Symbolic Integration of Elementary Functions (II)

1. Trace and Norm

Let $(E, ')$ be a differential extension of $(F, ')$ with $\mathrm{char}(F)=0$. Let $\alpha \in E$ be an algebraic element over F with the minimal polynomial $p = x^n + \sum_{i=0}^{n-1} p_i x^i \in F[x]$ s.t. $p(\alpha)=0$. Then $[F(\alpha):F]=n$ and $\{1, \alpha, \dots, \alpha^{n-1}\}$ is a basis of $F(\alpha)$ over F. Assume that $\lambda_1, \dots, \lambda_n$ be roots of p in $\bar{F}$. Then we have

$$\begin{cases} p_{n-1} = -(\lambda_1 + \cdots + \lambda_n) \\ p_0 = (-1)^n \lambda_1 \cdots \lambda_n . \end{cases}$$

For any $\beta \in F(\alpha)$, we define $\phi_\beta : F(\alpha) \longrightarrow F(\alpha)$ by

$$\phi_\beta(\gamma) = \beta\gamma, \ \forall \gamma \in F(\alpha).$$ Then ϕ_β is linear and we call ϕ_β the multiplition map associated with β. Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis of $F(\alpha)$ over F. Then

$$\phi_\beta\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} = M_\beta \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}, \quad \text{where } M_\beta \in F^{n\times n}.$$

The matrix M_β is called the matrix representation of β w.r.t. the basis $\{\alpha_1, \dots, \alpha_n\}$. We know that the matrix representations of β w.r.t. two different bases are similar.

<u>DEF1</u> Let $\beta \in F(\alpha)$ and M_β be a matrix representation of β w.r.t. some basis. We call $\mathrm{Tr}(M_\beta)$ the trace of β in $F(\alpha)$ over F, denoted by $\mathrm{Tr}_{F(\alpha)/F}(\beta)$ and call $\det(M_\beta)$ the norm of β in $F(\alpha)$ over F, denoted by $N_{F(\alpha)/F}(\beta)$.

<u>Remark</u> Since similar matrices have the same trace and norm, the above definitions of traces and norms are independent of bases.

Assume that $A, B \in F^{n\times n}$ are similar, i.e. $\exists$ invertible $P \in F^{n\times n}$ s.t. $A = P^{-1}BP$. Then

$$|\lambda I - A| = |\lambda I - P^{-1}BP| = |P^{-1}(\lambda I - B)P| = |\lambda I - B|$$

So the characteristic polynomials of A and B are same.

Write $|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = (\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$

Then we have

$$\begin{cases} Tr(A) = \lambda_1 + \cdots + \lambda_n \\ \det(A) = \lambda_1 \cdots \lambda_n \end{cases}$$

So $Tr(A)$ and $\det(A)$ are stable under similar transformations.

Theorem 2 Let α be algebraic over F with the minimal polynomial $p = \sum_{i=0}^{n} p_i x^i \in F[x]$. Let $\alpha_1 = \alpha, \alpha_2, \cdots, \alpha_n$ be the distinct roots of p in $\overline{F}$. and $\sigma_i : F(\alpha) \to \overline{F}$ be the F-embedding defined by $\sigma_i(\alpha) = \alpha_i$. Then $\sigma_i(\beta') = (\sigma_i(\beta))'$ for any $\beta \in F(\alpha)$.

proof It suffices to show that $\sigma_i(\alpha') = (\sigma_i(\alpha))'$.

Let $P_0 = \sum_{i=0}^{n} p_i' x^i$ and $P_1 = \sum_{i=1}^{n} i p_i x^{i-1}$. Then for any root α_i of p

$$\alpha_i' = -\frac{P_0(\alpha_i)}{P_1(\alpha_i)} = Q(\alpha_i), \text{ where } Q \in F[x] \text{ and } \deg_x(Q) \le n-1.$$

then for any $i \in \{1, 2, \cdots, n\}$, we have

$$\sigma_i(\alpha_1') = \sigma_i(Q(\alpha_1)) = Q(\sigma_i(\alpha_1)) = (\alpha_i)' = (\sigma_i(\alpha_1))'$$

proposition 3 Let $(F, ')$ be a diff. field of char. 0. and α be algebraic over F and $\alpha \neq 0$. Let $\beta_1, \beta_2 \in F(\alpha)$. Then

1) $Tr(\beta_1 + \beta_2) = Tr(\beta_1) + Tr(\beta_2)$ 2) $N(\beta_1\beta_2) = N(\beta_1)N(\beta_2)$

3) $\frac{N(\alpha)'}{N(\alpha)} = Tr\left(\frac{\alpha'}{\alpha}\right)$.

