Recall: • (**Differential Primitive Element Theorem**) Let (K, δ) be a differential field of characteristic 0 containing at least a nonconstant element. Assume $K\langle\alpha_1, \ldots, \alpha_n\rangle$ is differential algebraic over K. Then $\exists\, \xi \in K\langle\alpha_1, \ldots, \alpha_n\rangle$ s.t. $K\langle\alpha_1, \ldots, \alpha_n\rangle = K\langle\xi\rangle$. In particular, there exist $e_i \in K$ s.t. $K\langle\alpha_1, \ldots, \alpha_n\rangle = K\langle\sum_{i=1}^n e_i\alpha_i\rangle$.

• **Differential transcendence basis** of L/K: a subset A of L satisfying 1) A is δ-algebraically independent over K and 2) L is δ-algebraic over $K\langle A\rangle$.

Existence: Every δ-generating set of $L \supseteq K$ contains a δ-transcendence basis of L over K. And any two δ-transcendence bases of L over K are of the same size.

• **Differential transcendence degree** of L/K: the size of a δ-transcendence basis of L over K, denoted by δ-tr.deg(L/K). We have

(1) δ-tr.deg$(L/K) = \sup\{n \in \mathbb{N} \mid \exists\, a_1, \ldots, a_n \in L$ δ-algebraically independent over $K\}$.

(2) For $K \subseteq L \subseteq M$, δ-tr.deg$(M/K) = \delta$-tr.deg$(M/L) + \delta$-tr.deg(L/K).

4.4 Applications to differential varieties

Let (K, δ) be a δ-field of characteristic 0 and $(\bar{K}, \delta)$ a δ-closed field containing (K, δ).

4.4.1 Differential dimension polynomials of differential varieties

Let $V \subseteq \bar{K}^n$ be an irreducible δ-variety over K. Then $\mathbb{I}(V) \subset K\{y_1, \ldots, y_n\}$ is a prime differential ideal. The quotient ring $K\{y_1, \ldots, y_n\}/\mathbb{I}(V)$ is a differential domain, which we can write as $K\{\bar{y}_1, \ldots, \bar{y}_n\}$, where $\bar{y}_i$ is the residue class of y_i. It is called the **differential coordinate ring** of V and denoted by $K\{V\}$, We can consider its elements with $\bar{K}$-valued functions on V and so we call them differential polynomial functions on V. The field of fractions of the differential coordinate ring is called **the field of differential rational functions on** V, and is denoted by $K\langle V\rangle = K\langle\bar{y}_1, \ldots, \bar{y}_n\rangle$. Naturally, $K\langle V\rangle$ is a δ-field extension of K. Clearly, $(\bar{y}_1, \ldots, \bar{y}_n) \in (K\langle V\rangle)^n$ is a generic point of V. Indeed, given $f \in K\{y_1, \ldots, y_n\}$, $f(\bar{y}_1, \ldots, \bar{y}_n) = 0$ if and only if $f \in \mathbb{I}(V)$. Given any other generic point $(a_1, \ldots, a_n)$ of V, we have $K\langle V\rangle = K\langle\bar{y}_1, \ldots, \bar{y}_n\rangle \cong K\langle a_1, \ldots, a_n\rangle$ with $\bar{y}_i \leftrightarrow a_i$. In particular, δ-tr.deg$K\langle\bar{y}_1, \ldots, \bar{y}_n\rangle/K = \delta$-tr.deg$K\langle a_1, \ldots, a_n\rangle/K$.

In order to measure the "size" of a differential variety (i.e., the solution set of algebraic differential equations), we introduce the notion of differential dimension:

Definition 4.4.1. *Let $V \subseteq \mathbb{A}^n$ be an irreducible δ-variety over K. The* **differential dimension of** *V is defined as the δ-transcendence degree of the δ-field $K\langle V\rangle$ of δ-rational functions on V over K, denoted by δ-dim(V). That is,*

$$\delta\text{-dim}(V) := \delta\text{-tr.deg}K\langle V\rangle/K.$$

For an arbitrary V with irreducible components $V_1, \ldots, V_m$,

$$\delta\text{-dim}(V) := \max_i \delta\text{-dim}(V_i).$$

An equivalent definition of differential dimension in the language of differential ideals is given by Ritt:

Definition 4.4.2. *Let $P \subseteq K\{y_1, \ldots, y_n\}$ be a prime δ-ideal. A δ-variable set $U \subseteq \{y_1, \ldots, y_n\}$ is called a δ-independent set modulo P if $P \cap K\{U\} = \{0\}$. A* **parametric set of** *P is a maximal δ-independent set modulo P. The δ-dimension of P (or $\mathbb{V}(P)$) is defined to be the cardinal number of its parametric set.*

Exercise: Please show different parametric sets of a prime δ-ideal have the same cardinal number. And show Definition 4.4.1 and Definition 4.4.2 are equivalent for prime δ-ideals or irreducible δ-varieties.

Lemma 4.4.3. *Let V be a δ-variety and $W \subseteq V$ a δ-subvariety. Then $\delta\text{-}\dim(W) \leq \delta\text{-}\dim(V)$.*

Proof. First assume W and V are both irreducible. $W \subseteq V$ implies that $\mathbb{I}(W) \supseteq \mathbb{I}(V)$. Suppose $\delta\text{-}\dim(W) = d$ and $\{y_1, \ldots, y_d\}$ is a parametric set of $\mathbb{I}(W)$. Clearly, $\mathbb{I}(V) \cap \{y_1, \ldots, y_d\} = \{0\}$ and $\{y_1, \ldots, y_d\}$ is a δ-independent set modulo $\mathbb{I}(V)$ which could be extended to a parametric set of $\mathbb{I}(V)$. Thus, $\delta\text{-}\dim(V) = \delta\text{-}\dim(\mathbb{I}(V)) \geq d$.

Now let V and W be arbitrary. Let W_1 be an irreducible component of W with $\delta\text{-}\dim(W) = \delta\text{-}\dim(W_1)$. Then W_1 is contained in an irreducible component V_1 of V. By the above,

$$\delta\text{-}\dim(W) = \delta\text{-}\dim(W_1) \leq \delta\text{-}\dim(V_1) \leq \delta\text{-}\dim(V).$$

□

Exercise: Let $W \subseteq V$ be two irreducible δ-varieties with $\delta\text{-}\dim(W) = \delta\text{-}\dim(V)$. Is $W = V$?

It is true in the algebraic case but not valid in differential algebra:

Non-example: Let $W = \mathbb{V}(y') \subseteq \mathbb{A}^1$ and $V = \mathbb{V}(y'') \subseteq \mathbb{A}^1$. Then $W \subseteq V$ and $\delta\text{-}\dim(W) = \delta\text{-}\dim(V)$. But $W \neq V$.

This example shows that the differential dimension is not a fine enough measure of size of differential varieties, thus we need a more discriminating measure: the differential dimension polynomial of an irreducible δ-variety V or $\mathbb{I}(V)$. The idea of Hilbert polynomial for homogeneous ideals suggests that it might be a way to consider the truncated coordinate ring by order: Let $P \subseteq K\{y_1, \ldots, y_n\}$ be a prime δ-ideal. Denote $K[y_1^{[t]}, \ldots, y_n^{[t]}] = K[y_i^{(j)} : j \leq t, i = 1, \ldots, n]$ and let $P_t = P \cap K[y_1^{[t]}, \ldots, y_n^{[t]}]$. Then P_t is a prime algebraic ideal with dimension $\dim(P_t)$.

Recall that a polynomial $f \in \mathbb{R}[t]$ is said to be **numerical** if $f(s) \in \mathbb{Z}$ for sufficiently big $s \in \mathbb{N}$. Any $f \in \mathbb{R}[t]$ can be writen as

$$f = \sum_k a_k \binom{t+k}{k}$$

where $a_k \in \mathbb{R}$ and $\binom{t+k}{k} = (t+1)(t+2)\cdots(t+k)/k!$. f is numerical if and only if $a_k \in \mathbb{Z}$ for every k. We define $f \leqslant g$ to mean that $f(s) \leqslant g(s)$ for all sufficiently big $s \in \mathbb{N}$; this totally orders $\mathbb{R}[t]$ and well orders the set of all numerica polynomials which are ≥ 0.

