Chapter 4

Extensions of differential fields

4.1 Extensions of derivations

Let (K, δ) be a differential field of characteristic 0. Let x be an indeterminate over K. Then δ can be extended to a derivation δ_0 on $K[x]$ s.t. $\delta_0(x) = 0$ given by $\delta_0(\sum_{i=0}^{l} r_i x^i) = \sum_{i=0}^{l} \delta(r_i) x^i$. There is also a derivation on $K[x]$ s.t. $\frac{\mathrm{d}}{\mathrm{d}x}(K) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$ given by $\frac{\mathrm{d}}{\mathrm{d}x}(\sum_{i=0}^{l} r_i x^i) = \sum_{i=1}^{l} i r_i x^{i-1}$. Of course, $\frac{\mathrm{d}}{\mathrm{d}x}$ does not extend δ.

Lemma 4.1.1. *Any derivation δ_1 on $K[x]$ which extends δ is given by*

$$\delta_1 = \delta_0 + \delta_1(x)\tfrac{\mathrm{d}}{\mathrm{d}x}.$$

Conversely, by defining $\delta_1(x) = p(x) \in K[x]$, $\delta_1 = \delta_0 + p(x)\frac{\mathrm{d}}{\mathrm{d}x}$ is a derivation on $K[x]$ extending δ.

Proof. First suppose δ_1 is a derivation on $K[x]$ extending δ. Then $\forall\, f = \sum_{i=0}^{r} r_i x^i \in K[x]$, $\delta_1(f) = \sum_{i=0}^{r} \delta(r_i) x^i + \sum_{i=1}^{r} i r_i x^{i-1} \delta_1(x) = \delta_0(f) + \delta_1(x)\frac{\mathrm{d}}{\mathrm{d}x}(f)$. So $\delta_1 = \delta_0 + \delta_1(x)\frac{\mathrm{d}}{\mathrm{d}x}$. Now let $\delta_1 : K[x] \to K[x]$ be defined by $\delta_1(f) = \delta_0(f) + \delta_1(x)\frac{\mathrm{d}}{\mathrm{d}x}(f)$. Then $\forall\, a \in K$, $\delta_1(a) = \delta_0(a) + \delta_1(x)\frac{\mathrm{d}}{\mathrm{d}x}(a) = \delta(a)$;

$$\forall\, f, g \in K[x],\ \delta_1(f+g) = \delta_0(f+g) + \delta_1(x)\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \delta_1(f) + \delta_1(g),$$
$$\delta_1(fg) = \delta_0(fg) + \delta_1(x)\frac{\mathrm{d}}{\mathrm{d}x}(fg) = \delta_1(f)g + f\delta_1(g).$$

Thus, δ_1 is a derivation which extends δ. □

Theorem 4.1.2. *Let $K \subseteq L$ be fields of characteristic 0. Then any derivation on K could be extended to a derivation on L. This extension is unique if and only if L is algebraic over K.*

Proof. Let δ be a derivation on K. We first consider the case that $L = K(\alpha)$ for some $\alpha \in L$. If α is transcendental over K, then by Lemma 4.1.1, there exists a derivation δ_0 on $K[\alpha]$ extending δ on K, and by Lemma 1.1.2, δ_0 can be extended to a derivation on $L = K(\alpha)$. Otherwise, let α be algebraic over K and suppose $F(x) \in K[x]$ is the minimal polynomial of α over K. δ can be extended to a derivation δ_0 on $K[x]$ by setting $\delta_0(x) = 0$. By Lemma 4.1.1, $\delta_1 = \delta_0 + g(x)\frac{\mathrm{d}}{\mathrm{d}x}$ is a derivation on $K[x]$ where $g(x) \in K[x]$ is a polynomial to be determined. We want to choose $g(x)$ s.t. δ_1 maps the ideal $(F)_{K[x]}$ to itself. The condition for this is that $\delta_1(F)(\alpha) = 0$, ie., $\delta_0(F)(\alpha) + g(\alpha)\frac{\mathrm{d}F}{\mathrm{d}x}(\alpha) = 0$. Since $\frac{\mathrm{d}F}{\mathrm{d}x}(\alpha) \neq 0$,

$$g(\alpha) = -\frac{\delta_0(F)(\alpha)}{\frac{\mathrm{d}F}{\mathrm{d}x}(\alpha)} \in K(\alpha) = K[\alpha].$$

