<u>Recall</u> Let $V \subseteq \mathbb{A}^n$ be an irreducible δ-variety over (K,δ) with a generic point $\eta = (\eta_1, \dots, \eta_n)$. To measure the "size" of V we have introduced the following invariants of V:

- $\delta\text{-dim}(V) \triangleq \delta\text{.tr.deg}\, K\langle \eta_1, \dots, \eta_n \rangle / K$ (differential dimension of V)
- $\omega_V(t) \triangleq \text{tr.deg}\, K(\eta_1^{[t]}, \dots, \eta_n^{[t]})/K$ (diff dimension polynomial of V)

$$\overset{t \gg 0}{=\!=} \delta\text{-dim}(V) \cdot (t+1) + \text{ord}(V)$$

To compute $\omega_V(t)$, by Theorem 4.3.3, we need to compute a characteristic set $\mathcal{A}$ of $\mathbb{I}(V) \subseteq K\{y_1, \dots, y_n\}$ w.r.t. an orderly ranking, then $\omega_V(t) = (n - \delta\text{-dim}(V))(t+1) + \text{ord}(\mathcal{A})$. $\left(\text{ord}(\mathcal{A}) = \sum_{A \in \mathcal{A}} \text{ord}(\text{ld}(A))\right)$

<u>Theorem 4.3.7</u> Suppose (K,δ) contains a nonconstant elt. Let $P \subseteq K\{u_1, \dots, u_d, y_1, \dots, y_{n-d}\}$ be a prime δ-ideal with a parametric set $\{u_1, \dots, u_d\}$. Then $\exists\, a_1, \dots, a_{n-d} \in K$ s.t. $[P, w - a_1 y_1 - \cdots - a_{n-d} y_{n-d}] \subseteq K\{u_1, \dots, u_d, y_1, \dots, y_{n-d}, w\}$ has a characteristic set of the form

$$\begin{array}{l} \chi(u_1, \dots, u_d, w) \\ I_1(u_1, \dots, u_d, w)\, y_1 - T_1(u_1, \dots, u_d, w) \\ \quad\vdots \\ I_{n-d}(u_1, \dots, u_d, w)\, y_{n-d} - T_{n-d}(u_1, \dots, u_d, w) \end{array}$$

w.r.t. the elimination ranking $u_1 < \cdots < u_d < w < y_1 < \cdots < y_{n-d}$.

<u>Corollary 4.3.8</u> Let (K,δ) contain a nonconstant element. Let $V \subseteq \mathbb{A}^n$ be an irreducible δ-variety. Then V is δ-birationally equivalent to the general component of an irreducible δ-polynomial (i.e., an irreducible δ-variety of codim 1).

Proof. Suppose δ-$\dim(V)=d$ and $\{u_1,\cdots,u_d\}$ is a parametric set of $P=\mathbb{I}(V)\subseteq K\{u_1,\cdots,u_d,y_1,\cdots,y_{n-d}\}$. By Theorem 4.3.7, $\exists\, a_1,\cdots,a_{n-d}\in K$ s.t. $J_a=[P,\ w-a_1y_1-\cdots-a_{n-d}y_{n-d}]$ $\subseteq K\{u_1,\cdots,u_d,y_1,\cdots,y_{n-d},w\}$ has a characteristic set of the form $X(u_1,\cdots,u_d,w),\ I_1(u_1,\cdots,u_d,w)y_1-T_1(u_1,\cdots,u_d,w),\cdots,I_{n-d}y_{n-d}-T_{n-d}$ w.r.t. the elimination ranking $u_1<\cdots<u_d<w<y_1<\cdots<y_{n-d}$, where X is irreducible $(*)$.

Let $W=\mathbb{V}(\operatorname{sat}(X))\subseteq\mathbb{A}^{d+1}$ be the general component of X.

Define $\varphi: V\longrightarrow W$ by

$$\varphi(u_1,\cdots,u_d,y_1,\cdots,y_{n-d})=(u_1,\cdots,u_d,\ a_1y_1+\cdots+a_{n-d}y_{n-d})$$

and $\psi: W\dashrightarrow V$ by

$$\psi(u_1,\cdots,u_d,w)=\left(u_1,\cdots,u_d,\ \frac{T_1(u_1,\cdots,u_d,w)}{I_1(u_1,\cdots,u_d,w)},\cdots,\frac{T_{n-d}(u_1,\cdots,u_d,w)}{I_{n-d}(u_1,\cdots,u_d,w)}\right).$$

Let $\xi=(\bar{u}_1,\cdots,\bar{u}_d,\bar{y}_1,\cdots,\bar{y}_{n-d})$ be a generic point of V and $\eta=(\bar{u}_1,\cdots,\bar{u}_d,\bar{w})$ be a generic point of W.

It is easy to show that both φ and ψ are dominant, and $(\psi\circ\varphi)(\xi)=\xi$, $(\varphi\circ\psi)(\eta)=\eta$ from $(*)$. So V and W are δ-birationally equivalent. $\boxtimes$.

