Chapter 6 Decomposition Algorithms for Algebraic partial Differential Equations

Recall that a partial diff ring (R, Δ) is a commutative ring R with unity together with a finite set $\Delta = \{\delta_1, \dots, \delta_m\}$ of mutually commuting derivation operators (i.e., $\forall a \in R$, $\delta_i(\delta_j(a)) = \delta_j(\delta_i(a))$.)

Example: $\left(\mathbb{C}(x_1, \dots, x_m), \left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right\}\right)$

Notation. $\Theta = \{\delta_1^{i_1}\delta_2^{i_2}\cdots\delta_m^{i_m} \mid i_j \in \mathbb{N}, j=1,\dots,m\}$.

Let (K, Δ) be a diff field of char 0 with $\Delta = \{\delta_1, \dots, \delta_m\}$.

We consider the diff poly ring

$$K\{Y_1, \dots, Y_n\} = K[\Theta(Y_i) : i=1,\dots,n].$$

($K\{Y_1,\dots,Y_n\}$ is Ritt-Noetherian, i.e., for every radical Δ-ideal $J \subseteq K\{Y_1,\dots,Y_n\}$, $\exists f_1,\dots,f_l$ s.t. $J = \{f_1,\dots,f_l\}$.)

A diff ranking $\mathcal{R}$ on $K\{Y_1,\dots,Y_n\}$ is a total order on $\Theta(Y) = \{\theta(y_j) \mid \theta \in \Theta, j=1,\dots,n\}$ s.t.
$$\begin{cases} ① \ u < \theta(u), \ \forall \theta \in \Theta \setminus \{1\} \\ ② \ u < v \Rightarrow \theta(u) < \theta(v) \text{ for } \forall \theta \in \Theta. \end{cases}$$

<u>Remark</u>: In the ordinary diff case ($m=1$), there is a unique ranking on $K\{Y\}$; but here even for $n=1$ and $m=2$, there are uncountable number of rankings on $K\{Y\}$.

For example, we can define R_1: $\delta_1^{i_1}\delta_2^{i_2}(y) < \delta_1^{j_1}\delta_2^{j_2}(y) \Leftrightarrow (i_1, i_2) <_{lex} (j_1, j_2)$

and R_2: $\delta_1^{i_1}\delta_2^{i_2}(y) < \delta_1^{j_1}\delta_2^{j_2}(y) \Leftrightarrow i_1 + i_2 < j_1 + j_2$ or $i_1 + i_2 = j_1 + j_2$
& $(i_1, i_2) <_{lex} (j_1, j_2)$.

Fix a ranking R. Given a Δ-poly $f \in K\{y_1, \dots, y_n\} \setminus K$,

$$ld(f) = \max\{ u \in \Theta(Y) \mid \deg(f, u) > 0 \}.$$

Rewrite $f = I_0 (ld(f))^d + I_1 (ld(f))^{d-1} + \dots + I_{d-1} \, ld(f) + I_0$,

$$I_f := I_0$$

$$S_f := \frac{\partial f}{\partial\, ld(f)}$$

$$rk(f) := (ld(f), d) \quad (\text{or } ld(f)^d)$$

Concerning autoreduced sets, in the partial diff case here, we need Dickson's lemma to prove

i) Every autoreduced set is finite.

ii) Every nonempty set of autoreduced set has a minimal element.

iii) A strictly decreasing sequence of autoreduced sets has finite terms.

$\mathcal{A}$: $A_1 < A_2 < \dots < A_l$ (i.e., $rk(A_1) <_{lex} rk(A_2) <_{lex} \dots < rk(A_l)$)

Remark In an autoreduced set $\mathcal{A}$, there may be two Δ-polys whose leaders are derivatives of the same y_i.

Example ($m=2$): $\mathcal{A} = \delta_1(y_1), \delta_2(y_1)$.

We now give Δ-Reduction algorithms.

Lemma 6.1 Let $\mathcal{A}$ be an autoreduced set. There is a mechanical procedure to compute, for any $F \in K\{y_1, \dots, y_n\}$, the partial remainder $\tilde{F}$ of F with respect to A, and $s_A \in \mathbb{N}$ ($A \in \mathcal{A}$) such that the rank of $\tilde{F}$ is lower than or equal to that of F and

$$\prod_{A \in \mathcal{A}} S_A^{s_A} \cdot F \equiv \tilde{F} \mod [\mathcal{A}].$$

Moreover, $\prod_{A \in \mathcal{A}} S_A^{s_A} F - \tilde{F}$ can be written as a linear combination over $K\{y_1, \dots, y_n\}$ of derivatives $\theta(A)$ such that $\theta(\mathrm{ld}(A))$ is lower than or equal to the leader of F.