Proof. We only show 3). Let $p = x^n + \sum_{i=1}^{n} p_i x^i \in F[x]$ be the minimal polynomial of α. Then the multiplication matrix of α w.r.t. the basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is of the form

$$M_\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & \cdots & -p_{n-1} \end{pmatrix}$$

Then $|xI - M_\alpha| = p_0 + p_1 x + \cdots + p_{n-1} x^{n-1} + x^n$. Then

$$Tr(\alpha) = -p_{n-1} = \sigma_1(\alpha) + \cdots + \sigma_n(\alpha)$$

$$N(\alpha) = (-1)^n p_0 = \sigma_1(\alpha) \cdots \sigma_n(\alpha)$$

In general, we have $\forall \beta \in F(\alpha)$, (see Lang Algebra p. 284-288 Prop. 5.6)

$$Tr_{F(\alpha)/F}(\beta) = \sigma_1(\beta) + \cdots + \sigma_n(\beta)$$

$$N_{F(\alpha)/F}(\beta) = \sigma_1(\beta) \cdots \sigma_n(\beta)$$

Then

$$\frac{N(\alpha)'}{N(\alpha)} = \frac{\prod_{i=1}^{n} \sigma_i(\alpha)'}{\prod_{i=1}^{n} \sigma_i(\alpha)} = \sum_{i=1}^{n} \frac{(\sigma_i(\alpha))'}{\sigma_i(\alpha)} = \sum_{i=1}^{n} \frac{\sigma_i(\alpha')}{\sigma_i(\alpha)} = \sum_{i=1}^{n} \sigma_i\left(\frac{\alpha'}{\alpha}\right)$$

$$= Tr\left(\frac{\alpha'}{\alpha}\right)$$

Theorem 4 (Liouville's theorem)

Let $(F, ')$ be a diff. field of char 0, and $f \in F$. If there exists an elementary extension $E = F(t_1, \ldots, t_n)$ with $C_E = C_F$ and $g \in E$ s.t. $f = g'$, then $\exists\, v \in F$, $u_1, \ldots, u_m \in F^* = F \setminus \{0\}$, $c_1, \ldots, c_m \in C_F$ s.t.

$$f = v' + \sum_{i=1}^{m} c_i \frac{u_i'}{u_i}$$

Proof. We proceed by induction on n. When $n=0$, $E = F$ and the assertion holds by taking $v = g$ and $c_i = 0$ (for all $i = 1, \ldots, m$).

Assume that the assertion holds for $n \leq s-1$. We now consider the case $n=s$. Let $K = F(t_1)$. Then $E = K(t_2, \dots, t_s)$, which is $(s-1)$-tuple extension of K. We apply the hypothesis assumption to K and E. Then there exist $\tilde{v} \in K$, $\tilde{u}_1, \dots, \tilde{u}_m \in K \setminus \{0\}$, $\tilde{c}_1, \dots, \tilde{c}_m \in C_K$, s.t.

$$f = \tilde{v}' + \sum_{i=1}^{m} \tilde{c}_i \frac{\tilde{u}_i'}{\tilde{u}_i}$$

Note that $F \subseteq K \subseteq E$ and $C_F = C_E$ $\Rightarrow$ $C_K = C_E$

Case 1 t_1 is algebraic over F with $[K:F] = r$.

