初等函数的积分理论讲义

Let (R, D) be a differential ring and $C_R = \{ r \in R \mid Dr = 0 \}$ be the ring of constants of (R, D). We have the following facts:

1) Let 1 be the identity in R, $D(1) = 0$

2) For any $n \in \mathbb{N}$, $a \in R$, $D(a^n) = n a^{n-1} D(a)$

3) If b is invertible in R, then $D\left(\frac{a}{b}\right) = \frac{D(a)b - aD(b)}{b^2}$

4) If a_i is invertible in R, then for any $m_i \in \mathbb{Z}$,

$$\frac{D(a_1^{m_1} \cdots a_s^{m_s})}{a_1^{m_1} \cdots a_s^{m_s}} = \sum_{i=1}^{s} m_i \frac{Da_i}{a_i}$$

5) Let $Der(R) = \{ D : R \to R \mid D \text{ is a derivation on } R \}$. Then $Der(R)$ is a R-module, i.e., $r_1 D_1 + r_2 D_2 \in Der(R)$ for all $r_1, r_2 \in R$ and $D_1, D_2 \in Der(R)$.

Let (R, D) and $(\bar{R}, \bar{D})$ be differential rings. If $R \subseteq \bar{R}$ and $\bar{D}|_R = D$, then $(\bar{R}, \bar{D})$ is called a differential extension of (R, D).

Theorem 1

1) Let (R, D) be a differential integral domain. Then D can be uniquely extended to the quotient field F of R by

$$D\left(\frac{a}{b}\right) = \frac{D(a)b - aD(b)}{b^2}$$

2) Let (F, D) be a differential field and α be algebraic over F. Then D can be uniquely extension to the algebraic extension $F(\alpha)$.

3) Let (F, D) be a differential field and t be transcendental over F. Then D can be uniquely extended to $F(t)$ by fixing the value $Dt \in F(t)$.

Example 1 Let $F=\mathbb{Q}(x)$ and $\alpha\in\overline{F}$ satisfying that

$$4\alpha^2-9x=0$$

then $$\frac{d}{dx}(4\alpha^2-9x) = 8\alpha\frac{d\alpha}{dx}-9=0$$

$$\Rightarrow \quad \frac{d\alpha}{dx}=\frac{9}{8\alpha}$$

In general, we have

Theorem 2. Let (F,D) be a differential field of char. 0. and α be algebraic over F. Then $D\alpha\in F(\alpha)$

Proof Assume that $p\in F[x]$ be the minimal polynomial of α, i.e. $p(\alpha)=0$ and p is irreducible $/F$.

Write $p=p_dx^d+p_{d-1}x^{d-1}+\cdots+p_0$, with $p_i\in F$ and $p_0p_d\neq 0$

$$\underbrace{D(p(\alpha))}_{=0}=D(p)(\alpha)+p_x(\alpha)D(\alpha), \text{ where } D(p)=\sum_{i=0}^{d}D(p_i)x^i, \quad p_x=\sum_{i=1}^{d} i\,p_ix^{i-1}$$

Since p is irreducible, we have $\gcd(p,p_x)=1$, which implies

$ap+bp_x=1$ for some $a,b\in F[x]$.

$$D(\alpha)=-\frac{D(p)(\alpha)}{p_x(\alpha)}=-D(p)(\alpha)\cdot b(\alpha)\in F(\alpha)$$

Corollary 3 Let $\alpha(x)$ be an algebraic function over $\mathbb{C}(x)$. Then $\alpha(x)$ satisfies a nontrivial linear differential equation with coefficients in $\mathbb{C}[x]$, i.e.,

$$p_n\cdot\frac{d^n\alpha}{dx^n}+p_{n-1}\frac{d^{n-1}\alpha}{dx^{n-1}}+\cdots+p_0\alpha=0, \text{ where } p_i\in\mathbb{C}[x] \text{ and } p_n\neq 0.$$

Proof Since $\frac{d^i\alpha}{dx^i}\in\mathbb{C}(x)(\alpha)$ and $[\mathbb{C}(x)(\alpha):\alpha]=n<+\infty$, we have $\{\alpha,\frac{d\alpha}{dx},\ldots,\frac{d^n\alpha}{dx^n}\}$ is linearly dependent over $\mathbb{C}(x)$