Proof. If F is partially reduced w.r.t. $\mathcal{A}$, set $\tilde{F} = F$ and $s_A = 0$. Obviously, $\tilde{F}$ and the numbers s_A have the desired properties.

We suppose that F is not partially reduced w.r.t. $\mathcal{A}$.

Let $D(F,\mathcal{A}) = \{ \theta(u_A) \mid \theta \in \Theta \setminus \{1\}, A \in \mathcal{A}, \deg(F, \theta(u_A)) > 0 \}$. ($u_A = ld(A)$)

Then $D(F,\mathcal{A}) \neq \phi$. Let $v = v(F,\mathcal{A}) = \max D(F,\mathcal{A})$ (unique) and let

$$C \triangleq \max \{ A \in \mathcal{A} \mid \theta(u_A) = v \} \text{ (unique)}.$$

Since $\theta(C) = S_C v - T$ for $T \in K\{Y\}$ and the rank of T is lower than v. Letting $e = \deg(F, v)$, we may write

$$F = \sum_{i=0}^{e} J_i \cdot v^i, \text{ where } J_i \in K\{Y_1, \dots, Y_n\} \text{ are free of } v.$$

Then $S_C^e F = \sum_{i=0}^{e} S_C^{e-i} J_i (S_C v)^i \equiv \sum_{i=0}^{e} S_C^{e-i} J_i T^i \mod (\theta(C))$.

Obviously, the Δ-poly $G = \sum_{i=0}^{e} S_C^{e-i} J_i T^i$ cannot involve a proper derivative of any u_A as high as v, $\theta(u_C) = v \leq u_F$, and the rank of G is no higher than that of F. Then either $D(G,\mathcal{A}) \neq \phi$ or $v(G,\mathcal{A}) < v(F,\mathcal{A})$. Perform the above procedure for G. Since a strictly decreasing sequence of derivatives in $\Theta(Y)$ is finite, this procedure terminates with $D(\tilde{F},\mathcal{A}) = \phi$. And the obtained $\tilde{F}$ satisfies the desired properties.

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<u>Prop 6.2</u> Let $\mathcal{A}$ be an autoreduced set in $K\{Y\}$ and $F \in K\{Y\}$. There is a mechanical procedure to compute the remainder F_0 of F w.r.t. $\mathcal{A}$ (Δ-rem$(F,\mathcal{A})$) and $i_A, S_A \in \mathbb{N}$ $(A \in \mathcal{A})$ s.t.

the rank of F_0 is lower than or equal to that of F,

and $\prod_{A\in\mathcal{A}} I_A^{i_A} S_A^{s_A} \cdot F \equiv F_0 \mod [\mathcal{A}]$.

More precisely, $\prod_{A\in\mathcal{A}} I_A^{i_A} S_A^{s_A} \cdot F - F_0$ can be written as a linear combination over $K\{Y\}$ of derivatives $\theta(A)$ such that $A \in \mathcal{A}$ and $\theta(u_A) \le ld(F)$.

<u>Proof.</u> Let $\tilde{F}$ be the partial remainder of F w.r.t. $\mathcal{A}$ as computed in Lemma 6.1. Let $\mathcal{A} = A_1, \dots, A_\ell$ and $A_k = I_k u_{A_k}^{d_k} + I_{k1} u_{A_k}^{d_k-1} + \dots + I_{kd_k}$.

Let $i_\ell = \begin{cases} \deg(F, u_{A_\ell}) - d_\ell + 1 & \deg(F, u_{A_\ell}) \ge d_\ell \\ 0 & \text{otherwise} \end{cases}$

Then $I_{A_\ell}^{i_\ell} \tilde{F} \equiv \tilde{F}^{(\ell)} \mod (A_\ell)$ where $\tilde{F}^{(\ell)}$ is partially reduced w.r.t. $\mathcal{A}$, is reduced w.r.t. A_ℓ, and has rank lower than or equal to the rank of $\tilde{F}$.