Kolchin showed that for $t \gg 0$, $\dim(P_t)$ is a numerical polynomial. We state it with the language of δ-field extensions.

Theorem 4.4.4 (Kolchin)**.** *Let $P \subseteq K\{y_1, \ldots, y_n\}$ be a prime δ-ideal. There exists a unique numerical polynomial $\omega_P(t) \in \mathbb{R}[t]$ such that $\dim(P_t) = \omega_P(t)$ for all sufficiently big $t \in \mathbb{N}$, with the following properties:*

1) $\omega_P(t) = d(t+1) + s$ with $d = \delta\text{-}\dim(\mathbb{V}(P))$ and some $s \in \mathbb{N}$;

*2) (Computation of $\omega_P(t)$) Let $\mathcal{A} = A_1, \ldots, A_l$ be a characteristic set of P w.r.t. some **orderly** ranking and suppose $\mathrm{ld}(A_i) = y_{\sigma(i)}^{(s_i)}$. Then $\omega_P(t) = (n-l)(t+1) + \sum_{i=1}^{l} s_i$.*

3) $\omega_P(t) = n(t+1) \Leftrightarrow P = [0]$ (i.e., $\mathbb{V}(P) = \mathbb{A}^n$); $\omega_P(t) = 0 \Leftrightarrow \mathbb{V}(P)$ is a finite set.

Proof. Let $\eta = (\eta_1, \dots, \eta_n)$ be a generic point of P. Denote $\eta^{[t]} = (\eta_1, \dots, \eta_n, \eta_1', \dots, \eta_n', \dots, \eta_1^{(t)}, \dots, \eta_n^{(t)})$. Clearly, $\eta^{[t]}$ is a generic point of $P_t \subseteq K[y_1^{[t]}, \dots, y_n^{[t]}]$. So $\dim(P_t) = \text{tr.deg} K(\eta^{[t]})/K$.

For each $A \in \mathcal{A}$, $A(\eta) = 0$ and $\mathrm{I}_A(\eta) \neq 0$ imply that $u_A(\eta)$ is algebraic over $K(\eta_j^{(k)} : y_j^{(k)} < u_A, j = 1, \dots, n)$. Repeated differentiation shows that if v is any derivative of u_A, then $v(\eta)$ is algebraic over $K(\eta_j^{(k)} : y_j^{(k)} < v, j = 1, \dots, n)$. Let M denote the set of all derivatives $y_j^{(k)}$ that are not derivatives of any u_A ($A \in \mathcal{A}$) and let $M(t) = M \cap \{y_j^{(k)} : k \leq t, j = 1, \dots, n\}$. So, for $t \geq \max\{s_1, \dots, s_l\}$, we have that

$$K(\eta^{[t]}) \text{ is algebraic over } K\big((v(\eta))_{v \in M(t)}\big).^3 \qquad (*)$$

Thus, $\dim(P_t) = \text{tr.deg} K(\eta^{[t]})/K = \text{Card}(M(t))$. Since

$$M(t) = \{\underbrace{y_{\sigma(i)}, y_{\sigma(i)}', \dots, y_{\sigma(i)}^{(s_i-1)} : i = 1, \dots, l}_{\text{derivatives of leading variables}}\} \cup \{\underbrace{y_j, y_j', \dots, y_j^{(t)} : j \neq \sigma(1), \dots, \sigma(l)}_{\text{derivatives of parametric variables}}\},$$

$\text{Card}(M(t)) = (n-l)(t+1) + \sum_{i=1}^{l} s_i$. So $\dim(P_t) = (n-l)(t+1) + \sum_{i=1}^{l} s_i$ for $t \geq \max\{s_1, \dots, s_l\}$. Let $\omega_P(t) = (n-l)(t+1) + \sum_{i=1}^{l} s_i$, which is numerical and $\dim(P_t) = \omega_P(t)$ for $t \geq \max\{s_1, \dots, s_l\}$. This finishes the proof of the existence of $\omega_P(t)$ and 2).

To show 3), $\omega_P(t) = n(t+1) \iff M(t) = \{y_j^{(k)} : k \leq t, j = 1, \dots, n\} \iff P = [0]$; And $\omega_P(t) = 0 \iff M(t) = \emptyset \iff \text{ld}(\mathcal{A}) = \{y_1, \dots, y_n\} \iff \mathbb{V}(P)$ is a finite set.

