Recall: In order to measure the "size" of a differential variety (i.e., the solution set of alg diff equations), we introduced the notion of differential dimension:

Def 4.3.1 Let $V \subseteq \mathbb{A}^n$ be an irreducible δ-variety over K and $K\langle V \rangle$ be the diff (rational) function field of V. The differential dimension of V is defined as the diff transcendence degree of $K\langle V\rangle$ over K, that is,

$$\delta\text{-}\dim(V) = \delta.\text{tr.deg}\, K\langle V\rangle / K.$$

For an arbitrary V with irreducible components $V_1, \cdots, V_m$, $\delta\text{-}\dim(V) = \max_i \delta\text{-}\dim(V_i)$.

Remark: If V is irreducible with a generic point $\eta = (\eta_1, \cdots, \eta_n)$, then

1) $\delta\text{-}\dim(V) = \delta.\text{tr.deg}\, K\langle \eta \rangle / K$ (for $K\langle V\rangle = K\langle \bar{y}_1, \cdots, \bar{y}_n\rangle \cong K\langle \eta_1, \cdots, \eta_n\rangle$, $\bar{y}_i = y_i + \mathbb{I}(V)$)
2) $\delta\text{-}\dim(V) =$ the cardinal number of a parametric set of $\mathbb{I}(V)$ (Def 4.3.2 and Ex).

Lemma 4.3.2 Let V be a δ-variety and $W \subseteq V$ a δ-subvariety. Then $\delta\text{-}\dim(W) \leq \delta\text{-}\dim(V)$.

Proof. First assume W and V are both irreducible. $W \subseteq V$ implies that $\mathbb{I}(W) \supseteq \mathbb{I}(V)$. Suppose $\delta\text{-}\dim(W) = d$ and $\{y_1, \cdots, y_d\}$ is a parametric set of $\mathbb{I}(W)$. Clearly $\mathbb{I}(V) \cap \{y_1, \cdots, y_d\} = \{0\}$ and $\{y_1, \cdots, y_d\}$ is a δ-independent set modulo $\mathbb{I}(V)$ which could be extended to a parametric set of $\mathbb{I}(V)$. Thus, $\delta\text{-}\dim(V) = \delta\text{-}\dim(\mathbb{I}(V)) \geq d$.

Now, let V and and W be arbitrary. Let W_1 be an irreducible component of W with $\delta\text{-}\dim(W) = \delta\text{-}\dim(W_1)$. Then W_1 is contained in an irreducible component V_1 of V. By the above,

$$\delta\text{-}\dim(W) = \delta\text{-}\dim(W_1) \leq \delta\text{-}\dim(V_1) \leq \delta\text{-}\dim(V). \qquad \blacksquare$$

Exercise: Let $W \subseteq V$ be two irreducible δ-varieties with $\delta\text{-}\dim(W) = \delta\text{-}\dim(V)$. Is $W = V$?

It is true in the algebraic case but not valid in differential algebra:

<u>Non-example</u>: Let $W = \mathbb{V}(y') \subseteq \mathbb{A}^1$ and $V = \mathbb{V}(y'') \subseteq \mathbb{A}^1$. Then $W \subseteq V$ and $\delta\text{-dim}(W) = \delta\text{-dim}(V)$. But $W \neq V$.

This example shows that the differential dimension is not a fine enough measure of size, thus we need a more discriminating measure: the differential dimension polynomial of an irreducible δ-variety V or $\mathbb{I}(V)$.

The idea of Hilbert polynomial for homogenous ideals suggests that it might be a way to consider the truncated coordinate ring by order: Let $p \subseteq K\{y_1, \dots, y_n\}$ be a prime δ-ideal. Denote $K[y_1^{[t]}, \dots, y_n^{[t]}] = K[y_i^{(j)} : j \le t, i = 1, \dots, n]$ and let $p_t = p \cap K[y_1^{[t]}, \dots, y_n^{[t]}]$. Then p_t is a prime alg. ideal with dimension $\dim(p_t)$. Kolchin showed that for $t \gg 0$, $\dim(p_t)$ is a numerical polynomial. We state it with the language of (δ-)field extensions.