Then perform this pseudo reduction procedure for $\tilde{F}^{(l)}$ and continuing in this way, we can successively compute $i_{l-1}, \tilde{F}^{(l-1)}, i_{l-2}, \tilde{F}^{(l-2)}, \dots, i_1, \tilde{F}^{(1)}$ where $\tilde{F}^{(k)}$ is partially reduced w.r.t. $\mathcal{A}$, is reduced w.r.t. $A_k, \dots, A_l$, and has rank lower than or equal to the rank of $\tilde{F}$ and $I_k^{i_k} \cdots I_l^{i_l} \tilde{F} \equiv \tilde{F}^{(k)} \mod (A_k, \dots, A_l)$

Take $F_0 := \tilde{F}^{(1)}$ which satisfies the desired property. $\boxtimes$.

Coherence of Autoreduced Sets

In the ordinary diff case, an autoreduced set is a characteristic set of a prime δ-ideal if and only if it is irreducible. In the partial diff case, irreducible autoreduced set might be contradictory as shown in the following example:

Example 1 Let $K = (\mathbb{C}(x_1, x_2), \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\})$ and $\{A_1, A_2\} \subseteq K\{Y_1, Y_2\}$

$$A_1 = \frac{\partial}{\partial x_1}(Y_2) + \frac{\partial}{\partial x_1}(Y_1),$$

$$A_2 = \frac{\partial}{\partial x_2}(Y_2) + \frac{\partial}{\partial x_2}(Y_1) - Y_1$$

Take the elimination ranking $Y_1 < Y_2$ with

$$\frac{\partial^{l_1+l_2}(Y_1)}{\partial X_1^{l_1}\partial X_2^{l_2}} < \frac{\partial^{l_1'+l_2'}(Y_1)}{\partial X_1^{l_1'}\partial X_2^{l_2'}} \Leftrightarrow \begin{array}{l} l_2 > l_2' \text{ or} \\ l_2 = l_2' \ \& \ l_1 > l_1'. \end{array}$$

So A_1, A_2 is an irreducible autoreduced set.

But $\{A_1, A_2\}$ has no solutions. (For, $\frac{\partial A_1}{\partial X_2} - \frac{\partial A_2}{\partial X_1} = 1$)

A necessary and sufficient condition for the existence of a solution to a system of diff equations such as

$$\begin{cases} \dfrac{\partial Y}{\partial X_1} = f(X_1, X_2) \\ \dfrac{\partial Y}{\partial X_2} = g(X_1, X_2) \end{cases}$$

is that $\frac{\partial f}{\partial X_2} = \frac{\partial g}{\partial X_1}$.

Coherence is a property that generalizes this condition for systems given by autoreduced sets. Coherence implies formal integrability and is a cousin concept of Riquier's passivity (Riquier-Janet approach). Informally speaking,

Coherence $\longleftrightarrow$ passivity

Gröbner basis $\longleftrightarrow$ Involutive/Riquier bases.

Two derivatives $u, v \in \Theta(Y)$ have a common derivative if $\exists\, \phi, \varphi \in \Theta$ s.t. $\phi(u) = \varphi(v)$. This happens when u and v are derivatives of the same y_i. Assume $u = \delta_1^{e_1} \cdots \delta_m^{e_m}(y_i)$ and $v = \delta_1^{t_1} \cdots \delta_m^{t_m}(y_i)$. Then any $\delta_1^{g_1} \cdots \delta_m^{g_m}(y_i)$ with $g_i \geq \max\{e_i, t_i\}$ is a common derivative of u and v. If we take $g_i = \max\{e_i, t_i\}$, we obtain the lowest common derivative of u and v and denote it by $\mathrm{lcd}(u, v)$.

Def 6.3 Let a ranking be fixed. An autoreduced set $\mathcal{A} \subseteq K\{y_1, \ldots, y_n\}$ is called coherent if whenever $A, A' \in \mathcal{A}$ and v is a common derivative of u_A and $u_{A'}$, say $v = \theta(u_A) = \theta'(u_{A'})$, then

$$S_{A'}(\theta A) - S_A(\theta' A') \in (\Theta\mathcal{A}_{<v}) : H_{\mathcal{A}}^{\infty}$$

where $\Theta\mathcal{A}_{<v} = \{\tau(A) \mid \tau \in \Theta, A \in \mathcal{A}, \tau(u_A) < v\}$.

Testing coherence can be done with finitely many test. For each pair of Δ-polys A, A' in an autoreduced set, it is sufficient to look at the Δ-poly corresponding to the lowest common derivative between u_A and $u_{A'}$.

