Recall: Last week, we studied the notions of differential ranking, autoreduced set and characteristic set:

• A differential ranking $\mathscr{R}$ is a total ordering on $\Theta(Y) := \{y_i^{(k)} : k \geq 0, 1 \leq i \leq n\}$ satisfying (1) $v < \delta(v)$ and (2) $v < u \Rightarrow \delta(v) < \delta(u)$. It is a well-ordering.

• Given $f \in K\{Y\}$, the leader/initial/rank/separant of f is dentoed by $u_f, \mathrm{I}_f, \mathrm{S}_f, \mathrm{rk}(f)$. A polynomial g is partially reduced w.r.t. f if no proper derivative of u_f appears in g; and in addtion, if $\deg(g, u_f) < \deg(f, u_f)$, then g is reduced w.r.t f.

• An autoreduced set is a set $\mathcal{A} \subset K\{Y\}$ with each element reduced w.r.t. all the other elements. A characteristic set of a differential ieal I is an autoreduced set of lowest rank contained in I.

We start to introduce *pesudo-division* of differential polynomials:

Lemma 2.2.11. *Let $\mathcal{A} = A_1, \ldots, A_p$ be an autoreduced set in $K\{Y\}$ and $F \in K\{Y\}$. Then there exist $\tilde{F} \in K\{Y\}$ and $t_i \in \mathbb{N}$ satisfying*

1) $\tilde{F}$ is partially reduced with respect to $\mathcal{A}$ (i.e., $\tilde{F}$ is partially reduced w.r.t. each A_i),

2) the rank of $\tilde{F}$ is not higher than that of F,

3) $\prod_{i=1}^{p} \mathrm{S}_{A_i}^{t_i} F \equiv \tilde{F} \bmod [\mathcal{A}]$.

More precisely, $\prod_{i=1}^{p} \mathrm{S}_{A_i}^{t_i} F - \tilde{F}$ can be expressed as a linear combination of derivatives $\theta(A_i)$ with coefficients in $K\{Y\}$ such that $\theta(u_{A_i}) \leq u_F$.

Proof. If F is partially reduced with respect to $\mathcal{A}$, then set $\tilde{F} = F$ and $t_i = 0\,(i \leq p)$. Otherwise, F contains a proper derivative $\delta^k(u_{A_i})$ of the leader of some A_i. Let v_F be the maximal one among all such derivatives. We shall prove the lemma by induction on v_F. Suppose for all $G \in K\{Y\}$ that doesn't involve a proper derivative of any u_{A_i} of rank $\geq v_F$, the corresponding $\tilde{G}$ and natural numbers are defined satisfying the desired properties. There exists a unique $A \in \mathcal{A}$ such that $v_F = \delta^k(u_A)$ for some $k > 0$. If $A = \sum_{i=0}^{d} I_i {u_A}^i$, then

$$\delta^k(A) = \mathrm{S}_A \delta^k(u_A) + T \text{ with } T \text{ having lower rank than } \delta^k(u_A) = v_F.$$

Denoting $l = \deg(F, v_F)$ and write F as $F = \sum_{i=0}^{l} J_i {v_F}^i$ where $J_0, \ldots, J_l$ don't involve proper derivatives of any u_{A_i} of rank $\geq v_F$. Then we have

$$\mathrm{S}_A{}^l F = \sum_{i=0}^{l} J_i \mathrm{S}_A{}^{l-i} (\mathrm{S}_A v_F)^i \equiv \sum_{i=0}^{l} J_i \mathrm{S}_A{}^{l-i} (-T)^i \bmod (\delta^k(A)).$$

Clearly, $G = \sum_{i=0}^{l} J_i \mathrm{S}_A{}^{l-i} (-T)^i$ doesn't involve proper derivatives of any u_{A_i} of rank $\geq v_F$. By the induction hypothesis, $\exists\ \tilde{G}$ partially reduced with respect to $\mathcal{A}$ and $k_i \in \mathbb{N}$ such that $\prod_{i=1}^{p} \mathrm{S}_{A_i}^{k_i} G \equiv \tilde{G} \bmod [\mathcal{A}]$. Now it suffices to set $\tilde{F} = \tilde{G}$, $t_i = \begin{cases} k_i, & A_i \neq A \\ k_i + l, & A_i = A \end{cases}$. □

Remark: $\tilde{F}$ constructed by the process in the proof is called the **partial remainder** of F w.r.t $\mathcal{A}$.