<u>Theorem 4.3.3 (Kolchin)</u> Let $p \subseteq K\{y_1, \dots, y_n\}$ be a prime δ-ideal with a generic point $\eta = (\eta_1, \dots, \eta_n)$. Then there exists a numerical polynomial $\omega_p(t) \in \mathbb{R}[t]$ with the following properties:

1) For sufficiently large $t \in \mathbb{N}$, $\dim(p_t) = \omega_p(t)$;
2) $\omega_p(t) = d(t+1) + s$ with $d = \delta\text{-dim}(\mathbb{V}(p))$ and some $s \in \mathbb{N}$;
3) (Computation of $\omega_p(t)$) Let $\mathcal{A} = A_1, \dots, A_\ell$ be a characteristic set of p w.r.t. some orderly ranking of $\Theta(Y) = \{\delta^k y_j : k \in \mathbb{N}, j = 1, \dots, n\}$ and suppose $\mathrm{ld}(A_i) = y_{\sigma(i)}^{(s_i)}$. Then $\omega_p(t) = (n - \ell)(t+1) + \sum_{i=1}^{\ell} s_i$.
4) $\omega_p(t) = n(t+1) \iff p = [0]$ (i.e. $\mathbb{V}(p) = \mathbb{A}^n$);
   $\omega_p(t) = 0 \iff \mathbb{V}(p)$ is a finite set.

Def. 4.3.4 Let $V \subseteq \mathbb{A}^n$ be an irreducible differential variety $/K$ and $p = \mathbb{I}(V)$. The above $\omega_p(t)$ is defined as the differential dimension polynomial of p or V, also denoted by $\omega_V(t)$.

Proof of Theorem 4.3.3: Denote $\eta^{[t]} = (\eta_1, \dots, \eta_n, \eta_1', \dots, \eta_n', \dots, \eta_1^{(t)}, \dots, \eta_n^{(t)})$.

Clearly, $\eta^{[t]}$ is a generic point of $p_t \subseteq K[Y_1^{[t]}, \dots, Y_n^{[t]}]$.

So $\dim(p_t) = \mathrm{tr.deg}\, K(\eta^{[t]})/K$.

Recall that $u_A = \mathrm{lead}(A)$, $I_A S_A \notin p$.

For each $A \in \mathcal{A}$, $A(\eta) = 0$ and $I_A(\eta) \neq 0$ imply that

$$u_A(\eta) \text{ is algebraic over } \mathcal{F}(\eta_j^{(k)} : Y_j^{(k)} < u_A, j = 1, \dots, n).$$

Repeated differentiation shows that if v is any derivative of u_A, then $v(\eta)$ is algebraic over $\mathcal{F}(\eta_j^{(k)} : Y_j^{(k)} < v, j = 1, \dots, n)$.

Let M denote the set of all derivatives $Y_j^{(k)}$ that are not derivatives of any u_A $(A \in \mathcal{A})$ and let $M(t) = M \cap \{Y_j^{(k)} : k \le t; j = 1, \dots, n\}$.

So, for $t \geq \max\{s_1, \dots, s_\ell\}$, we have that

$$K(\eta^{[t]}) \text{ is algebraic over } K\big((v(\eta))_{v \in M(t)}\big). \quad (*)$$

(Arrange $\{Y_j^{(k)} : k \le t, j = 1, \dots, n\} \setminus M(t)$ in increasing order: $u_1 < u_2 < \cdots$. From the above, u_1 is alg over $K\big((v(\eta))_{v \in M(t)}\big)$ and $(*)$ can be shown by induction.)

Thus, $\dim(p_t) = \mathrm{tr.deg}\, K(\eta^{[t]})/K = \mathrm{Card}(M(t))$.

Since $M(t) = \underbrace{\{Y_{\sigma(i)}, Y_{\sigma(i)}', \dots, Y_{\sigma(i)}^{(s_i - 1)} : i = 1, \dots, \ell\}}_{\text{derivatives of leading variables}} \cup \underbrace{\{Y_j, Y_j', \dots, Y_j^{(t)} : j \neq \sigma(1), \dots, \sigma(\ell)\}}_{\text{parametric variable parts}}$,

$\mathrm{Card}(M(t)) = (n - \ell)(t+1) + \sum_{i=1}^{\ell} s_i$. So $\dim(p_t) = (n-\ell)(t+1) + \sum_{i=1}^{\ell} s_i$ for $t \geq \max\{s_1, \dots, s_\ell\}$. Denote $\omega_p(t) = (n-\ell)(t+1) + \sum_{i=1}^{\ell} s_i$. This finishes the proof of 1) and 3).