So we can select a $g(x) \in K[x]$ with the desired property. With this $g(x)$, δ_1 maps $(F)_{K[x]}$ to itself, so it can induce a map

$$\bar{\delta_1} : K[x]/(F)_{K[x]} \longrightarrow K[x]/(F)_{K[x]}$$

with $\bar{\delta_1}(f(x) + (F)_{K[x]}) = \delta_1(f(x)) + (F)_{K[x]}$ which is derivation on $K[x]/(F)_{K[x]}$. Since $K[\alpha] \cong K[x]/(F)_{K[x]}$, by defining $\bar{\delta_1}(f(\alpha)) = \delta_1(f)(\alpha)$ for each $f(\alpha) \in K[\alpha]$, $\bar{\delta_1}$ gives a derivation on $K(\alpha) = K[\alpha]$. (Note that $\bar{\delta_1}(\alpha) = g(\alpha) = -\delta_0(F)(\alpha)/F'(\alpha)$.)

For the general case, let $U = \{(K_1, \delta_1) : K \subseteq K_1 \subseteq L \text{ and } \delta_1 \mid_K = \delta\}$. Then U is nonempty for $(K, \delta) \in U$. Let $(K_1, \delta_1) \subseteq (K_2, \delta_2) \subseteq \cdots \subseteq (K_n, \delta_n) \subseteq \cdots$ be an ascending chain in U. Then $\big(\bigcup_i K_i, D\big)$ with $\forall\, a \in K_i$, $D(a) = \delta_i(a)$ is in U. By Zorn's lemma, there exists a maximal element (M, δ_M) in U. Clearly, $M = L$.

<u>Uniqueness</u> If L is not algebraic over K, then $\exists\, \alpha \in L$ transcendental over K. By Lemma 4.1.1, for any $g(\alpha) \in K[\alpha]$, $\delta_1 = \delta_0 + g(\alpha)\frac{\mathrm{d}}{\mathrm{d}\alpha}$ is a derivation on $K[\alpha]$ extending δ. So there will be more than one derivation on $L \supset K(\alpha)$ which extends δ. If L is algebraic over K, for each $\alpha \in L$, let $F(x) = \sum_{i=0}^{d} r_i x^i \in K[x]$ be the minimal polynomial of α over K. Suppose D is a derivation on L which extends δ on K. $F(\alpha) = 0 \Rightarrow 0 = D(F(\alpha)) = D(\sum_{i=0}^{d} r_i\alpha^i) = \sum_{i=0}^{d} \delta(r_i)\alpha^i + (\sum_{i=1}^{d} ir_i\alpha^{i-1})D(\alpha)$ $\Rightarrow D(\alpha) = -(\sum_{i=0}^{d} \delta(r_i)\alpha^i)/(\sum_{i=1}^{d} ir_i\alpha^{i-1})$. Thus, D is the unique derivation on L which extends δ. □

Corollary 4.1.3. *If $K \subseteq L$ are fields of characteristic 0 and δ is a derivation on L s.t. $\delta(K) \subseteq K$. If $\alpha \in L$ is algebraic over K, then $\delta(\alpha) \in K(\alpha)$. In particular, if $\alpha \in L$ is algebraic over a constant subfield of L, then α is a constant.*

Proof. Let $F(x) = \sum_{i=0}^{d} r_i x^i \in K[x]$ be the minimal polynomial of α over K. By the proof of Theorem 4.1.2, $\delta(\alpha) = -(\sum_{i=0}^{d} \delta(r_i)\alpha^i)/(\sum_{i=1}^{d} ir_i\alpha^{i-1}) \in K(\alpha)$. If $\delta(r_i) = 0$ for $i = 0, \ldots, d$, then $\delta(\alpha) = 0$. □

With the language of differential polynomials, Definition 2.1.1 can be restated as:

Definition 4.1.4. *Let $K \subseteq L$ be differential field extensions and $\alpha \in L$. If there exists $p(y) \in K\{y\}\backslash\{0\}$ s.t. $p(\alpha) = 0$, then α is said to be* ***differential algebraic*** *over K. Otherwise, α is called* ***differentially transcendental*** *over K. Let $\alpha_1, \ldots, \alpha_n \in K$. We call $\alpha_1, \ldots, \alpha_n$* ***differentially algebraically dependent*** *over K if there exists a nonzero $F(y_1, \ldots, y_n) \in K\{y_1, \ldots, y_n\}$ such that $F(\alpha_1, \ldots, \alpha_n) = 0$. Otherwise, they are said to be* ***differentially algebraically independent*** *over K or a set of* **differential indeterminates** *over K.*

Lemma 4.1.5. *Let $K \subseteq L$ be differential fields of characteristic 0 and $\alpha \in L$. Then α is differential algebraic over $K \Leftrightarrow \mathrm{tr.deg} K\langle\alpha\rangle/K < \infty$.*

Proof. "$\Rightarrow$" Suppose α is differential algebraic over K. Let $A(y) \in K\{y\}$ be a characteristic set of $\mathbb{I}(\alpha) \subseteq K\{y\}$.[1] Assume $\mathrm{ord}(A) = n$. We claim that $\mathrm{tr.deg}K\langle\alpha\rangle/K = n$.

Clearly, $\alpha, \alpha', \ldots, \alpha^{(n-1)}$ are algebraically independent over K and $\alpha^{(n)}$ is algebraic over $K(\alpha, \alpha', \ldots, \alpha^{(n-1)})$. And $A(\alpha) = 0 \Rightarrow \mathrm{S}_A(\alpha)\cdot\alpha^{(n+1)} + T_A(\alpha) = 0$, where $T_A(\alpha) \in K(\alpha, \ldots, \alpha^{(n)}) \Rightarrow \alpha^{(n+1)} = -\frac{T_A(\alpha)}{\mathrm{S}_A(\alpha)} \in K(\alpha, \alpha', \ldots, \alpha^{(n)})$. $\Rightarrow \forall\, k \in \mathbb{N}$, $\alpha^{(n+k)} \in K(\alpha, \alpha', \ldots, \alpha^{(n)})$. So $K\langle\alpha\rangle = K(\alpha, \alpha', \ldots, \alpha^{(n)})$ and $\mathrm{tr.deg}K\langle\alpha\rangle/K = n$.

"$\Leftarrow$" $n = \mathrm{tr.deg}K\langle\alpha\rangle/K < \infty$ impies that $\alpha, \alpha', \alpha'', \ldots, \alpha^{(n)}$ are algebraically dependent over K. So α is differential algebraic over K. □

Remark:

1) If α is differential algebraic over K and $f(y) \neq 0$ is a differential polynomial of minimal order which vanishes at α, then $\mathrm{tr.deg}K\langle\alpha\rangle/K = \mathrm{ord}(f)$.

2) The result "$\Rightarrow$" is false in the partial differential case $(K, \{\delta_1, \ldots, \delta_m\})$, where $\mathrm{tr.deg}K\langle\alpha\rangle/K$ might be infinity but the differential type[2] of $K\langle\alpha\rangle$ is $\leq m-1$.

Example: $K = (\mathbb{R}(x), \frac{\mathrm{d}}{\mathrm{d}x})$, $L = (K\langle \mathrm{e}^x, \sin(x)\rangle, \frac{\mathrm{d}}{\mathrm{d}x})$. Since $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$ and $(\frac{\mathrm{d}}{\mathrm{d}x})^2(\sin(x)) = -\sin(x)$, both e^x and $\sin(x)$ are differentially algebraic over K. Note that $\mathrm{tr.deg}K\langle \mathrm{e}^x\rangle/K = 1$, and $\mathrm{tr.deg}K\langle\sin(x)\rangle/K = 1$ (for $\mathbb{I}\big(\sin x) = \mathrm{sat}((z')^2 + z^2)\big)$.

We say $L \supseteq K$ is differential algebraic over K, if each element $a \in L$ is differential algebraic over K. Note that every differential field extension with finite transcendence degree is differential algebraic over K. But the converse doesn't hold.