Then there exist r F-embedding $\sigma_j : F(t_1) \longrightarrow \bar{F}$. $(j=1, \dots, r)$

Since $f \in F$, we have

$$\sigma_j(f) = f = \sigma_j\left(\tilde{v}' + \sum_{i=1}^{m} \tilde{c}_i \frac{\tilde{u}_i'}{\tilde{u}_i}\right)$$

$$= \sigma_j(\tilde{v})' + \sum_{i=1}^{m} \tilde{c}_i \, \sigma_j\left(\frac{\tilde{u}_i'}{\tilde{u}_i}\right)$$

$$rf = \sum_{j=1}^{r} \sigma_j(\tilde{v})' + \sum_{i=1}^{m} \sum_{j=1}^{r} \tilde{c}_i \, \sigma_j\left(\frac{\tilde{u}_i'}{\tilde{u}_i}\right)$$

$$= \left(\mathrm{Tr}(\tilde{v})\right)' + \sum_{i=1}^{m} \tilde{c}_i \frac{N(\tilde{u}_i)'}{N(\tilde{u}_i)} \quad \text{by prop. 3 (3)}$$

$$\Rightarrow f = v' + \sum_{i=1}^{m} c_i \frac{u_i'}{u_i}, \quad \text{where } v = \frac{\mathrm{Tr}(\tilde{v})}{r} \in F, \quad c_i = \frac{\tilde{c}_i}{r} \in C_F$$

and $u_i = N(\tilde{u}_i) \in F^\times$

Case 2 t_1 is transcendental over F.

If t_1 is either exponential or logarithmic over F, then

$$t_1' \in F[t_1], \quad f = \tilde{v}' + \sum_{i=1}^{m} \tilde{c}_i \frac{\tilde{u}_i'}{\tilde{u}_i}, \quad \tilde{v}, \tilde{u}_i \in F(t_1).$$

W.L.O.G. we may assume that $\tilde{u}_i \in F[t_1]$ is irreducible if $\tilde{u}_i \notin F$.
(using the logarithmic derivative formula)

<u>Claim 1</u> For any normal $p \in F[t_1]$, we have

1) $\mathrm{ord}_p(\tilde{v}) \geq 0$

2) $\mathrm{ord}_p(\tilde{u}_i) = 0$

<u>Proof of the claim</u> 1) Suppose that $\mathrm{ord}_p(\tilde{v}) < 0$. Then $\mathrm{ord}_p(\tilde{v}') = \mathrm{ord}_p(\tilde{v}) - 1 < -1$. Note that $\mathrm{ord}_p(f) = 0$ and

$$\mathrm{ord}_p\left(\frac{\tilde{u}_i'}{\tilde{u}_i}\right) = \begin{cases} \geq 0 & \text{if } \mathrm{ord}_p(\tilde{u}_i) = 0 \\ -1 & \text{if } \mathrm{ord}_p(\tilde{u}_i) \neq 0 \end{cases}$$

which implies that $\mathrm{ord}_p(\tilde{v}') = \mathrm{ord}_p\left(f - \sum_{i=1}^{m} \tilde{c}_i \frac{\tilde{u}_i'}{\tilde{u}_i}\right) \geq -1$, a contradiction!

2) Suppose that $\mathrm{ord}_p(\hat{u}_i) \neq 0$ for some $i \in \{1, \dots, m\}$. Then $\mathrm{ord}_p\left(\hat{c}_i \frac{\hat{u}_i'}{\hat{u}_i}\right) = -1$. By 1), we have $\mathrm{ord}_p(f - \hat{v}') > -1$, but $\mathrm{ord}_p\left(\sum_{i=1}^{m} \hat{c}_i \frac{\hat{u}_i'}{\hat{u}_i}\right) = -1$, a contradiction!