Example 2 Let $F = \mathbb{C}(x)$, $D = \frac{d}{dx}$. We first show that $t = \exp(x)$ is transcendental over F. Note that

$$Dt = t$$

Suppose that $\exp(x)$ is algebraic over F. Then there exists an irreducible polynomial $P = p_0 + p_1 Y + \cdots + Y^n \in F[Y]$ s.t. with $p_0 \neq 0$

$$t^n + p_{n-1} t^{n-1} + \cdots + p_0 = 0$$

$$\Rightarrow \quad n t^{n-1} Dt + D(p_{n-1}) t^{n-1} + (n-1) p_{n-1} D(t) t^{n-2} + \cdots + D(p_0) = 0$$

$$\Rightarrow \quad n t^n + \left(D(p_{n-1}) + (n-1) p_{n-1} \right) t^{n-1} + \cdots + D(p_0) = 0$$

$$\Rightarrow \quad \frac{D(p_0)}{p_0} = n \neq 0$$

Claim $D(y) = ny$ has no nonzero solution in $\mathbb{C}(x)$.

If $f = \frac{p(x)}{q(x)}$ is a nonzero solution of $D(y) = ny$, then

$$D(f) = \frac{D(p)q - pD(q)}{q^2} = \frac{np}{q} \Rightarrow (D(p) - np) q = p D(q) \quad (*)$$

Suppose that $q \notin \mathbb{C}$. Then $q = (x-\lambda)^m \bar{q}$, $\bar{q}(\lambda) \neq 0$.

$$D(q) = m (x-\lambda)^{m-1} \bar{q} + (x-\lambda)^m D(\bar{q}) \Rightarrow (x-\lambda)^{m-1} \mid D(q)$$

but $(x-\lambda)^m \nmid D(q)$.

But by $(*)$, we have $q \mid p D(q)$, $q \mid D(q)$ (since $\gcd(p,q) = 1$)

$\Rightarrow$ $(x-\lambda)^m \mid Dq$, a contradiction.

$\Rightarrow$ $t = \exp(x)$ is transcendental over $\mathbb{C}(x)$

Remark By a similar argument, we can prove that $t = \log(x)$ is also transcendental $/\mathbb{C}(x)$.

Let $t=\exp(x)$. We can extend $D=\frac{d}{dx}$ on $\mathbb{C}(x)$ to $\mathbb{C}(x)(t)$ by defining $Dt=t$.

Let $t=\log(x)$, we can extend $D=\frac{d}{dx}$ on $\mathbb{C}(x)$ to $\mathbb{C}(x)(t)$ by defining $Dt=\frac{1}{x}$.

- Elementary extensions. (All fields are of char 0)

<u>DEFinition 4</u> Let (E,D) be a differential extension of (F,D). Let $t\in E$. We say that t is <u>algebraic</u> over F, if $\exists\, p\in F[x]\setminus F$ s.t. $p(t)=0$; t is <u>exponential</u> over F, if $\exists\, a\in F$ s.t. $Dt=D(a)\cdot t$; t is <u>logarithmic</u> over F, if $\exists\, a\in F\setminus\{0\}$, s.t. $D(t)=\frac{D(a)}{a}$.

t is said to be <u>elementary</u> over F if t is algebraic, exponential, or logarithmic over F. E is said to be <u>elementary</u> over F if $E=F(t_1,\dots,t_n)$ and t_i is elementary over $F(t_1,\dots,t_{i-1})$ for all $i=1,\dots s$. If $F=\mathbb{C}(x)$, any element of E is called an <u>elementary function</u> over $\mathbb{C}(x)$.

<u>Example 1</u> Let $F=\mathbb{C}(x)$, $D=\frac{d}{dx}$ consider the function

$$f(x)=\frac{\pi}{\sqrt{\log\left(\exp\left(\sqrt{\frac{1}{2x^2+1}}\right)^2+x^2+1\right)}}$$

We show that $f(x)$ is elementary over $\mathbb{C}(x)$. Let $E=\mathbb{C}(x)(t_1,t_2,t_3,t_4)$ with

$$t_1=\sqrt{\frac{1}{2x^2+1}},\quad t_2=\exp(t_1),\quad t_3=\log(t_2^2+x^2+1),\quad t_4=\sqrt{t_3}$$

Then $f=\frac{\pi}{t_4}\in E$.