<u>Def 6.4</u> Let A and B be two Δ-polys in an autoreduced set. We define the Δ-polynomial of A and B, $\Delta(A, B)$, as follows:

$$\Delta(a,b) = \begin{cases} S_B \cdot \theta(A) - S_A \cdot \tau(B), & \text{if } \operatorname{lcd}(u_A, u_B) = \theta(u_A) = \tau(u_B); \\ 0, & \text{if } u_A \text{ and } u_B \text{ have no common derivatives.} \end{cases}$$

<u>Lemma 6.5</u> If a, b are elts of a Δ-ring R. For any $\theta \in \Theta$ with $\operatorname{ord}(\theta) = e$,

$$a^{e+1}\theta(b) \in (\tau(ab) \mid \tau \in \Theta, \tau | \theta).$$

proof. If $e=0$, it is trivial. Sps the property is true for all θ of order $\leq e$. Let $\theta \in \Theta$ with $\operatorname{ord}(\theta) = e+1$. Then $\exists\, \delta_i \in \Delta$ and $\theta' \in \Theta$ s.t. $\theta = \delta_i \theta'$. Since $\operatorname{ord}(\theta') = e$, $a^{e+1}\theta'(b) \in (\tau(ab) \mid \tau | \theta')$. That is,

$a^{e+1}\theta'(b) = \sum_{\tau | \theta'} C_\tau \cdot \tau(ab) \Rightarrow \delta_i(a^{e+1}\theta'(b)) = a^{e+1}\theta(b) +$
$(e+1)a^e \cdot \delta_i(a)\theta'(b) = \sum_{\tau|\theta'} [\delta_i(C_\tau)\tau(ab) + C_\tau\,\delta_i\tau(ab)]$. Thus,
$a^{e+2}\theta(b) \in (\tau(ab) \mid \tau | \theta)$. $\boxtimes$.

Prop 6.6 Let $\mathcal{A}$ be an autoreduced set.
If for all $A, B \in \mathcal{A}$, we have $\Delta(A,B) \in (\Theta\mathcal{A}_{<v}) : H_{\mathcal{A}}^{\infty}$
where $v = \mathrm{lcd}(u_A, u_B)$, then $\mathcal{A}$ is coherent.

Proof. First we show for any $v \in \Theta(Y)$,

if $f \in (\Theta\mathcal{A}_{<v}) : H_{\mathcal{A}}^{\infty}$, then

$\theta f \in (\Theta\mathcal{A}_{<\theta v}) : H_{\mathcal{A}}^{\infty}$ for any $\theta \in \Theta$.

Indeed, if $f \in (\Theta\mathcal{A}_{<v}) : H_{\mathcal{A}}^{\infty}$, then $\exists\, m \in \mathbb{N}$ s.t.

$H_{\mathcal{A}}^{m} \cdot f \in (\Theta\mathcal{A}_{<v})$, i.e. $H_{\mathcal{A}}^{m} \cdot f = \sum_{\tau(u_A) < v} C_{\tau,A}\,\tau(A)$.

By Lemma 6.5, $(H_{\mathcal{A}}^{m})^{\mathrm{ord}(\theta)+1} \cdot \theta(f) \in (\theta'(H_{\mathcal{A}}^{m} f) : \theta' | \theta)$.
For any $\theta' | \theta$, $\theta'(H_{\mathcal{A}}^{m} f) = \sum_{\tau(u_A) < v} \theta'(C_{\tau,A}\tau(A)) \in (\theta''(A) \mid \theta''(u_A) < \theta(v), A \in \mathcal{A})$.
Hence, $\theta(f) \in (\Theta\mathcal{A}_{<\theta(v)}) : H_{\mathcal{A}}^{\infty}$.

Now, sps $\Delta(A,B) \in (\textcircled{H} A_{<v}) : H_A^\infty$ ($v = \mathrm{lcd}(u_A, u_B)$) to show A is coherent.

Let $w = \theta_1(v)$ for some $\theta_1 \in \textcircled{H}$ be a common derivative of u_A and u_B. That is, $w = \theta_1\theta'(u_A) = \theta_1\theta''(u_B)$ and $\Delta(A,B) = S_B\theta'(A) - S_A\theta''(B) \in (\textcircled{H} A_{<v}) : H_A^\infty$.

So $\theta_1(\Delta(A,B)) \in (\textcircled{H} A_{<\theta_1 v}) : H_A^\infty$.