It remains to show $\delta\text{-dim}(P) = n - l$ to complete the proof of 1). Assume $d = \delta\text{-dim}(P) = \delta\text{-tr.deg} K\langle\eta\rangle/K$. W.L.O.G, let $\eta_1, \dots, \eta_d$ be a differential transcendence basis of $K\langle\eta\rangle$ over K. Thus, $\omega_P(t) = \text{tr.deg} K(\eta_1^{[t]}, \dots, \eta_n^{[t]})/K = (n-l)(t+1) + \sum_{i=1}^{l} s_i \geq \text{tr.deg} K(\eta_1^{[t]}, \dots, \eta_d^{[t]})/K = d(t+1)$, and $n - l \geq d$ follows. Conversely, let $\{z_1, \dots, z_{n-l}\} = \{y_1, \dots, y_n\} \backslash \{y_{\sigma(1)}, \dots, y_{\sigma(l)}\}$. Since any nonzero polynomial in $K\{z_1, \dots, z_{n-l}\}$ is reduced w.r.t. $\mathcal{A}$, we have $K\{z_1, \dots, z_{n-l}\} \cap P = \{0\}$. So $\{z_1, \dots, z_{n-l}\}$ is an independent set modulo P and can be enlarged to be a parametric set of P. Thus, $n - l \leq \delta\text{-dim}(P) = d$. Hence, $n - l = d = \delta\text{-dim}(P)$. □

Definition 4.4.5. *Let $V \subseteq \mathbb{A}^n$ be an irreducible differential variety over K and $P = \mathbb{I}(V)$. The above $\omega_P(t)$ is defined as the* **differential dimension polynomial** *of P or V, also denoted by $\omega_V(t)$.*

The δ-dimension polynomial of an irreducible δ-variety $V \subseteq \mathbb{A}^n$ is of the form

$$\omega_V(t) = d(t+1) + s, \text{ where } d = \delta\text{-dim}(V) \text{ and } s \in \mathbb{N}.$$

The number s is defined as the **order** of V, denoted by $\text{ord}(V)$. The order is the rigorous definition for the notion "the number of arbitrary constants" of the solution of algebraic differential equations.

For an autoreduced set $\mathcal{A} = A_1, \dots, A_p$ under an arbitrary ranking, if $\text{ld}(A_i) = y_{k_i}^{(s_i)}$, we define the order of $\mathcal{A}$ as $\text{ord}(\mathcal{A}) = \sum_{i=1}^{p} s_i$. By the proof of the Theorem 4.4.4, we have

Corollary 4.4.6. *Let $P \subseteq K\{y_1, \dots, y_n\}$ be a prime δ-ideal and $\mathcal{A} = A_1, \dots, A_l$ be a characteristic set of P w.r.t. some* **orderly** *ranking. Then $\delta\text{-dim}(P) = n - \text{Card}(\mathcal{A})$ and $\text{ord}(P) = \text{ord}(\mathcal{A})$.*

[3] Arrange $\{y_j^{(k)} : k \leq t, j = 1, \dots, n\} \backslash M(t)$ in increasing order: $u_{A_1} < \cdots$. From the above, u_{A_1} is algebraic over $K\big((v(\eta))_{v \in M(t)}\big)$ and $(*)$ can be shown by induction.

Remark: In the partial differential case, $(K, \{\delta_1, \ldots, \delta_m\})$, the differential dimension polynomial of V has the form

$$\omega_V(t) = a_m \binom{t+m}{m} + a_{m-1}\binom{t+m-1}{m-1} + \cdots + a_1(t+1) + a_0,$$

where $a_m = \delta\text{-dim}(V)$. And the proof of the partial differential analogue of Theorem 4.4.4 is more complicated.

Example: Let $W = \mathbb{V}(y') \subseteq \mathbb{A}^1$ and $V = \mathbb{V}(y'') \subseteq \mathbb{A}^1$. $W \subsetneq V$ but $\delta\text{-dim}(W) = \delta\text{-dim}(V)$. Note that $\omega_W(t) = 1 < \omega_V(t) = 2$.