To show 4), $\omega_p(t) = n(t+1) \Leftrightarrow M(t) = \{Y_j^{(k)} : k \le t; j = 1, \dots, n\} \Leftrightarrow p = [0]$;

And $\omega_p(t) = 0 \Leftrightarrow M(t) = \emptyset \Leftrightarrow \mathrm{ld}(\mathcal{A}) = \{Y_1, \dots, Y_n\} \Leftrightarrow \mathbb{V}(p)$ is a finite set.

It suffices to show $\delta\text{-dim}(p) = n-l$ to complete the proof of 2).
Assume $d = \delta\text{-dim}(p) = \delta\text{-tr.deg}\, K\langle\eta\rangle/K$. W.L.O.G. let
$\eta_1, \ldots, \eta_d$ be a diff transcendence basis of $K\langle\eta\rangle$ over K.
Thus, $\omega_p(t) = \text{tr.deg}\, K(\eta_1^{[t]}, \ldots, \eta_n^{[t]})\, K = (n-l)(t+1) + \sum_{j=1}^{l} s_j$

$$\geq \text{tr.deg}\, K(\eta_1^{[t]}, \ldots, \eta_d^{[t]})/K = d(t+1),$$

and $n-l \geq d$ follows.

Conversely, let $\{z_1, \ldots, z_{n-l}\} = \{Y_1, \ldots, Y_n\} \setminus \{Y_{\sigma(1)}, \ldots, Y_{\sigma(l)}\}$. Since any nonzero poly in $K\{z_1, \ldots, z_{n-l}\}$ is reduced w.r.t. $\mathcal{A}$, we have $K\{z_1, \ldots, z_{n-l}\} \cap p = \{0\}$. So $\{z_1, \ldots, z_{n-l}\}$ is an independent set modulo p and can be enlarged to be a parametric set of p. Thus, $n-l \leq \delta\text{-dim}(p) = d$.
Hence, $n-l = d = \delta\text{-dim}(p)$. $\boxtimes$.

Remark: 1) The δ-dimension polynomial of an irreducible δ-variety $V \subseteq \mathbb{A}^n$ is of the form

$$\omega_V(t) = d(t+1) + s, \text{ where } d = \delta\text{-dim}(V).$$

The number s is defined as the order of V, denoted by $\text{ord}(V)$. The order is the regorous definition for the notion "the number of arbitrary constants" of the solution set of algebraic differential equations.

2) In the partial differential case, $(K, \{\delta_1, \ldots, \delta_m\})$, we have the similar notion of differential dimension polynomial. There, $\omega_V(t) = a_m\binom{t+m}{m} + a_{m-1}\binom{t+m-1}{m-1} + \cdots + a_1(t+1) + a_0$, where $a_m = \delta\text{-dim}(V)$.

Example: Let $W = \mathbb{V}(y') \subseteq \mathbb{A}^1$ and $V = \mathbb{V}(y'') \subseteq \mathbb{A}^1$. $W \subsetneqq V$ but δ-dim$(W) = \delta$-dim$(V) = 0$. Note that $\omega_W(t) = 1$ and $\omega_V(t) = 2$.

The next Proposition shows that δ-dimension poly is a finer measure than δ-dimension.

Prop 4.3.5. Let $W, V \subseteq \mathbb{A}^n$ be irreducible δ-varieties and $W \subsetneqq V$. Then $\omega_W(t) < \omega_V(t)$.

Proof. Let $P_1 = \mathbb{I}(W)$ and $P_2 = \mathbb{I}(V)$. Then $W \subsetneqq V$ implies that $P_1 \supsetneqq P_2$.

So for t sufficiently large ($t \gg 0$),

$$P_1 \cap K[y_1^{[t]}, \dots, y_n^{[t]}] \supsetneqq P_2 \cap K[y_1^{[t]}, \dots, y_n^{[t]}],$$

Consequently,

$$\begin{aligned} \omega_W(t) &= \dim P_1 \cap K[y_1^{[t]}, \dots, y_n^{[t]}] \\ &> \dim P_2 \cap K[y_1^{[t]}, \dots, y_n^{[t]}] \\ &= \omega_V(t) \quad \text{for } t \gg 0. \end{aligned}$$

$\boxtimes$.

We end this section by showing that an irreducible δ-variety is differentially birationally equivalent to an irreducible δ-variety of codimension 1.