Thus, $S_B \cdot \theta_1\theta'(A) - S_A\, \theta_1\theta''(B)$

$$= \theta_1\left(S_B \cdot \theta'(A) - S_A\, \theta''(B)\right) - \sum_{\substack{\tau \mid \theta_1 \\ \tau \neq \theta_1}} \binom{\theta_1}{\tau} \frac{\theta_1}{\tau}(S_B) \cdot \tau\theta'(A)$$

$$+ \sum_{\substack{\tau \mid \theta_1 \\ \tau \neq \theta_1}} \binom{\theta_1}{\tau} \frac{\theta_1}{\tau}(S_A) \cdot \tau\theta''(B) \qquad \begin{pmatrix} \tau\theta'(u_A) < w = \theta_1(v) \\ \tau\theta''(u_B) < w = \theta_1(v) \end{pmatrix}$$

$$\in (\textcircled{H} A_{<\underset{w}{\underset{\shortparallel}{\theta_1 v}}}) : H_A^\infty.$$

So A is coherent. $\boxed{\Sigma}$.

The simplest test for coherence is thus the following. It gives only a sufficient condition

Prop 6.7 Let $\mathcal{A}$ be an autoreduced set.
If for all $A, B \in \mathcal{A}$, $\Delta\text{-rem}(\Delta(A,B), \mathcal{A}) = 0$,
then $\mathcal{A}$ is coherent.

Proof. Note that $\mathrm{ld}(\Delta(A,B)) < V = \mathrm{lcd}(u_A, u_B)$.

By prop 6.2, $\exists\, H \in H_{\mathcal{A}}$ s.t.

$$H \cdot \Delta(A,B) \in \left(\Theta \mathcal{A}_{\leq \mathrm{ld}(\Delta(A,B))} \right)$$

$$\subseteq \left(\Theta \mathcal{A}_{< V} \right). \qquad \blacksquare$$

Thm 6.8 (Rosenfeld's lemma)

Let $\mathcal{A}$ be a coherent autoreduced set in $K\{Y\}$. Let $f \in [\mathcal{A}] : H_{\mathcal{A}}^{\infty}$. If f is partially reduced w.r.t. $\mathcal{A}$, then $f \in (\mathcal{A}) : H_{\mathcal{A}}^{\infty}$.

Proof. Since $f \in [\mathcal{A}] : H_{\mathcal{A}}^{\infty}$, there exists a finite subset D of $\Theta^{+} \times \mathcal{A}$ s.t. for some $H \in H_{\mathcal{A}}^{\infty}$ we can

Write $Hf = \sum_{(\theta,A)\in D \subseteq \Theta^+ \times \mathcal{A}} C_{\theta,A}\,\theta(A) + \sum_{A\in\mathcal{A}} g_A \cdot A$ (1)

for some $C_{\theta,A}$ and $g_A \in K\{Y\}$, where $\Theta^+ = \{\theta\in\Theta \mid \mathrm{ord}(\theta)>0\}$.

Assume f is partially reduced w.r.t. $\mathcal{A}$. If $D=\phi$, then $f\in(\mathcal{A}):H_{\mathcal{A}}^{\infty}$. Assume, for contradiction, that there is no relation of type (1) with $D=\phi$ for f. And for each relation of type (1), set v to be the highest derivative in $\Theta^+(ld(\mathcal{A}))$ that appears effectively in the right hand side. Among all the possible relationships (1) that can be written, take one for which v is minimal.

Let $E=\{(\theta,A)\in D \mid \theta(u_A)=v\}$ and select one, say $(\theta',A')\in E$. Then

$$S_{A'}\cdot Hf = \sum_{(\theta,A)\in E} C_{\theta,A} S_{A'}\theta(A) + S_{A'}\cdot\sum_{(\theta,A)\in D\setminus E} C_{\theta,A}\,\theta(A) + \sum_{A\in\mathcal{A}} S_{A'} g_A\cdot A$$

$$= \sum_{(\theta,A)\in E} C_{\theta,A}\left(S_{A'}\theta(A) - S_A\theta'(A')\right) + \left(\sum_{(\theta,A)\in E} C_{\theta,A} S_A\right)\cdot\theta'(A')$$

$$+ S_{A'}\cdot\left[\sum_{(\theta,A)\in D\setminus E} C_{\theta,A}\,\theta(A) + \sum_{A\in\mathcal{A}} g_A\cdot A\right]. \quad (2)$$

Since $\mathcal{A}$ is coherent, $S_{A'}\theta(A) - S_A\theta'(A') \in (\Theta\mathcal{A}_{<v}):H_{\mathcal{A}}^{\infty}$.

