\mathcal{A}$.

"$\Leftarrow$" We first show $\mathrm{Sat}(\mathcal{A})$ is a prime δ-ideal.

Given $f_1, f_2 \in K\{Y\}$ with $f_1 f_2 \in \mathrm{Sat}(\mathcal{A})$. Let $r_i = \delta\text{-rem}(f_i, \mathcal{A})$.

Then $\exists\, m_1, m_2 \in \mathbb{N}$ s.t. $H_{\mathcal{A}}^{m_1} \cdot f_1 \equiv r_1 \mod [\mathcal{A}]$ and

$$H_{\mathcal{A}}^{m_2} \cdot f_2 \equiv r_2 \mod [\mathcal{A}].$$

$\Rightarrow r_1 r_2 \in \mathrm{sat}(\mathcal{A})$ and $r_1 r_2$ is partially reduced w.r.t. $\mathcal{A}$.

By Lemma 5.2.1, $r_1 r_2 \in (\mathcal{A}):H_{\mathcal{A}}^{\infty}$. Since $(\mathcal{A}):H_{\mathcal{A}}^{\infty}$ is prime, $r_1 \in (\mathcal{A}):H_{\mathcal{A}}^{\infty}$ or $r_2 \in (\mathcal{A}):H_{\mathcal{A}}^{\infty}$. So $f_1 \in \mathrm{sat}(\mathcal{A})$ or $f_2 \in \mathrm{sat}(\mathcal{A})$ and $\mathrm{sat}(\mathcal{A})$ is a prime δ-ideal.

It remains to show that $\mathcal{A}$ is a char set of $\mathrm{sat}(\mathcal{A})$. Let $f \in \mathrm{sat}(\mathcal{A})$ and $r = \delta\text{-rem}(f, \mathcal{A})$. Then by Lemma 5.2.1, $r \in (\mathcal{A}):H_{\mathcal{A}}^{\infty}$. Since $(\mathcal{A}):H_{\mathcal{A}}^{\infty}$ contains no nonzero poly reduced w.r.t. $\mathcal{A}$, $r=0$. Thus, $\mathcal{A}$ is a characteristic set of $\mathrm{sat}(\mathcal{A})$. 証.

<u>Remark</u> Given an autoreduced set $\mathcal{A} \subseteq K\{Y\}$, denote V to be the set of all derivatives appearing effectively in $\mathcal{A}$. By the proof of Lemma 6.2.2,

$\mathcal{A}$ is a δ-char set of a prime δ-ideal

$\Leftrightarrow$ $\mathcal{A}$ is a char set of a prime algebraic ideal in $K[V]$.