<u>Case 2.1</u> t_1 is logarithmic over F, i.e., $t_1' = u_0'/u_0$ for some $u_0 \in F$. In this case, we first show that $t_1' \neq u'$ for any $u \in F$. If $t_1' = u'$ for some $u \in F$, then $(t_1 - u)' = 0 \Rightarrow t_1 - u \in C_{F(t_1)} = C_F \Rightarrow t_1 \in F$, which is contradicts with the assumption that t_1 is transcendental $/F$. Then all of the irreducible polynomials in $F[t_1]$ are normal. By Claim 1, we have $\hat{u}_i \in F$ for all $i = 1, \dots, m$ and $\hat{v} \in F[t_1]$. Since $f - \sum_{i=1}^{m} \hat{c}_i \frac{\hat{u}_i'}{\hat{u}_i} \in F$, we have

$$\hat{v} = c_0 t_1 + v_0, \text{ with } c_0 \in C_F \text{ and } v \in F$$

Then
$$f = \hat{v}' + \sum_{i=1}^{m} \hat{c}_i \frac{\hat{u}_i'}{\hat{u}_i} = c_0 \frac{u_0'}{u_0} + v' + \sum_{i=1}^{m} \hat{c}_i \frac{\hat{u}_i'}{\hat{u}_i}$$

Hence the assertion holds.

<u>Case 2.2</u> t_1 is exponential over F, i.e., $t_1' = \frac{u_0'}{u_0} t_1$ for some $u_0 \in F \setminus \{0\}$. We first show that $\frac{t_1'}{t_1} \neq \frac{1}{\ell} \frac{a'}{a}$ for any $\ell \in \mathbb{N} \setminus \{0\}$ and $a \in F \setminus \{0\}$. If $\frac{t_1'}{t_1} = \frac{1}{\ell} \frac{a'}{a}$, then $(t_1^\ell - a)' = 0 \Rightarrow t_1^\ell - a \in C_F \Rightarrow t_1$ is algebraic $/F$ which contradicts with the assumption that t_1 is transcendental $/F$.

Then the only irreducible special polynomial in $F[t_1]$ is t_1.

By claim 1, we have $\hat{v} \in F[t_1, t_1^{-1}]$ and at most one of $\hat{u}_i$'s is equal to t_1, say $\hat{u}_1 = t_1$. Then

$$\sum_{i=1}^{m} \hat{c}_i \frac{\hat{u}_i'}{\hat{u}_i} = \hat{c}_1 \frac{t_1'}{t_1} + \sum_{i=2}^{m} \hat{c}_i \frac{\hat{a}_i'}{\hat{a}_i} = (\hat{c}_1 a)' + \sum_{i=2}^{m} \hat{c}_i \frac{\hat{u}_i'}{\hat{a}_i} \in F.$$

Since $f \in F$, we have $(\hat{v})' = f - \sum_{i=1}^{m} \hat{c}_i \frac{\hat{u}_i'}{\hat{u}_i} \in F$.

We claim that $\hat{v} \in F$. Suppose that $\hat{v} = \sum_{-d \le i \le d} a_i t_1^i \notin F$.

Then $\exists -d \le i \le d$, i s.t. $a_i \neq 0$. Then

$$\hat{v}' = \sum_{-d \le i \le d} (a_i' + i a' a_i) t_1^i$$

Since $\hat{v}' \in F$ and t_1 is transcendental over F, we have

$$a_i' + i a' a_i = 0 \implies a' = \frac{t_1'}{t_1} = -\frac{1}{i} \frac{a_i'}{a_i}$$

which is an contradiction!

Hence $$f = (\underbrace{\hat{v} + \hat{c}_1 a}_{\in F})' + \sum_{i=2}^{m} \hat{c}_i \frac{\hat{u}_i'}{\hat{u}_i}, \quad \hat{u}_i \in F.$$

This completes the proof.

<u>Remark</u> There is a stronger version of Liouville's theorem saying that if f has an elementary integral in $E = F(t_1, \dots, t_n)$, then $\exists\, v \in F$, $c_1, \dots, c_m \in \overline{C_F}$, and $u_1, \dots, u_m \in F(c_1, \dots, c_m)$ s.t.

$$f = v' + \sum_{i=1}^{m} c_i \frac{u_i'}{u_i}.$$

See Bronstein's Book: Symbolic Integration (Theorem 5.5.3)

But we will consider complex functions which is defined over $\mathbb{C}$. And we knew that $\mathbb{C}$ is algebraically closed. So we can always assume that no new constants is needed to express the integral.