Let us recall the *pseudo reduction algorithm* in commutative algebra:

Let D be an integral domain and we consdier the polynomial ring $D[v]$ (v is an indeterminate over D). Let $F, A \in D[v]$ be of respective degrees $d_F, d_A (\geq 0)$ and assume

$$A = I_{d_A} v^{d_A} + \cdots + I_1 v + I_0$$

with $I_i \in D$. Let $e = \max\{d_F - d_A + 1, 0\}$. Then we can compute unique $Q, R \in D[v]$ with $\deg(R, v) < \deg(A, v)$ such that

$$I_{d_A}^e F = QA + R.$$

Theorem 2.2.12. *Let $\mathcal{A} = A_1, \ldots, A_p$ be an autoreduced set in $K\{y_1, \ldots, y_n\}$. If $F \in K\{y_1, \ldots, y_n\}$, then $\exists\ F_0 \in K\{y_1, \ldots, y_n\}$ (called the* **differential remainder** *of F w.r.t. $\mathcal{A}$) and $r_i, t_i \in \mathbb{N}$ s.t.*

1) F_0 is reduced w.r.t $\mathcal{A}$,

2) The rank of F_0 is no higher than the rank of F,

3) $\prod_{i=1}^{p} \mathrm{S}_{A_i}^{t_i} \mathrm{I}_{A_i}^{r_i} F \equiv F_0 \mod [\mathcal{A}]$.

Proof. Let $\tilde{F}$ be the partial remainder of F with respect to $\mathcal{A}$ and $\prod_{i=1}^{p} \mathrm{S}_{A_i}^{t_i} F \equiv \tilde{F} \mod [\mathcal{A}]$. Let $r_p = \max\{0, \deg(F, u_{A_p}) - \deg(A_p, u_{A_p}) + 1\}$. Then $\exists\, F_{p-1} \in K\{Y\}$ partially reduced with respect to $\mathcal{A}$ and reduced with respect to A_p such that $\mathrm{I}_{A_p}^{r_p} \tilde{F} \equiv F_{p-1} \mod (A_p)$. If $p = 1$, then we are done. Otherwise, we can find r_{p-1} and $F_{p-2} \in K\{Y\}$ partially reduced with respect to $\mathcal{A}$ and reduced with respect to A_{p-1}, A_p s.t. $\mathrm{I}_{A_{p-1}}^{r_{p-1}} \mathrm{I}_{A_p}^{r_p} \tilde{F} \equiv F_{p-2} \mod (A_{p-1}, A_p)$ and is not higher than $\tilde{F}$. Continuing in this way, we get F_0 satisfying the desired properties.

□

Remark: The reduction procedures above could be summarized in an algorithm, called the *Ritt-Kolchin algorithm* to compute the δ-remainder of a δ-polynomial F with respect to an autoreduced set $\mathcal{A}$. Denote F_0 above by δ-rem$(F, \mathcal{A})$, or $F \underset{\mathcal{A}}{\rightarrow} F_0$.

Example: Consider $K\{y_1, y_2\}$ and fix the orderly ranking with $y_1 > y_2$.

(1) Let $f = y_1$ and $\mathcal{A} = A_1 = y_2 y_1$. Here $f \underset{\mathcal{A}}{\rightarrow} 0$, and $\mathrm{I}_{A_1} f \in [\mathcal{A}]$.

(2) Let $f = y_1' + 1$ and $\mathcal{A} = A_1 = y_2 y_1^2$. $u_{A_1} = y_1$ and $\mathrm{S}_{A_1} = 2y_2 y_1$. Clearly, f is not partially reduced with respect to $\mathcal{A}$. Note that $\delta(A_1) = 2y_2 y_1 y_1' + y_2' y_1^2$. The partial remainder of f with respect to $\mathcal{A}$ is $2y_2 y_1 - y_2' y_1^2 = \tilde{f}$ and $\mathrm{S}_{A_1} f - \tilde{f} = A_1' \in [\mathcal{A}]$.