The next proposition shows that δ-dimension polynomial is a finer measure than δ-dimension.

Proposition 4.4.7. *Let $W, V \subseteq \mathbb{A}^n$ be irreducible δ-varieties and $W \subsetneq V$. Then $\omega_W(t) < \omega_V(t)$.*

Proof. Let $P_1 = \mathbb{I}(W)$ and $P_2 = \mathbb{I}(V)$. Then $W \subsetneq V$ implies that $P_1 \supsetneq P_2$. So for all sufficiently big t, $P_1 \cap K[y_1^{[t]}, \ldots, y_n^{[t]}] \supsetneq P_2 \cap K[y_1^{[t]}, \ldots, y_n^{[t]}]$, consequently,

$$\begin{aligned}\omega_W(t) &= \dim P_1 \cap K[y_1^{[t]}, \ldots, y_n^{[t]}] \\ &< \dim P_2 \cap K[y_1^{[t]}, \ldots, y_n^{[t]}] \\ &= \omega_V(t) \quad \text{for } t \gg 0.\end{aligned}$$

□

4.4.2 Relative orders and differential resolvents

In this section, we will show that an irreducible δ-variety is differentially birationally equivalent to an irreducible δ-variety of codimension one.

Let $P \subseteq K\{y_1, \ldots, y_n\}$ be a prime δ-ideal with a generic point $(\xi_1, \ldots, \xi_n)$. Let $U = \{y_{i_1}, \ldots, y_{i_d}\}$ be a parametric set of P. The **relative order**[4] of P or $\mathbb{V}(P)$ w.r.t. U, denoted by $\mathrm{ord}_U P$, is defined as

$$\mathrm{ord}_U(P) = \mathrm{tr.deg}\, K\langle \xi_1, \ldots, \xi_n\rangle / K\langle \xi_{i_1}, \ldots, \xi_{i_d}\rangle.$$

If $\mathcal{A}$ is a characteristic set of P w.r.t. any elimination ranking and $U = \{y_{i_1}, \ldots, y_{i_d}\}$ is the set of non-leading variables of $\mathcal{A}$, then U is a parametric set of P and the relative order of P w.r.t.U is equal to $\mathrm{ord}(\mathcal{A})$.

Theorem 4.4.8. *Suppose (K, δ) contains a nonconstant element. Let $P \subseteq K\{u_1, \ldots, u_d, y_1, \ldots, y_{n-d}\}$ be a prime δ-ideal with a parametric set $\{u_1, \ldots, u_d\}$. Then $\exists\, a_1, \ldots, a_{n-d} \in K$ s.t. $[P, \omega - a_1 y_1 - \cdots - a_{n-d} y_{n-d}] \subseteq K\{u_1, \ldots, u_d, y_1, \ldots, y_{n-d}, \omega\}$ has a characteristic set of the form*

$$\begin{aligned}&X(u_1, \ldots, u_d, \omega)\\ &I_1(u_1, \ldots, u_d, \omega) y_1 - T_1(u_1, \ldots, u_d, \omega)\\ &\qquad\vdots\\ &I_{n-d}(u_1, \ldots, u_d, \omega) y_{n-d} - T_{n-d}(u_1, \ldots, u_d, \omega)\end{aligned}$$

w.r.t. the elimination ranking $u_1 < \cdots < u_d < \omega < y_1 < \cdots < y_{n-d}$. Moreover, $\mathrm{ord}(X, \omega) = \mathrm{ord}_U(P)$.

[4] In Chapter 5, we shall show how relative order and differential dimension can read off a characteristic set under arbitry ranking.