Applications of Liouville's Theorem.

We now can show that

$$\exp(x^2), \quad \frac{1}{\log(x)}, \quad \frac{\exp(x)}{x}, \quad \exp(\exp(x)), \quad \log(\log(x))$$

have no elementary integral.

Example 1 $\quad f = \exp(x^2)$. $\quad \frac{f'}{f} = 2x$

Let $F = \mathbb{C}(x)(f)$, which is elementary over $\mathbb{C}(x)$.

If f has an elementary integral, then $\exists$ $v \in F$, $c_1, \dots, c_m \in C_F = \mathbb{C}$ and $u_1, \dots, u_m \in F$ s.t.

Note that f is transcendental over $\mathbb{C}(x)$.

$$f = v' + \sum_{i=1}^{m} c_i \frac{u_i'}{u_i}$$

By the order estimate, we have $\forall$ normal $p \in \mathbb{C}(x)[f]$

$$\mathrm{ord}_p(v) \geq 0 \quad \text{and} \quad \mathrm{ord}_p(u_i) = 0$$

Then $v \in \mathbb{C}(x)[f, f^{-1}]$. Since $\deg_f\left(f - \sum_{i=1}^{m} c_i \frac{u_i'}{u_i}\right) = 1$, we have

and at most one of u_i's is equal to t,

$$v = a \cdot f$$

$$\Rightarrow \quad f = (af)' = \left(a' + a\frac{f'}{f}\right) f$$

$$\Rightarrow \quad 1 = a' + 2x\, a \quad \text{for } a \in \mathbb{C}(x)$$

By the order estimate, we have shown that the equation

$$1 = y' + 2xy \quad \text{has no solution in } \mathbb{C}(x)$$

So $f = \exp(x^2)$ has no elementary integral.

Example 2 $\quad f = \frac{1}{\log(x)}$

Let $F = \mathbb{C}(x)(t)$, with $t = \log(x)$, which is transcendental over $\mathbb{C}(x)$. If f has an elementary integral over F, then

$\exists\ v \in F$, $u_1, \dots, u_m \in F \setminus \{0\}$, and $c_1, \dots, c_m \in \mathbb{C}$ s.t.

$$\frac{1}{t} = v' + \sum_{i=1}^{m} c_i \frac{u_i'}{u_i}$$

By the order estimate, we can show that

$\mathrm{ord}_p(v) \geq 0$ and $u_i = t$ or $\mathrm{ord}_p(u_i) = 0$ for any irreducible polynomial $p \in \mathbb{C}(x)[t]$, which is also normal.

In fact, if $p \neq t$, then $\mathrm{ord}_p(v') = \begin{cases} \geq 0 & \mathrm{ord}_p(v) \geq 0 \\ < -1 & \mathrm{ord}_p(v) < 0 \end{cases}$

but $\mathrm{ord}_p\left(\frac{1}{t} - \sum_{i=1}^{m} c_i \frac{u_i'}{u_i}\right)$ is either ≥ 0 or -1. Then we have $\mathrm{ord}_p(v) \geq 0$. If $\mathrm{ord}_p(u_i) \neq 0$ for some $i \in \{1, \dots, m\}$, then $\mathrm{ord}_p\left(\frac{1}{t} - \sum_{i=1}^{m} c_i \frac{u_i'}{u_i}\right) = -1$, contradicts with $\mathrm{ord}_p(v') \geq 0$. Then $\mathrm{ord}_p(u_i) = 0$ for all $i \in \{1, \dots, m\}$.

If $p = t$, $\mathrm{ord}_p(\frac{1}{t}) = -1$, which implies that $u_i = t$ for some i.

Then.

$$\frac{1}{t} = v' + c_i \frac{t'}{t} + \sum_{\substack{j=1 \\ j \neq i}}^{m} c_j \frac{u_j'}{u_j}, \quad c_i \in \mathbb{C}$$

$$\Rightarrow \quad 1 = c_i t' = c_i \cdot \frac{1}{x}, \text{ which is an contradiction!}$$