Since

$$\mathrm{I}_{A_1} \tilde{f} - \mathrm{I}_{\tilde{f}} A_1 = y_2(2y_2 y_1 - y_2' y_1^2) - (-y_2') y_2 y_1^2 = 2y_2^2 y_1$$

is reduced with respect to $\mathcal{A}$, $f \underset{\mathcal{A}}{\rightarrow} 2y_2^2 y_1$ and $\mathrm{I}_{A_1} \mathrm{S}_{A_1} f - 2y_2^2 y_1 = -y_2' A_1 + \mathrm{I}_{A_1} A_1' \in [\mathcal{A}]$.

Theorem 2.2.13. *Let $\mathcal{A}$ be an autoreduced set of a proper differential ideal $I \subseteq K\{y_1, \ldots, y_n\}$. Then the following are equivalent:*

(1) $\mathcal{A}$ is a characteristic set of I.

(2) $\forall f \in I$, δ-rem$(f, \mathcal{A})$=0.

(3) I doesn't contain a nonzero differential polynomial reduced with respect to $\mathcal{A}$.

Proof. (2) $\Leftrightarrow$ (3) is obvious.

"(1) $\Rightarrow$ (3)" Suppose $f \in I\backslash\{0\}$ is reduced with respect to $\mathcal{A} = A_1, \ldots, A_p$. Let $k \in \mathbb{N}$ be maximal such that $\mathrm{rk}(A_k) < \mathrm{rk}(f)$. Then $A_1, \ldots, A_k, f$ is an autoreduced set lower than $\mathcal{A}$. (Here, in the case $\mathrm{rk}(f) < \mathrm{rk}(A_1)$, take $k = 0$ and $\{f\}$ is an autoreduced set $< \mathcal{A}$.) Thus, we get a contradiction, and (3) is valid.

"(3) $\Rightarrow$ (1)" Assume (3) is valid. Suppose $\mathcal{A} = A_1, \ldots, A_p$ is not a characteristic set of I. Then $\exists\ \mathcal{B} = B_1, \ldots, B_q$, an autoreduced set of I of lower rank than $\mathcal{A}$. Thus, by definition, either (1) $\exists\ k \leq \min\{p, q\}$ such that for $i < k$, $\mathrm{rk}(A_i) = \mathrm{rk}(B_i)$ and $A_k > B_k$, or (2) $q > p$ and for $i \leq p$, $\mathrm{rk}(A_i) = \mathrm{rk}(B_i)$. Then either B_k or B_{p+1} is nonzero and reduced with respect to $\mathcal{A}$. □

Remark: By Theorem 2.2.13, if $\mathcal{A} = A_1, \ldots, A_p$ is a characteristic set of $I \subseteq K\{Y\}$, then $\mathrm{I}_{A_i}, \mathrm{S}_{A_i} \notin I$ $(i = 1, \ldots, p)$.

A characteristic set of I can be obtained by the following procedure (non-constructive) : choose $A_1 \in I$ of minimal rank. Choose A_2 of minimal rank in the set $\{f \in I \mid f$ is reduced with respect to $A_1\}$. Then A_1, A_2 is autoreduced. Choose A_3 of minimal rank in the set $\{f \in I \mid f$ is reduced with respect to $A_1, A_2\}$. Then A_1, A_2, A_3 is autoreduced. Continue like this. The process must terminate for an autoreduced set is finite. In the end, we will obtain an autoreduced set $\mathcal{A} := A_1, \ldots, A_p$ of I such that no polynomial in I is reduced with respect to $\mathcal{A}$. Clearly, $\mathcal{A}$ is a characteristic set of I.

Lemma 2.2.14. *Let $\mathcal{A}$ be a characteristic set of a proper differential ideal $I \subseteq K\{Y\}$. Denote $\mathrm{H}_{\mathcal{A}}^{\infty}$ to be the multiplicative set generated by initials and separants of elements in $\mathcal{A}$ and set*

$$\mathrm{sat}(\mathcal{A}) := [\mathcal{A}] : \mathrm{H}_{\mathcal{A}}^{\infty} = \{f \in K\{Y\} \mid \exists\, M \in \mathrm{H}_{\mathcal{A}}^{\infty}, Mf \in [\mathcal{A}]\}.$$

Then $I \subseteq \mathrm{sat}(\mathcal{A})$. Furthermore, if I is prime, $I = \mathrm{sat}(\mathcal{A})$.