DEF 5 Let (F, D) be a differential field. and $f \in F$. If there exists an elementary extension (E, D) of (F, D) and $g \in E$ s.t. $f = D(g)$, then we say that f is elementarily integrable over F

Problem Given $f \in F = \mathbb{C}(x)(t_1, \dots, t_n)$, an elementary extension of $\mathbb{C}(x)$, decide whether f is elementarily integrable over F.

We will show that the elementary functions:

$$\exp(x^2), \quad \frac{\exp(x)}{x}, \quad \frac{1}{\log(x)}, \quad \sin(x^2)$$

have no elementary (indefinite) integrals (即这些函数是"初等积不出来的")

- Special and normal polynomials.

DEF 6 Let (k, D) be a differential field, t be transcendental $/k$. Assume that $Dt \in k[t]$. and $p \in k[t]$. We call p a <u>special polynomial</u> if $\gcd(p, Dp) = p$. and a <u>normal polynomial</u> if $\gcd(p, Dp) = 1$.

Remark If p is irreducible, then p is either special or normal. If p is not irreducible, then p can be neither special nor normal.

Example 7 1) Let $k = \mathbb{C}(x)$, $t = \tan(x)$. Then $Dt = 1 + t^2$. Let $p_1 = 1 + t^2$, we have $Dp_1 = 2t(1+t^2)$, so p_1 is special. Let $p_2 = t^2$ we have $Dp_2 = 2t(1+t^2)$. Then p_2 is neither special nor normal. $p_3 = t + t^2$ is normal.

2) $K=\mathbb{C}(x)$, $t=\exp(x)$. Then $p_1=t$ is special and $p_2=1+t$ is normal

3) $K=\mathbb{C}(x)$, $t=\log(x)$. Then all monic irreducible polynomials p are normal since $\deg_t D(p) < \deg_t p$.

- order functions.

DEF. 7 Let K be a field of char. 0. and t be transcendental $/K$. Let $f \in K(t)$ and $p \in K[t]$ be irreducible. Then $f = p^m g$, for some $m \in \mathbb{Z}$, $g = \frac{a}{b} \in K(t)$ with $a, b \in K[t]$ and $\gcd(a,b)=1$ such that $p \nmid ab$, we call m the order of f at p, denoted by $\mathrm{ord}_p(f)$.

Lemma 8 Let $p \in K[t]$ be irreducible and $f, g \in K(t)$. Then

1) $\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$

2) $\mathrm{ord}_p(f+g) \geq \min\{\mathrm{ord}_p(f), \mathrm{ord}_p(g)\}$. The equality holds when $\mathrm{ord}_p(f) \neq \mathrm{ord}_p(g)$.

Remark If p is not irreducible, then 1) may not be true. For example $p = t^2$, $f=g=t$, $\mathrm{ord}_p(fg) \geq \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$

Lemma 9. Let (K, D) be a differential field and t be transcendental over K with $Dt \in K[t]$. Let $f \in K(t)$ and p be irreducible in $K[t]$

Then 1) $\mathrm{ord}_p(Df) \geq \mathrm{ord}_p(f) - 1$;

2) If p is a normal polynomial, then

$$\mathrm{ord}_p(Df) = \begin{cases} \geq 0 &, \mathrm{ord}_p(f) = 0, \\ \mathrm{ord}_p(f) - 1, & \mathrm{ord}_p(f) \neq 0. \end{cases}$$

proof. 1) Write $f = p^m g = p^m \frac{a}{b}$, $\gcd(p, ab) = 1$, $m = ord_p(f)$

If $m = 0$, then $ord_p(Df) = ord_p(D(\frac{a}{b})) = ord_p\left(\frac{D(a)b - aD(b)}{b^2}\right) \geq 0 \geq -1$

If $m \neq 0$, then

$$Df = (mp^{m-1}Dp)\frac{a}{b} + p^m D(\frac{a}{b}) = p^{m-1}\left(mD(p)\frac{a}{b} + pD(\frac{a}{b})\right)$$

Since $\gcd(p, ab) = 1$, $ord_p(\frac{a}{b}) = 0$. $\Rightarrow ord_p(mD(p)\frac{a}{b}) \geq 0$ since $ord_p(Dp) \geq 0$