Lemma 4.1.6. *Let $L \supseteq K$ be a differential field extension and $a, b \in L$. If a and b are differential algebraic over K, then $a+b, ab, \delta(a)$ and a^{-1} $(a \neq 0)$ are differential algebraic over K. In particular, a differential field extension generated by differential algebraic elements is differential algebraic over K and the set of all elements in L which are differential algebraic over K is a differential algebraic differential field extension of K.*

Proof. Since $\mathrm{tr.deg}K\langle a\rangle/K < \infty$ and $\mathrm{tr.deg}K\langle b\rangle/K < \infty$, we have $\mathrm{tr.deg}K\langle a, b\rangle/K = \mathrm{tr.deg}K\langle a\rangle/K + \mathrm{tr.deg}K\langle a\rangle\langle b\rangle/K\langle a\rangle < \infty$. So $a+b, ab, \delta(a)$ and a^{-1} $(a \neq 0)$ are differential algebraic over K. □

Lemma 4.1.7. *Let $K \subseteq L \subseteq M$ be differential fields. Then M is differential algebraic over K $\Leftrightarrow$ M is differential algebraic over L and L is differential algebraic over K.*

Proof. "$\Rightarrow$" Valid by definition.

"$\Leftarrow$" For any $a \in M$, a is differential algebraic over L, so $\exists\, p(y) \in L\{y\}\backslash\{0\}$ s.t. $p(a) = 0$. Denote the coefficient set of $p(y)$ to be $\{b_1, \ldots, b_t\} \subseteq L$. Then $\mathrm{tr.deg}K\langle b_1, \ldots, b_t, a\rangle/K = \mathrm{tr.deg}K\langle b_1, \ldots, b_t\rangle/K + \mathrm{tr.deg}K\langle b_1, \ldots, b_t, a\rangle/K\langle b_1, \ldots, b_t\rangle < \infty$. Thus, $\mathrm{tr.deg}K\langle a\rangle/K < \infty$ and a is differential algebraic over K. □

4.2 Differential primitive theorem

It is a well-known theorem of algebra that a finite algebraic extension of a field K of characteristic 0 has a primitive element ω:

$$K(a_1, \ldots, a_n) = K(\omega).$$

[1] $A(y)$ is of minimal order and minimal degree under the desired order.

[2] Differential type is the degree of differential dimension polynomial of $\mathbb{I}(\alpha)$

In this section, we treat analogous problem for arbitrary differential field of characteristic 0.

Note that $\mathbb{Q}\langle\pi, \mathrm{e}\rangle$ is a finitely generated differential extension field of $\mathbb{Q}$ ($\delta(\pi) = \delta(\mathrm{e}) = 0$). Clearly, $\mathbb{Q}\langle\pi, \mathrm{e}\rangle \neq \mathbb{Q}\langle\omega\rangle$ for any $\omega \in \mathbb{Q}\langle\pi, \mathrm{e}\rangle$. So to derive an analog of primitive element theorem in differential algebra, we need some restrictions. For the ordinary differential fields, the mild condition is that (K, δ) contains a non-constant element (i.e., $\exists\, \eta \in K$ s.t. $\eta' \neq 0$).

We need two lemmas for preparation to state the main theorem. Throughout this section, (K, δ) is a fixed differential field of characteristic 0 containing a non-constant.

A set of elements $\eta_1, \ldots, \eta_s$ of K is called **linearly dependent** if there exists a relation

$$c_1\eta_1 + \cdots + c_s\eta_s = 0,$$

where the c_i's are constant elements in K, not all zero.

The **Wronskian determinant** of $\eta_1, \ldots, \eta_s$ is defined as

$$\mathrm{wr}(\eta_1, \ldots, \eta_s) = \begin{vmatrix} \eta_1 & \cdots & \eta_s \\ \eta_1' & \cdots & \eta_s' \\ \cdots & \cdots & \cdots \\ \eta_1^{(s-1)} & \cdots & \eta_s^{(s-1)} \end{vmatrix}.$$

Lemma 4.2.1. *A set of elements $\eta_1, \ldots, \eta_s$ of K is linearly dependent if and only if*

$$wr(\eta_1, \ldots, \eta_s) = \begin{vmatrix} \eta_1 & \cdots & \eta_s \\ \eta_1' & \cdots & \eta_s' \\ \cdots & \cdots & \cdots \\ \eta_1^{(s-1)} & \cdots & \eta_s^{(s-1)} \end{vmatrix} = 0 \qquad (*)$$