Proof. For each $f \in I$, by Theorem 2.2.13, δ-rem$(f, \mathcal{A}) = 0$. Thus, $\exists\, i_A, t_A \in \mathbb{N}\,(A \in \mathcal{A})$ s.t. $\prod_{A \in \mathcal{A}} \mathrm{I}_A^{i_A} \mathrm{S}_A^{t_A} f \in [\mathcal{A}]$. That is, $f \in \mathrm{sat}(\mathcal{A})$.

Suppose I is prime. For each $f \in \mathrm{sat}(\mathcal{A})$, $\exists\ i_A, t_A$ s.t. $\prod_{A \in \mathcal{A}} \mathrm{I}_A^{i_A} \mathrm{S}_A^{t_A} f \in [\mathcal{A}] \subseteq I$. Since $\mathrm{I}_A, \mathrm{S}_A$ are not in I, $f \in I$ and $I = \mathrm{sat}(\mathcal{A})$ follows. □

Exercise: Develop a division algorithm as follows:

Input: $f \in K\{Y\}$ and an autoreduced set $\mathcal{A} = A_1, \ldots, A_p$ w.r.t. a fixed ranking.

Output: $g \in K\{Y\}$, the differential remainder of f w.r.t. $\mathcal{A}$. That is, g is reduced w.r.t. $\mathcal{A}$ and there exist $i_k, j_k \in \mathbb{N}$ s.t. $\mathrm{I}_{A_1}^{i_1} \cdots \mathrm{I}_{A_p}^{i_p} \mathrm{S}_{A_1}^{j_1} \cdots \mathrm{S}_{A_p}^{j_p} f - g \in [\mathcal{A}]$.

2.3 The Ritt-Raudenbush basis theorem

In the end of section 2.1, we gave an example showing that a differential ideal in $K\{Y\}$ might not be differentially finitely generated. For example,

$$I = [y^2, (y')^2, \ldots, (y^{(k)})^2, \ldots]$$

and

$$J = [yy', y'y'', \ldots, y^{(k)}y^{(k+1)}, \ldots]$$

are not differentially finitely generated. But note that $\{I\} = \{y\}$ and $\{J\} = \{yy'\}$ are differentially finitely generated as radical differential ideals. In this section, we will show every radical differential ideal in $K\{Y\}$ is differentially finitely generated as radical differential ideals.

Definition 2.3.1. *A differential ring is called* **Ritt-Noetherian** *if the set of radical differential ideals satisfies the ascending chain condition (ACC).*

Lemma 2.3.2. *Let (R, δ) be a differential ring. Then R is Ritt-Noetherian $\Leftrightarrow$ every radical differential ideal I of R is finitely generated as a radical differential ideal. (i.e. $\exists\, f_1, \ldots, f_s \in I$ s.t. $I = \{f_1, \ldots, f_s\}$).*

Proof. "$\Rightarrow$" Let I be an arbitrary radical differential ideal of R. Suppose I is not finitely generated as a radical differential ideal. Then we can construct a strict increasing sequence of radical differential ideals, i.e., $\{a_1\} \subsetneqq \{a_1, a_2\} \subsetneqq \cdots \subsetneqq \{a_1, a_2, \ldots, a_p\} \subsetneqq \cdots$.
"$\Leftarrow$" Let $I_1 \subseteq I_2 \subseteq \cdots$ be sequence of radical differential ideals. Take $I = \bigcup_{i=1}^{\infty} I_i$. Then I is a radical differential ideal. Thus, $\exists\, f_1, \ldots, f_s \in I$ s.t. $I = \{f_1, \ldots, f_s\}$. Since each $f_i \in I$, $\exists\, m \in \mathbb{N}$ s.t. $f_i \in I_m$ ($\forall\, i = 1, \ldots, s$). So $\{f_1, \ldots, f_s\} \subseteq I_m \subseteq I \Rightarrow I_m = I_{m+j} = \{f_1, \ldots, f_s\}$ for $j \in \mathbb{N}$. □

Lemma 2.3.3. *Let R be a diffeential ring with $\mathbb{Q} \subset R$. Let $S \subset R$ be a subset and $a \in R$ such that the radical differential ideal $\{S, a\}$ has a finite set of generators as a radical differential ideal. Then, there exists $s_1, \ldots, s_p \in S$ such that $\{S, a\} = \{s_1, \ldots, s_p, a\}$.*

Proof. By hypothesis, $\exists\, h_1, \ldots, h_l$ s.t. $\{a, S\} = \{h_1, \ldots, h_l\}$. For each i, $h_i \in \{a, S\} \Rightarrow \exists\, m_i$ s.t. $h_i^{m_i} \in [a, S]$. So $\exists\, s_1, \ldots, s_p \in S$ s.t. for each i, $h_i^{m_i} \in [a, s_1, \ldots, s_p]$. Thus, $h_i \in \{a, s_1, \ldots, s_p\} \subset \{a, S\} \Rightarrow \{h_1, \ldots, h_l\} \subseteq \{a, s_1, \ldots, s_p\} \subseteq \{a, S\}$.