$D(\frac{a}{b}) = \frac{bDa - aDb}{b^2} \Rightarrow ord_p(D(\frac{a}{b})) \geq 0 \Rightarrow ord_p(pD(\frac{a}{b})) \geq 1$

$$\Rightarrow ord_p\left(mD(p)\frac{a}{b} + pD(\frac{a}{b})\right) \geq 0$$

$$\Rightarrow ord_p(Df) \geq m-1 = ord_p(f) - 1$$

2) If $ord_p(f) = 0$, then $ord_p(Df) \geq 0$

If $ord_p(f) \neq 0$, then $ord_p(Df) \geq ord_p(f) - 1$ by 1)

Since p is normal, $\gcd(p, Dp) = 1 \Rightarrow ord_p(Dp) = 0$

$$\Rightarrow ord_p((mp^{m-1}Dp)\frac{a}{b}) = m-1.$$

Since $ord_p(p^m D(\frac{a}{b})) \geq m$. Then $ord_p(Df) = m-1$

Remark Lemma 9 is very useful in the following discussions.

In particular, we have $ord_p(\frac{Df}{f}) = -1$ if $ord_p(f) \neq 0$.

and $ord_p(Df) \neq -1$.

prop. 10 Let (K, D) be a differential field and F be a differential extension of K. If $t \in F$ is such that $a = \frac{t'}{t} \in K$ and $a \neq \frac{1}{n}\frac{u'}{u}$ for all $n \in \mathbb{N}$, $u \in K \setminus \{0\}$, then t is transcendental over K, $C_{K(t)} = C_K$ and t is the only irreducible special polynomial in $K[t]$.

<u>Proof.</u> Assume that t is algebraic over K. Then $\exists$ irreducible $p \in K[x]$ s.t.

$$t^n + p_{n-1}t^{n-1} + \cdots + p_0 = 0, \quad p_i \in K, \ p_0 \neq 0$$

By $Dt = at$, we have

$$ant^n + (p_{n-1}' + p_{n-1}(n-1)a)t^{n-1} + \cdots + p_0' = 0$$

Then $an = \frac{p_0'}{p_0} \Rightarrow a = \frac{1}{n}\frac{p_0'}{p_0}$ $\quad p_0 \in K \setminus \{0\}$ $\rightarrow\leftarrow$

So t is transcendental over K.

If $C_{K(t)} \neq C_K$, then $\exists \frac{p}{q} \in K$, $\gcd(p,q)=1$, satisfies that

$$D\left(\frac{p}{q}\right) = \frac{D(p)q - pD(q)}{q^2} = 0 \Rightarrow D(p)q = pD(q) \Rightarrow \begin{matrix} p \mid Dp \\ \text{and } q \mid Dq \end{matrix}$$

$\Rightarrow$ p, q are both special.

<u>Claim</u> special polynomials are of the form at^m, $a \in K$, $m \in \mathbb{N}$.

Let $p = p_n t^n + \cdots + p_0$ be a special polynomial with $p_n \neq 0$ and $p_i \neq 0$ for some $i \in \{0, \ldots, n-1\}$

Then $Dp = (Dp_n + p_n n a)t^n + \cdots + Dp_0$.

By $p \mid Dp$, we have

$$\frac{Dp_n + np_n a}{p_n} = \frac{Dp_i + np_i a}{p_i} \quad \text{for some } i$$

$$\Rightarrow (n-i)a = \frac{Dp_i}{p_i} - \frac{Dp_n}{p_n} = \frac{D(p_i/p_n)}{p_i/p_n} \Rightarrow a = \frac{1}{n-i}\frac{D(p_i/p_n)}{p_i/p_n} \rightarrow\leftarrow$$

Then the only irreducible special polynomial is t.

If $f = \frac{p}{q} \in C_{K(t)}$, then p, q are both special

$\Rightarrow$ $f = a t^m$, $a \in K$ and $m \in \mathbb{Z}$.