Proof. "$\Rightarrow$" Suppose $\eta_1, \ldots, \eta_s$ are linearly dependent. Then $\exists\, c_1, \ldots, c_s$, constants of K, not all zero s.t. $c_1\eta_1 + \cdots + c_s\eta_s = 0$. Differentiate the relation $s-1$ times, we get a system of linear equations for c's:

$$\begin{cases} c_1\eta_1 + \cdots + c_s\eta_s = 0 \\ c_1\eta_1' + \cdots + c_s\eta_s' = 0 \\ \qquad \cdots\cdots \\ c_1\eta_1^{(s-1)} + \cdots + c_s\eta_s^{(s-1)} = 0 \end{cases}$$

has a nonzero solution. So $(*)$ holds.

"$\Leftarrow$" Suppose we have $(*)$. We now show $\eta_1, \ldots, \eta_s$ are linearly dependent by induction on s. If $s = 1$, $\eta_1 = 0 \Rightarrow \eta_1$ is linearly dependent. Suppose it is valid for the case $\leq s-1$ and we treat for the case s. If $\mathrm{wr}(\eta_1, \ldots, \eta_{s-1}) = \begin{vmatrix} \eta_1 & \cdots & \eta_{s-1} \\ \eta_1' & \cdots & \eta_{s-1}' \\ \cdots & \cdots & \cdots \\ \eta_1^{(s-2)} & \cdots & \eta_{s-1}^{(s-2)} \end{vmatrix} = 0$, by the induction hypothesis, $\eta_1, \ldots, \eta_{s-1}$ are linearly dependent, so $\eta_1, \ldots, \eta_s$ are linearly dependent too.

So it suffices to consider the case $\mathrm{wr}(\eta_1, \ldots, \eta_{s-1}) \neq 0$. By $(*)$, $\exists\, c_1, \ldots, c_s \in K$, not all zero s.t.

$$c_1\eta_1^{(j)} + \cdots + c_s\eta_s^{(j)} = 0 \quad (**) \quad \text{for } j = 0, \ldots, s-1.$$

Since $\mathrm{wr}(\eta_1, \ldots, \eta_{s-1}) \neq 0$, $c_s \neq 0$. By dividing c_s on both sides when necessary, we can take $c_s = 1$. For $j = 0, \ldots, s-2$, differentiate $(**)_j$ and then subtract the equation $(**)_{j+1}$, then we have

$$c_1'\eta_1^{(j)} + \cdots + c_{s-1}'\eta_{s-1}^{(j)} = 0 \text{ for } j = 0, \ldots, s-2.$$

Since $\mathrm{wr}(\eta_1, \ldots, \eta_{s-1}) \neq 0$, we have $c_i' = 0$ for $i = 1, \ldots, s-1$. Thus, $\eta_1, \ldots, \eta_s$ are linearly dependent. □

Lemma 4.2.2. *Let K be a nonconstant differential field of characteristic 0. If G is a nonzero differential polynomial in $K\{y_1, \ldots, y_n\}$, there exist elements $\eta_1, \ldots, \eta_n$ in K such that $G(\eta_1, \ldots, \eta_n) \neq 0$.*

Proof. It suffices to treat a differential polynomial in a single indeterminate y (the case $n = 1$). Take a nonconstant $\xi \in K$. Fix any $r \in \mathbb{N}$.

Claim: If $G \in K\{y\}$ is a nonzero differential polynomial of order $\leq r$, there exists

$$\eta = c_0 + c_1\xi + \cdots + c_r\xi^r$$

where all the c_i's are constants in K, satisfying $G(\eta) \neq 0$.
Suppose the claim is false and let H be a nonzero differential polynomial of lowest rank which vanishes for every element $c_0 + c_1\xi + \cdots + c_r\xi^r$ (c_i are constants from K). Let $\mathrm{ord}(H, y) = s$. Then $0 < s \leq r$. Introduce algebraic indeterminates $z_0, \ldots, z_r$ with $z_i' = 0$. Then $\overline{H} = H(z_0 + z_1\xi + \cdots + z_r\xi^r) \in K[z_0, \ldots, z_r]$ is the zero polynomial. Take the partial derivative of $\overline{H}$ w.r.t. $z_0, \ldots, z_s$, then