□

Theorem 2.3.4. *Let (K, δ) be a differential field with $\mathbb{Q} \subseteq K$. The differential polynomial ring $K\{y_1, \ldots, y_n\}$ is Ritt-Noetherian.*

Proof. By Lemma 2.3.2, it suffices to prove that every radical differential ideal of $K\{y_1, \ldots, y_n\}$ is finitely generated as radical differential ideals. Suppose the contrary and $\exists$ a radical differential ideal of $K\{y_1, \ldots, y_n\}$ that is not finitely generated. By Zorn's lemma, $\exists$ a maximal radical differential ideal $J \subseteq K\{y_1, \ldots, y_n\}$ that is not finitely generated.

Claim: J is a prime differential ideal.

Proof of the claim. Suppose the contrary, then $\exists\, a, b \in K\{y_1, \ldots, y_n\}$ s.t. $a, b \notin J$ but $ab \in J$. Since $\{a, J\} \supsetneqq J$ and $\{b, J\} \supsetneqq J$, $\{a, J\}$ and $\{b, J\}$ are finitely generated as radical differential ideals. Then by Lemma 2.3.3, $\exists\, f_1, \ldots, f_s, g_1, \ldots, g_t \in J$ s.t. $\{a, J\} = \{a, f_1, \ldots, f_s\}$ and $\{b, J\} = \{b, g_1, \ldots, g_t\}$. Hence,

$$\begin{aligned} J^2 &\subseteq \{a, J\} \cdot \{b, J\} = \{a, f_1, \ldots, f_s\} \cdot \{b, g_1, \ldots, g_t\} \\ &\subseteq \{ab, ag_j, bf_i, f_i g_j : 1 \le i \le s, 1 \le j \le t\} \triangleq P \\ &\subseteq J. \end{aligned}$$

For each $f \in J$, $f^2 \in J^2 \subseteq P \Rightarrow f \in P \Rightarrow J = P = \{ab, ag_j, bf_i, f_i g_j : 1 \le i \le s, 1 \le j \le t\}$, contradicting the hypothesis that J is not finitely generated. The claim thus is proved.

Fix a ranking on $\Theta(Y)$ and take a characteristic set $\mathcal{A}$ of J under this ranking. Let $\mathcal{A} = A_1, \ldots, A_p$ and denote $\mathrm{I} \triangleq \prod_{i=1}^{p} \mathrm{I}_{A_i}, \mathrm{S} \triangleq \prod_{i=1}^{p} \mathrm{S}_{A_i}$. Since J is prime, $J = \mathrm{sat}(\mathcal{A}) = [\mathcal{A}] : \mathrm{H}_{\mathcal{A}}^{\infty} \subseteq \{\mathcal{A}\} : (\mathrm{IS})$. Since $\mathrm{I}_{A_i}, \mathrm{S}_{A_i} \notin J$ for each i, $\mathrm{IS} \notin J$. Thus $\{J, \mathrm{IS}\}$ is finitely generated as a radical differential ideal. That is, $\exists\, h_1, \ldots, h_l \in J$ s.t. $\{J, \mathrm{IS}\} = \{h_1, \ldots, h_l, \mathrm{IS}\}$. Thus,

$$\begin{aligned} J^2 &\subseteq J \cdot \{J, \mathrm{IS}\} = J \cdot \{h_1, \ldots, h_l, \mathrm{IS}\} \\ &\subseteq \{h_1, \ldots, h_l, \mathcal{A}\} (\text{for } \mathrm{IS} \cdot J \subseteq \{\mathcal{A}\}) \\ &\subseteq J. \end{aligned}$$

Hence, $J = \{h_1, \ldots, h_l, A_1, \ldots, A_p\}$, which leads to a contradiction. So every radical differential ideal of $K\{y_1, \ldots, y_n\}$ is finitely generated as a radical differential ideal. □