Proof. Let $\eta = (\bar{u}_1, \ldots, \bar{u}_d, \bar{y}_1, \ldots, \bar{y}_{n-d})$ be a generic point of P. Introduce $n-d$ new differential indeterminates $\lambda_1, \ldots, \lambda_{n-d}$ over $K\langle\eta\rangle$. Let

$$J = [P, \omega - \lambda_1 y_1 - \cdots - \lambda_{n-d} y_{n-d}] \subseteq K\{u_1, \ldots, u_d, y_1, \ldots, y_{n-d}, \lambda_1, \ldots, \lambda_{n-d}, \omega\}.$$

Then J is a prime δ-ideal with a generic point

$$\xi = (\bar{u}_1, \ldots, \bar{u}_d, \bar{y}_1, \ldots, \bar{y}_{n-d}, \lambda_1, \ldots, \lambda_{n-d}, \lambda_1 \bar{y}_1 + \cdots + \lambda_{n-d} \bar{y}_{n-d}).$$

Since $\delta\text{-}\dim(P) = d$, $\delta\text{-tr.deg}K\langle\eta\rangle/K = d$ and

$$\begin{aligned}\delta\text{-tr.deg}K\langle\xi\rangle/K &= \delta\text{-tr.deg}K\langle\eta\rangle/K + \delta\text{-tr.deg}K\langle\eta\rangle\langle\lambda_1, \ldots, \lambda_{n-d}\rangle/K\langle\eta\rangle \\ &= d + n - d = n.\end{aligned}$$

So $J_\lambda = J \cap K\{u_1, \ldots, u_d, \lambda_1, \ldots, \lambda_{n-d}, \omega\} \neq [0]$ and $\{u_1, \ldots, u_d, \lambda_1, \ldots, \lambda_{n-d}\}$ is a parametric set of J_λ. Let $\{R(u_1, \ldots, u_d, \lambda_1, \ldots, \lambda_{n-d}, \omega)\}$ be a characteristic set of J_λ w.r.t. the elimination ranking $u_1 < \cdots < u_d < \lambda_1 < \cdots < \lambda_{n-d} < \omega$. Denote $s = \mathrm{ord}(R, \omega) \geq 0$. Since $R(\bar{u}_1, \ldots, \bar{u}_d, \lambda_1, \ldots, \lambda_{n-d}, \lambda_1\bar{y}_1 + \cdots + \lambda_{n-d}\bar{y}_{n-d}) = 0$, for $j = 1, \ldots, n-d$, take the partial derivative of this identity w.r.t. $\lambda_j^{(s)}$ on both sides, then we obtain

$$\overline{\frac{\partial R}{\partial \lambda_j^{(s)}}} + \overline{\frac{\partial R}{\partial \omega^{(s)}}} \cdot \bar{y}_j = 0, \tag{4.1}$$

where $\overline{\frac{\partial R}{\partial \lambda_j^{(s)}}}$ and $\overline{\frac{\partial R}{\partial \omega^{(s)}}}$ are obtained from $\frac{\partial R}{\partial \lambda_j^{(s)}}$ and $\frac{\partial R}{\partial \omega^{(s)}}$ by substituting $(u_1, \ldots, u_d, \lambda_1, \ldots, \lambda_{n-d}, \omega) = (\bar{u}_1, \ldots, \bar{u}_d, \lambda_1, \ldots, \lambda_{n-d}, \lambda_1\bar{y}_1 + \cdots + \lambda_{n-d}\bar{y}_{n-d})$. Note that $\frac{\partial R}{\partial \omega^{(s)}} \notin J_\lambda$, so $\overline{\frac{\partial R}{\partial \omega^{(s)}}} \neq 0$. As $\overline{\frac{\partial R}{\partial \omega^{(s)}}} \in K\{\eta\}\{\lambda_1, \ldots, \lambda_{n-d}\}$ is nonzero, by the non-vanishing theorem of nonzero polynomials, $\exists\, a_1 \ldots, a_{n-d} \in K$ s.t. $\overline{\frac{\partial R}{\partial \omega^{(s)}}}|_{\lambda_i = a_i} \in K\{\eta\}\backslash\{0\}$. Let $I(u_1, \ldots, u_d, \omega) = \frac{\partial R}{\partial \omega^{(s)}}|_{\lambda_i = a_i} \in K\{u_1, \ldots, u_d, \omega\}$. Then $I(\bar{u}_1, \ldots, \bar{u}_d, a_1\bar{y}_1 + \cdots + a_{n-d}\bar{y}_{n-d}) = \overline{\frac{\partial R}{\partial \omega^{(s)}}}|_{\lambda_i = a_i} \neq 0$.