We now identify elements of the differential coordinate ring $K\{V\} = K\{y_1, \dots, y_n\}/\mathbb{I}(V)$ with K-valued functions on V and call them diff polynomial functions on V. And each elt of $K\langle V\rangle = \mathrm{Frac}(K\{V\})$ can be identified as a differential rational function on V. If $\eta \in V$, $f/g \in K\langle V\rangle$ is defined at η if $g(\eta) \neq 0$ ($f, g \in K\{V\}$).

Def 4.3.6 Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be irreducible δ-varieties over K. A differential rational map $\varphi : V \dashrightarrow W$ is a family $(f_1, \cdots, f_m) \in K\langle V\rangle^m$ such that $\varphi(\eta) = (f_1(\eta), \cdots, f_m(\eta)) \in W$ whenever the coordinate functions $f_1, \cdots, f_m$ are defined at η.
φ is called dominate if the Kolchin closure of $\varphi(V)$ is W (Or equivalently, φ maps a generic point of V to that of W).

And, φ is called a differential birational map if φ is dominate and there is a dominant differential rational map $\psi : W \dashrightarrow V$, called the generic inverse of φ such that

- if φ is defined at η and ψ is defined at $\varphi(\eta)$, then $\psi(\varphi(\eta)) = \eta$;
- if ψ is defined at ζ and φ is defined at $\psi(\zeta)$, then $\varphi(\psi(\zeta)) = \zeta$.

In this case, we also call V and W are δ-birationally equivalent.

Theorem 4.3.7 Suppose (K, δ) contains a nonconstant elt. Let $P \subseteq K\{u_1, \cdots, u_d, y_1, \cdots, y_{n-d}\}$ be a prime δ-ideal with a parametric set $\{u_1, \cdots, u_d\}$. Then $\exists\, a_1, \cdots, a_{n-d} \in K$ s.t. $[P, w - a_1 y_1 - \cdots - a_{n-d} y_{n-d}] \subseteq K\{u_1, \cdots, u_d, y_1, \cdots, y_{n-d}, w\}$ has a characteristic set of the form

$$\chi(u_1, \cdots, u_d, w)$$
$$I_1(u_1, \cdots, u_d, w) y_1 - T_1(u_1, \cdots, u_d, w)$$
$$\vdots$$
$$I_{n-d}(u_1, \cdots, u_d, w) y_{n-d} - T_{n-d}(u_1, \cdots, u_d, w)$$

w.r.t. the elimination ranking $u_1 < \cdots < u_d < w < y_1 < \cdots < y_{n-d}$.

Proof. Let $\eta=(\bar{u}_1,\cdots,\bar{u}_d,\bar{y}_1,\cdots,\bar{y}_{n-d})$ be a generic point of $\mathcal{P}$. Introduce $n-d$ new differential indeterminates $\lambda_1,\cdots,\lambda_{n-d}$ over $K\langle\eta\rangle$. Let $\mathcal{J}=[\mathcal{P}, w-\lambda_1 y_1-\cdots-\lambda_{n-d}y_{n-d}]\subseteq K\{u_1,\cdots,u_d,y_1,\cdots,y_{n-d},\lambda_1,\cdots,\lambda_{n-d},w\}$. Then $\mathcal{J}$ is a prime δ-ideal with a generic point

$$\zeta=(\bar{u}_1,\cdots,\bar{u}_d,\bar{y}_1,\cdots,\bar{y}_{n-d},\lambda_1,\cdots,\lambda_{n-d},\lambda_1\bar{y}_1+\cdots+\lambda_{n-d}\bar{y}_{n-d}).$$

Since $\delta\text{-}\dim(\mathcal{P})=d$, $\delta.\mathrm{tr.deg}\,K\langle\eta\rangle/K=d$ and

$$\begin{aligned}\delta.\mathrm{tr.deg}\,K\langle\zeta\rangle/K &= \delta.\mathrm{tr.deg}\,K\langle\eta\rangle/K+\delta.\mathrm{tr.deg}\,K\langle\eta\rangle\langle\lambda_1,\cdots,\lambda_{n-d}\rangle/K\langle\eta\rangle\\ &= d+n-d=n.\end{aligned}$$