Theorem 2.3.5. *Let R be a differential ring which is Ritt-Noetherian and $\mathbb{Q} \subseteq R$. Then for every radical differential ideal $I \subsetneqq R$, there exist a finite number of prime differential ideals $P_1, \ldots, P_l$ s.t.*

$$I = \bigcap_{i=1}^{l} P_i. \tag{2.1}$$

Moreover, if (2.1) is irredundant ($\forall\, i, \bigcap_{j \neq i} P_j \nsubseteq P_i$), then this set of prime ideals is unique. In this case, $P_1, \ldots, P_l$ are called prime components *of I.*

Proof. Suppose the statement is false, i.e., the set $U = \{I \mid I \subsetneqq K\{y_1, \ldots, y_n\}$ is a radical differential ideal and I is not a finite intersection of prime differential ideals$\}$ is not empty. Since R is Ritt-Noetherian, every ascending chain of radical differential ideals has an upper bound in U. By Zorn's lemma, U has a maximal element $J \in U$. Clearly, J is not prime. So $\exists\, a, b \notin J$ but $ab \in J$. Thus, $\{J, a\} \supsetneqq J$ and $\{J, b\} \supsetneqq J$. Also, $\{J, a\} \neq R$. Indeed, if not, then $1 \in \{J, a\}$. Since $\mathbb{Q} \subseteq R, 1 \in [J, a]$ and $1 = f + \sum *\delta^k(a)$, where $f \in J$. By $ab \in J$ and J is radical, $b\delta^k(a) \in J\, \forall\, k \in \mathbb{N}$. So $b = fb + \sum *b\delta^k(a) \in J$, contradicting to $b \notin J$. Similarly, $\{J, b\} \neq R$ could be shown.

By the maximality of J, $\exists\, P_1^a, \ldots, P_l^a, P_{l+1}^b, \ldots, P_{l+t}^b$ prime differential ideals in R s.t.

$$\{J, a\} = P_1^a \cap \cdots \cap P_l^a \text{ and}$$
$$\{J, b\} = P_{l+1}^b \cap \cdots \cap P_{l+t}^b.$$

Now show $J = \{J, a\} \cap \{J, b\}$. Indeed, let $f \in \{J, a\} \cap \{J, b\}$, then $f^2 \in \{J, a\} \cdot \{J, b\} \subseteq \{J, ab\} \subseteq J \Rightarrow f \in J$. Thus, $J = \{J, a\} \cap \{J, b\} = P_1^a \cap \cdots \cap P_l^a \cap P_{l+1}^b \cap \cdots \cap P_{l+t}^b$, contradicting to the hypothesis $J \in U$. So the first statement is valid.

Uniqueness. Suppose $I = \bigcap_{i=1}^{l} P_i = \bigcap_{j=1}^{t} Q_j$ be irredundant intersections. For each $j = 1, \ldots, t$, $\bigcap_{i=1}^{l} P_i \subseteq Q_j$. Then $\exists\, i_0 \in \{1, \ldots, l\}$ s.t. $P_{i_0} \subseteq Q_j$. Indeed, suppose the contrary, then $\exists\, f_i \in P_i \backslash Q_j$ for each $i = 1, \ldots, l$. Thus, $f_1 f_2 \cdots f_l \in \bigcap_{i=1}^{l} P_i \subseteq Q_j$, which yields a contradiction. Similarly, $\exists\, j_0 \in \{1, \ldots, t\}$ s.t. $Q_{j_0} \subseteq P_{i_0} \subseteq Q_j$. Since $I = \bigcap_{j=1}^{t} Q_j$ is irredundant, $j_0 = j$ and $P_{i_0} = Q_j$. Thus, $l = t$ and $\exists$ a permutation $\sigma \in S_l$ s.t. $P_i = Q_{\sigma(i)}$. □

Corollary 2.3.6. *Every proper radical differential ideal $I \subsetneqq K\{y_1, \ldots, y_n\}$ ($\mathrm{char}(K) = 0$) can be written as a finite intersection of prime differential ideals. If $I = \bigcap_{i=1}^{l} P_i$ is irredundant, P_i are called prime components of I.*

Example: $I = \{y'^2 - 4y\} \subseteq \mathbb{Q}\{y\}$. Then $I = \{y'^2 - 4y, y'' - 2\} \cap \{y\}$ (Chapter 3).