$$\begin{cases} \dfrac{\partial \overline{H}}{\partial z_0} = \overline{\dfrac{\partial H}{\partial y}} & = 0 \\ \dfrac{\partial \overline{H}}{\partial z_1} = \overline{\dfrac{\partial H}{\partial y}}\xi + \overline{\dfrac{\partial H}{\partial y'}}\xi' + \cdots + \overline{\dfrac{\partial H}{\partial y^{(s)}}}\xi^{(s)} & = 0 \\ \qquad\qquad \cdots\cdots & \\ \dfrac{\partial \overline{H}}{\partial z_s} = \overline{\dfrac{\partial H}{\partial y}}\xi^s + \overline{\dfrac{\partial H}{\partial y'}}(\xi^s)' + \cdots + \overline{\dfrac{\partial H}{\partial y^{(s)}}}(\xi^s)^{(s)} & = 0, \end{cases}$$

where $\overline{\frac{\partial H}{\partial y^{(j)}}} = \frac{\partial H}{\partial y^{(j)}}(z_0 + \cdots + z_r\xi^r)$. So

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \xi & \xi' & \cdots & \xi^{(s)} \\ \cdots & \cdots & \cdots & \cdots \\ \xi^s & (\xi^s)' & \cdots & (\xi^s)^{(s)} \end{pmatrix} \begin{pmatrix} \overline{\frac{\partial H}{\partial y}} \\ \overline{\frac{\partial H}{\partial y'}} \\ \vdots \\ \overline{\frac{\partial H}{\partial y^{(s)}}} \end{pmatrix} = 0$$

Since $\frac{\partial H}{\partial y^{(s)}}$ is of lower rank than H, $\overline{\frac{\partial H}{\partial y^{(s)}}} \neq 0$. Thus,

$$\begin{vmatrix} \xi' & (\xi^2)' & \cdots & (\xi^s)' \\ \xi'' & (\xi^2)'' & \cdots & (\xi^s)'' \\ \cdots & \cdots & \cdots & \cdots \\ \xi^{(s)} & (\xi^2)^{(s)} & \cdots & (\xi^s)^{(s)} \end{vmatrix} = \mathrm{wr}(\xi', (\xi^2)', \ldots, (\xi^s)') = 0.$$

So $\exists\, c_1, \ldots, c_s$ constants of K, not all zero s.t. $c_1\xi' + c_2(\xi^2)' + \cdots + c_s(\xi^s)' = 0$. Then $c_1\xi + c_2\xi^2 + \cdots + c_s\xi^s = c_0$ with c_0 a constant. Thus ξ is algebraic over the constant field of K. By Corollary 4.1.3, $\xi' = 0$, a contradiction to the hypothesis $\xi' \neq 0$. So we can find some $\eta = c_0 + c_1\xi + \cdots + c_r\xi^r$ with c_i constants s.t. $G(\eta) \neq 0$. □

Remark:

1) Lemma 4.2.2 is false without the restriction that (K, δ) contains at least a nonconstant element. A non-example: $K = \mathbb{Q}$, $G(y) = y'$.

2) For the partial differential case $(K, \{\delta_1, \ldots, \delta_m\})$, the condition that "$\exists\, \xi \in K$ s.t. $\xi' = 0$" should be replaced by

$$\text{“ } \exists\, \xi_1, \ldots, \xi_m \in K \text{ s.t. } \begin{vmatrix} \delta_1(\xi_1) & \cdots & \delta_1(\xi_m) \\ \delta_2(\xi_1) & \cdots & \delta_2(\xi_m) \\ \cdots & \cdots & \cdots \\ \delta_m(\xi_1) & \cdots & \delta_m(\xi_m) \end{vmatrix} \neq 0. \text{ ”}$$

The lemma is called " non-vanishing of differential polynomials ".

3) Lemma 4.2.2 is the differential analog of the following result in Algebra:

" Let K be an infinite field. Then for any nonzero polynomial $f \in K[y_1, \ldots, y_n]$, there exists $(a_1, \ldots, a_n) \in K^n$ s.t. $f(a_1, \ldots, a_n) \neq 0$. "