Let $J_a = [P, \omega - a_1 y_1 - \cdots - a_{n-d} y_{n-d}] \subseteq K\{u_1, \ldots, u_d, y_1, \ldots, y_{n-d}, \omega\}$. Then J_a is a prime δ-ideal with a generic point

$$\xi_a = (\bar{u}_1, \ldots, \bar{u}_d, \bar{y}_1, \ldots, \bar{y}_{n-d}, a_1\bar{y}_1 + \cdots + a_{n-d}\bar{y}_{n-d}).$$

Clearly, $I(u_1, \ldots, u_d, \omega) \notin J_a$. Let $T_j(u_1, \ldots, u_d, \omega) = -\frac{\partial R}{\partial \lambda_j^{(s)}}\Big|_{\lambda_i = a_i, i=1,\ldots,n-d}$. By (4.1),

$$I(u_1, \ldots, u_d, \omega) y_j - T_j(u_1, \ldots, u_d, \omega) \in J_a.$$

Since $\delta\text{-tr.deg}K\langle\xi_a\rangle/K = d$, $J_a \cap K\{u_1, \ldots, u_d, \omega\} \neq [0]$ with a parametric set $\{u_1, \ldots, u_d\}$. So its characteristic set consists of a single δ-polynomial. Let $X(u_1, \ldots, u_d, \omega)$ be an irreducible polynomial constituting a characteristic set of $J_a \cap K\{u_1, \ldots, u_d, \omega\}$ w.r.t the elimination ranking $\mathscr{R} : u_1 < \cdots < u_d < \omega$. For each j, take the differential remainder of $Iy_j - T_j$ w.r.t X (under $\mathscr{R}$). Since $I \notin J_a \cap K\{u_1, \ldots, u_d, \omega\}$, $\delta\text{-rem}(Iy_j - T_j, X)$ is of the form $I_j y_j - \bar{T}_j$ where $I_j, \bar{T}_j \in K\{u_1, \ldots, u_d, \omega\}$, $I_j \notin J_a$.

<u>Claim:</u> $X(u_1, \ldots, u_d, \omega), I_1 y_1 - \bar{T}_1, \ldots, I_{n-d} y_{n-d} - \bar{T}_{n-d}$ is a characteristic set of J_a w.r.t. the elimination ranking $u_1 < \cdots < u_d < \omega < y_1 < \cdots < y_{n-d}$. Indeed, for all $f \in J_a$, first perform the Ritt-Kolchin reduction process for f w.r.t. $I_1 y_1 - \bar{T}_1, \ldots, I_{n-d} y_{n-d} - \bar{T}_{n-d}$, then we get $f_0 \in J_a \cap K\{u_1, \ldots, u_d, \omega\}$, thus f_0 could be reduced to 0 by X. Thus, we have proved the claim.

It remains to show that $\mathrm{ord}(X,\omega) = \mathrm{ord}_U(P)$. Since $K\langle\eta\rangle = K\langle\bar{u}_1,\ldots,\bar{u}_d, a_1\bar{y}_1+\cdots+a_{n-d}\bar{y}_{n-d}\rangle$,

$$\begin{aligned}
\mathrm{ord}_U(P) &= \mathrm{tr.deg}\, K\langle\eta\rangle/K\langle\bar{u}_1,\ldots,\bar{u}_d\rangle \\
&= \mathrm{tr.deg}\, K\langle\bar{u}_1,\ldots,\bar{u}_d, a_1\bar{y}_1+\cdots+a_{n-d}\bar{y}_{n-d}\rangle/K\langle\bar{u}_1,\ldots,\bar{u}_d\rangle \\
&= \mathrm{ord}(X,\omega).
\end{aligned}$$

□

Remark:

1) The above irreducible $X(u_1,\ldots,u_d,\omega)$ is called a **differential resolvent** of P or $\mathbb{V}(P)$.

2) With the obtained $a_1,\ldots,a_{n-d}$, we have $K\langle\bar{u}_1,\ldots,\bar{u}_d,\bar{y}_1,\ldots,\bar{y}_{n-d}\rangle = K\langle\bar{u}_1,\ldots,\bar{u}_d, a_1\bar{y}_1+\cdots+a_{n-d}\bar{y}_{n-d}\rangle$. (Proposition 4.2.14) In the case $d=0$, this is the differential primitive element theorem.