So $\mathcal{J}_\lambda=\mathcal{J}\cap K\{u_1,\cdots,u_d,\lambda_1,\cdots,\lambda_{n-d},w\}\neq[0]$ and $\{u_1,\cdots,u_d,\lambda_1,\cdots,\lambda_{n-d}\}$ is a parametric set of $\mathcal{J}_\lambda$. Let $\{R(u_1,\cdots,u_d,\lambda_1,\cdots,\lambda_{n-d},w)\}$ be a characteristic set of $\mathcal{J}_\lambda$ w.r.t. the elimination ranking $u_1<\cdots<u_d<\lambda_1<\cdots<\lambda_{n-d}<w$. Denote $s=\mathrm{ord}(R,w)\geq 0$. Since $R(\bar{u}_1,\cdots,\bar{u}_d,\lambda_1,\cdots,\lambda_{n-d},\lambda_1\bar{y}_1+\cdots+\lambda_{n-d}\bar{y}_{n-d})=0$, for $j=1,\cdots,n-d$, take the partial derivative of this identity w.r.t. $\lambda_j^{(s)}$ on both sides, then we obtain

$$\overline{\frac{\partial R}{\partial\lambda_j^{(s)}}}+\overline{\frac{\partial R}{\partial w^{(s)}}}\cdot\bar{y}_j=0 \qquad (1)$$

where $\overline{\frac{\partial R}{\partial\lambda_j^{(s)}}}$ and $\overline{\frac{\partial R}{\partial w^{(s)}}}$ are obtained from $\frac{\partial R}{\partial\lambda_j^{(s)}}$ and $\frac{\partial R}{\partial w^{(s)}}$ by substituting $(u_1,\cdots,u_d,\lambda_1,\cdots,\lambda_{n-d},w)=(\bar{u}_1,\cdots,\bar{u}_d,\lambda_1,\cdots,\lambda_{n-d},\lambda_1\bar{y}_1+\cdots+\lambda_{n-d}\bar{y}_{n-d})$. Note that $\frac{\partial R}{\partial w^{(s)}}\notin\mathcal{J}_\lambda$, so $\overline{\frac{\partial R}{\partial w^{(s)}}}\neq 0$. As $\overline{\frac{\partial R}{\partial w^{(s)}}}\in K\{\eta\}\{\lambda_1,\cdots,\lambda_{n-d}\}$ is nonzero, by the non-vanishing theorem of nonzero polynomials, $\exists\, a_1,\cdots,a_{n-d}\in K$ (K contains nonconstant elements) s.t. $\overline{\frac{\partial R}{\partial w^{(s)}}}\Big|_{\lambda_i=a_i}\in K\{\eta\}\setminus\{0\}$. Let $I(u_1,\cdots,u_d,w)=\frac{\partial R}{\partial w^{(s)}}\Big|_{\lambda_i=a_i,\,i=1,\cdots,n-d}\in K\{u_1,\cdots,u_d,w\}$. Then $I(\bar{u}_1,\cdots,\bar{u}_d,a_1\bar{y}_1+\cdots+a_{n-d}\bar{y}_{n-d})=\overline{\frac{\partial R}{\partial w^{(s)}}}\Big|_{\lambda_i=a_i}\neq 0$.

Let $J_a = [P, w - a_1 y_1 - \cdots - a_{n-d} y_{n-d}] \subseteq K\{u_1, \cdots, u_d, y_1, \cdots, y_{n-d}, w\}$. Then J_a is a prime δ-ideal with a generic point $\zeta_a = (\bar{u}_1, \cdots, \bar{u}_d, \bar{y}_1, \cdots, \bar{y}_{n-d}, a_1 \bar{y}_1 + \cdots + a_{n-d} \bar{y}_{n-d})$. Clearly, $I(u_1, \cdots, u_d, w) \notin J_a$. Let $T_j(u_1, \cdots, u_d, w) = \frac{\partial R}{\partial \lambda_j^{(s)}}\Big|_{\lambda_i = a_i, i=1,\cdots,n-d}$. By (1),

$$I(u_1, \cdots, u_d, w)\, y_j - T_j(u_1, \cdots, u_d, w) \in J_a.$$

Since $\delta.\text{tr.deg}\, K\langle \zeta_a \rangle / K = d$, $J_a \cap K\{u_1, \cdots, u_d, w\} \neq [0]$ with a parametric set $\{u_1, \cdots, u_d\}$. So its characteristic set consists of a single δ-polynomial. Let $X(u_1, \cdots, u_d, w)$ be an irr. poly consistuting a characteristic set of $J_a \cap K\{u_1, \cdots, u_d, w\}$ w.r.t. the elimination ranking R: $u_1 < \cdots < u_d < w$. For each j, take the differential remainder of $I y_j - T_j$ w.r.t. X (under R). Since $I \notin J_a \cap K\{u_1, \cdots, u_d, w\}$, $\delta\text{-rem}(I y_j - T_j, X)$ is of the form

$I_j y_j - T_j^0$ where $I_j, T_j^0 \in K\{u_1, \cdots, u_d, w\}$, $I_j \notin J_a$.