So from (2), there exist $H_1 \in H_{\mathcal{A}}^{\infty}$, $\beta_{\theta',A'}$, g_A', $\beta_{\theta,A} \in K\{Y\}$ s.t.

$$H_1 f = \beta_{\theta',A'} \cdot \theta'(A') + \sum_{\substack{(\theta,A)\in\Theta^+\times\mathcal{A} \\ \theta(u_A)<v}} \beta_{\theta,A} \cdot \theta(A) + \sum_{A\in\mathcal{A}} g_A' \cdot A. \quad (3)$$

Note that $\theta'(A') = S_{A'} \cdot v + T$ with T free of v.

Substituting $v = -\frac{T}{S_{A'}}$ in (3) and multiplying a suitable power of $S_{A'}$ to clear denominators, we have

$$H_2 f = \sum_{\substack{(\theta,A)\in\Theta^+\times\mathcal{A} \\ \theta(u_A)<v}} \beta'_{\theta,A}\, \theta(A) + \sum_{A\in\mathcal{A}} g_A'' \cdot A,$$

for some $H_2 \in H_{\mathcal{A}}^{\infty}$, $\beta'_{\theta,A}$ and $g_A'' \in K\{Y\}$, which is a relation of type (1) for f in which either $D=\phi$ or v is replaced by a derivative lower than v. This contradiction completes the proof. $\boxtimes$.

Theorem 6.9 Let $\mathcal{A}$ be an autoreduced set in $K\{Y\}$. If $\mathcal{A}$ is a characteristic set of a prime Δ-ideal $P \subseteq K\{Y\}$, then $P = [\mathcal{A}] : H_{\mathcal{A}}^{\infty}$, $\mathcal{A}$ is coherent, and $(\mathcal{A}) : H_{\mathcal{A}}^{\infty}$ is a prime ideal not containing a nonzero elt reduced w.r.t. $\mathcal{A}$. Conversely, if $\mathcal{A}$ is a coherent autoreduced set s.t. $(\mathcal{A}) : H_{\mathcal{A}}^{\infty}$ is prime and doesn't contain a nonzero element reduced w.r.t. $\mathcal{A}$, then $\mathcal{A}$ is a characteristic set of a prime Δ-ideal of $K\{Y\}$.

Proof. First, suppose $\mathcal{A}$ is a Δ-char set of a prime Δ-ideal P. Then $H_{\mathcal{A}} \notin P$ and $P = [\mathcal{A}] : H_{\mathcal{A}}^{\infty}$ follows.

Since $\Delta(A, B) \in P$ for any $A, B \in \mathcal{A}$, $\Delta\text{-rem}(\Delta(A,B), \mathcal{A}) = 0$ and by prop 6.7, $\mathcal{A}$ is coherent. Let V be the minimal subset of $\Theta(Y)$ s.t. $\mathcal{A} \subseteq K[V]$. Then by Rosenfeld's lemma,

$$[\mathcal{A}] : H_{\mathcal{A}}^{\infty} \cap K[V] = (\mathcal{A}) : H_{\mathcal{A}}^{\infty} \cap K[V],$$ which is prime. Thus, $(\mathcal{A}) : H_{\mathcal{A}}^{\infty} = \left((\mathcal{A}) : H_{\mathcal{A}}^{\infty} \cap K[V] \right)_{K\{Y\}}$ is prime and $(\mathcal{A}) : H_{\mathcal{A}}^{\infty}$ contains no nonzero elt reduced w.r.t. $\mathcal{A}$.