If $m \neq 0$, then $Df = 0$

$$\Rightarrow \frac{Df}{f} = \frac{Da}{a} + m \frac{Dt}{t} = 0$$

$$\Rightarrow \frac{Dt}{t} = -\frac{1}{m} \frac{Da}{a} = \frac{1}{m} \frac{D(\frac{1}{a})}{1/a} \rightarrow \Leftarrow$$

$\Rightarrow$ $f \in C_K$. $\Rightarrow$ $C_{K(t)} = C_K$.

<u>Corollary 11</u> $\exp(f(x))$ is transcendental over $\mathbb{C}(x)$ if $f \in \mathbb{C}(x) \setminus \mathbb{C}$

<u>proof</u> Let $t = \exp(f(x))$, $f \in \mathbb{C}(x) \setminus \mathbb{C}$. Then

$\frac{Dt}{t} = Df(x))$. If $D(f) = \frac{1}{n} \frac{Dg}{g}$ for some $n \in \mathbb{N}$ and $g \in \mathbb{C}(x)$ $n \neq 0$ $g \neq 0$

Let p be any irreducible polynomial in $\mathbb{C}[x]$. Then

$\text{ord}_p(D(f))$ is either ≥ 0 or < -1, but

$$\text{ord}_p\left(\frac{Dg}{g}\right) = \begin{cases} \geq 0 & \text{ord}_p(g) = 0 \\ -1 & \text{ord}_p(g) \neq 0 \end{cases}$$

$\Rightarrow$ $\text{ord}_p(g) = 0$ for all p $\Rightarrow$ $g \in \mathbb{C}$

$\Rightarrow$ $D(f) = 0$ $\Rightarrow$ $f \in \mathbb{C}$ $\rightarrow \Leftarrow$

$\Rightarrow$ t is transcendental by <u>prop 10</u>

Prop 12 Let (K, D) be a differential field and F be an differential extension of K. Let $t \in F$ be such that $Dt \in K$ and $Dt \neq Du$ for any $u \in K$. Then t is transcendental $/K$, $C_{K(t)} = C_K$ and all irreducible polynomial in $K[t]$ are normal.

Proof Assume that t is algebraic$/K$. Then $\exists$ irreducible $p \in K[x]$ s.t. $p(t) = 0$, i.e. $t^n + p_{n-1}t^{n-1} + \cdots + p_0 = 0$

By $Dt = a \in K$, we have

$$(na + D(p_{n-1}))t^{n-1} + p_{n-1}(n-1)at^{n-2} + \cdots + Dp_0 = 0$$

$$\Rightarrow \quad na + D(p_{n-1}) = 0 \Rightarrow Dt = a = D(-\tfrac{1}{n}p_{n-1}) \rightarrow \Leftarrow$$

Let $p = t^n + p_{n-1}t^{n-1} + \cdots + p_0 \in K[t]$ be an irreducible polynomial

$D(p) = (na + D(p_{n-1}))t^{n-1} + \cdots + Dp_0 \Rightarrow \deg_t(Dp) < \deg_t(p)$

Then $\gcd(p, D(p)) = 1$.

Assume that $f = \frac{a}{b} \in C_{K(t)}$ with $\gcd(a,b) = 1$. Then

$$Df = \frac{D(a)b - bD(a)}{b^2} = 0 \Rightarrow D(a)b - bD(a) = 0 \Rightarrow D(a)b = aD(b)$$

$\Rightarrow a \mid D(a)$ and $b \mid D(b)$ $\Rightarrow a, b$ are both special

$\Rightarrow a, b \in K \Rightarrow f \in C_K \Rightarrow C_{K(t)} = C_K$.

Corollary 13 $\log(f(x))$ is transcendental over $\mathbb{C}(x)$ if $f \in \mathbb{C}(x) \setminus \mathbb{C}$.

Proof Let $t = \log(f)$. Then $Dt = \frac{Df(x)}{f(x)}$. Claim $Dt \neq Dg$ for any $g \in \mathbb{C}(x)$. Otherwise, $\frac{Df}{f} = Dg$. Since $f \in \mathbb{C}(x) \setminus \mathbb{C}$, there exists irreducible $p \in \mathbb{C}[x]$ s.t. $\mathrm{ord}_p(f) \neq 0$. $\Rightarrow \mathrm{ord}_p(\frac{Df}{f}) = -1$.
But $\mathrm{ord}_p(Dg) \neq -1$ for any p. $\rightarrow \Leftarrow$.