<u>Claim:</u> $X(u_1, \cdots, u_d, w)$, $I_1 y_1 - T_1^0, \cdots, I_{n-d} y_{n-d} - T_{n-d}^0$ is a characteristic set of J_a w.r.t. the elimination ranking $u_1 < \cdots < u_d < w < y_1 < \cdots < y_{n-d}$.

Indeed, for $\forall f \in J_a$, first perform the Ritt-Kolchin reduction process for f w.r.t. $I_1 y_1 - T_1^0, \cdots, I_{n-d} y_{n-d} - T_{n-d}^0$, then we get $f_0 \in J_a \cap K\{u_1, \cdots, u_d, w\}$, thus f_0 could be reduced to 0 by X. Thus, we have proved the theorem.

Remark: 1) The above irreducible $X(u_1, \cdots, u_d, w)$ is called a differential resolvent of P or $\mathbb{V}(P)$.

2) With the obtained $a_1, \cdots, a_{n-d}$, we have

$K\langle \bar{u}_1, \cdots, \bar{u}_d, \bar{y}_1, \cdots, \bar{y}_{n-d}\rangle = K\langle \bar{u}_1, \cdots, \bar{u}_d, a_1\bar{y}_1 + \cdots + a_{n-d}\bar{y}_{n-d}\rangle$.
(Prop 4.2.14) In the case $d=0$, this is the primitive thm.

<u>Corollary 4.3.8</u> Let (K, δ) contain a nonconstant element. Let $V \subseteq \mathbb{A}^n$ be an irreducible δ-variety. Then V is δ-birationally equivalent to the general component of an irreducible δ-polynomial (i.e., an irreducible δ-variety of codim 1).

<u>Proof</u>. Suppose δ-$\dim(V) = d$ and $\{u_1, \cdots, u_d\}$ is a parametric set of $P = \mathbb{I}(V) \subseteq K\{u_1, \cdots, u_d, y_1, \cdots, y_{n-d}\}$. By Theorem 4.3.7, $\exists\, a_1, \cdots, a_{n-d} \in K$ s.t. $J_a = [P, w - a_1y_1 - \cdots - a_{n-d}y_{n-d}] \subseteq K\{u_1, \cdots, u_d, y_1, \cdots, y_{n-d}, w\}$ has a characteristic set of the form $X(u_1, \cdots, u_d, w),\ I_1(u_1, \cdots, u_d, w)y_1 - T_1(u_1, \cdots, u_d, w), \cdots, I_{n-d}y_{n-d} - T_{n-d}$ w.r.t. the elimination ranking $u_1 < \cdots < u_d < w < y_1 < \cdots < y_{n-d}$, where X is irreducible $(*)$.

Let $W = \mathbb{V}(\mathrm{sat}(X)) \subseteq \mathbb{A}^{d+1}$ be the general component of X.
Define $\varphi: V \longrightarrow W$ by

$$\varphi(u_1, \cdots, u_d, y_1, \cdots, y_{n-d}) = (u_1, \cdots, u_d, a_1y_1 + \cdots + a_{n-d}y_{n-d})$$

and $\psi: W \dashrightarrow V$ by

$$\psi(u_1, \cdots, u_d, w) = \left(u_1, \cdots, u_d, \frac{T_1(u_1, \cdots, u_d, w)}{I_1(u_1, \cdots, u_d, w)}, \cdots, \frac{T_{n-d}(u_1, \cdots, u_d, w)}{I_{n-d}(u_1, \cdots, u_d, w)}\right).$$

Let $\xi = (\bar{u}_1, \cdots, \bar{u}_d, \bar{y}_1, \cdots, \bar{y}_{n-d})$ be a generic point of V and $\eta = (\bar{u}_1, \cdots, \bar{u}_d, \bar{w})$ be a generic point of W.
It is easy to show that both φ and ψ are dominant, and $(\psi \circ \varphi)(\xi) = \xi$, $(\varphi \circ \psi)(\eta) = \eta$ from $(*)$. So V and W are δ-birationally equivalent. $\boxtimes$.