Conversely, sps $\mathcal{A}$ is a coherent autoreduced set and $(\mathcal{A}):H_{\mathcal{A}}^{\infty}$ is prime which doesn't contain a nonzero elt reduced w.r.t. $\mathcal{A}$. To show $\mathrm{Sat}(\mathcal{A})=[\mathcal{A}]:H_{\mathcal{A}}^{\infty}$ is prime and $\mathcal{A}$ is a char set of $\mathrm{Sat}(\mathcal{A})$. For $f_1, f_2 \in K\{Y\}$ with $f_1 f_2 \in \mathrm{Sat}(\mathcal{A})$, let $r_1 = \Delta\text{-rem}(f_1, \mathcal{A})$ and $r_2 = \Delta\text{-rem}(f_2, \mathcal{A})$. Then $r_1 r_2 \in \mathrm{Sat}(\mathcal{A})$. Since $r_1 r_2$ is partially reduced w.r.t. $\mathcal{A}$, $r_1 r_2 \in (\mathcal{A}):H_{\mathcal{A}}^{\infty}$ by Rosenfeld's lemma. Since $(\mathcal{A}):H_{\mathcal{A}}^{\infty}$ is prime, $r_1 \in (\mathcal{A}):H_{\mathcal{A}}^{\infty}$ or $r_2 \in (\mathcal{A}):H_{\mathcal{A}}^{\infty}$. So $f_1 \in \mathrm{Sat}(\mathcal{A})$ or $f_2 \in \mathrm{Sat}(\mathcal{A})$. Thus, $\mathrm{Sat}(\mathcal{A})$ is a prime Δ-ideal. Given any $f \in \mathrm{Sat}(\mathcal{A})$, let $r = \Delta\text{-rem}(f, \mathcal{A})$. Since $r \in \mathrm{Sat}(\mathcal{A})$ is reduced w.r.t. $\mathcal{A}$, $r \in (\mathcal{A}):H_{\mathcal{A}}^{\infty}$ and thus $r=0$. Thus, $\mathcal{A}$ is a char set of $\mathrm{Sat}(\mathcal{A})$, a prime Δ-ideal. $\boxtimes$.

<u>Remark:</u> An autoreduced set $\mathcal{A}$ is a char set of a prime Δ-ideal $\Longleftrightarrow$ $\mathcal{A}$ is irreducible and coherent.

$\Sigma \subseteq K\{Y\}$: a finite set of nonzero Δ-polys.

<u>Well-ordering principle</u>

$$\begin{array}{llll} \Sigma_0 = \Sigma & \Sigma_1 = \Sigma_0 \cup R_0 & \cdots & \Sigma_e = \Sigma_{e-1} \cup R_{e-1} \\ \mathcal{A}_0 = b.s.(\Sigma_0) > & \mathcal{A}_1 = b.s.(\Sigma_1) > & \cdots > & \mathcal{A}_e \\ R_0 \neq \phi & R_1 \neq \phi & \cdots & R_e = \phi \end{array}$$

Here, $\mathcal{A}_i$ is a minimal autoreduced set contained in Σ_i,

$R_i = \{ \Delta\text{-rem}(g, \mathcal{A}_i) \mid g \in \Sigma_i \setminus \mathcal{A}_i$ or $g = \Delta(A, B)$ for $A, B \in \mathcal{A}_i \} \setminus \{0\}$.

Since $\mathcal{A}_0 > \mathcal{A}_1 > \mathcal{A}_2 > \cdots$, $\exists\, e \in \mathbb{N}$ s.t. $R_e = \phi$,

$\mathcal{A} = \mathcal{A}_e$ is a coherent autoreduced set in $[\Sigma]$

satisfying $\Delta\text{-rem}(\Sigma, \mathcal{A}) = \{0\}$.

As in the ordinary diff case, we have the following

zero decomposition theorems:

<u>Theorem 6.10</u> (Zero decomposition theorem: weak form)

There is an algorithmic procedure to compute for any finite

$\Sigma \subseteq K\{Y\}$ a finite set of coherent autoreduced sets $\mathcal{A}_1, \ldots, \mathcal{A}_\ell$

such that $\mathbb{V}(\Sigma) = \bigcup_{i=1}^{\ell} \mathbb{V}(\mathcal{A}_i / H_{\mathcal{A}_i})$,

where $\Delta\text{-rem}(\Sigma, \mathcal{A}_i) = \{0\}$ for each i.

<u>Theorem 6.11</u> (Irreducible Decomposition Theorem: partial diff case) There is an algorithmic procedure which permits to detect whether $\mathbb{V}(\Sigma)=\phi$ for any finite subset $\Sigma \subseteq K\{Y\}$ or in the nonempty case, to decompose

$$\mathbb{V}(\Sigma) = \bigcup_{i=1}^{l} \mathbb{V}(\mathcal{A}_i/H_{\mathcal{A}_i}) = \bigcup_{i=1}^{l} \mathbb{V}(\mathrm{Sat}(\mathcal{A}_i))$$

in which each $\mathcal{A}_i$ is an irreducible coherent autoreduced set.