Example: Let $K=(\mathbb{Q}(t),\frac{d}{dt})$ and $V=\mathbb{V}(y_1',y_2')\subseteq\mathbb{A}^2(\overline{K})$.

Introduce new δ-indeterminates w,λ_1,λ_2 and consider $J=[y_1',y_2',\ w-\lambda_1y_1-\lambda_2y_2]\subseteq K\{w,y_1,y_2\}$.

To eliminate y_1,y_2 in order to get $R(w)\in K\{w\}$, we have

$$R(w,\lambda_1,\lambda_2)=\begin{vmatrix} w & -\lambda_1 & -\lambda_2\\ w' & -\lambda_1' & -\lambda_2'\\ w'' & -\lambda_1'' & -\lambda_2''\end{vmatrix}=(\lambda_1\lambda_2'-\lambda_1'\lambda_2)w''-(\lambda_1\lambda_2''-\lambda_1''\lambda_2)w'+(\lambda_1'\lambda_2''-\lambda_2'\lambda_1'')w.$$

$$S_R Y_1 + \frac{\partial R}{\partial \lambda_1''} = S_R Y_1 + (\lambda_2 w' - \lambda_2' w) \quad \text{with } S_R = \lambda_1 \lambda_2' - \lambda_1' \lambda_2$$

$$S_R Y_2 + \frac{\partial R}{\partial \lambda_2''} = S_R Y_2 - (\lambda_1 w' - \lambda_1' w)$$

Choose $\lambda_1 = 1$, $\lambda_2 = t$, then $\overline{S_R} = 1 \neq 0$. So
$X(w) = w''$, $Y_1 + (tw' - w)$, $Y_2 - w'$.

Let $W = \mathbb{V}(w'') \subseteq \mathbb{A}^1$. Then V and W are δ-birationally equivalent. Indeed, let $\phi: V \longrightarrow W$, $(Y_1, Y_2) \mapsto Y_1 + tY_2$ and $\psi: W \to V$, $w \mapsto (w - tw', w')$

$$\psi \circ \phi(Y_1, Y_2) = \psi(Y_1 + tY_2) = (t(Y_1 + tY_2)' - (Y_1 + tY_2), -(Y_1 + tY_2)') = (Y_1, Y_2)$$

and $\phi \circ \psi(w) = w - tw' + tw' = w$.

Note that $X(w)$ is a δ-resolvent of V, and if c_1, c_2 are algebraic indeterminates with $c_1' = c_2' = 0$, then

$$\mathbb{Q}(t)\langle c_1, c_2 \rangle = \mathbb{Q}(t)(c_1 + tc_2).$$

Chapter 6 Algorithms and open problems in differential algebra

§6.1 Well-ordering theorem for differential polynomials

Let (K, δ) be a differential field of char 0 and consider the differential polynomial ring $K\{Y\} \triangleq K\{y_1, \dots, y_n\}$.

We have introduced the theory of diff characteristic sets in §2.1, in this section we focus on the computational aspects. Before that, we first recall the basic notions and results on diff char sets.

A ranking $\mathcal{R}$ is a total ordering on $\Theta(Y) = \{\delta^i y_j : i \in \mathbb{N}, j = 1, \dots, n\}$ satisfying 1) $u < \delta(u)$ and 2) $u < v \Rightarrow \delta(u) < \delta(v)$.

Two important rankings:

1) Elimination ranking: $y_i < y_j \Rightarrow \delta^k y_i < \delta^l y_j$ for all $k, l \in \mathbb{N}$.
2) Orderly ranking: $k < l \Rightarrow \delta^k y_i < \delta^l y_j$ for all $i, j \in \{1, \dots, n\}$

Fix an arbitrary ranking R. Given $f \in K\{Y\} \setminus K$, the leader of f w.r.t. R is $u_f = \max\{\delta^i y_j \mid \deg(f, \delta^i y_j) > 0\}$; If $\deg(f, u_f) = d$, then the rank of f is $rk(f) = (u_f, d)$, the initial of f is $I_f = \mathrm{coeff}(f, u_f^d)$, and the separant of f is $S_f = \frac{\partial f}{\partial u_f}$.

Given $f, g \in K\{Y\} \setminus K$. $f < g$ if $rk(f) <_{lex} rk(g)$. Set $a (\in K) < f$.

f is partially reduced w.r.t. g if any proper derivative of g doesn't appear in f; f is reduced w.r.t. g if f is partially reduced w.r.t. g and $\deg(f, u_g) < \deg(g, u_g)$. ($f$ isn't reduced w.r.t. $a \in K$)

$\mathcal{A} \subseteq K\{Y\}$ is autoreduced if any $A \in \mathcal{A}$ is reduced w.r.t. any other element in $\mathcal{A}$.

Fact: Each autoreduced set of $K\{Y\}$ is finite.

Write $\mathcal{A} = A_1, \dots, A_p$ with $rk(A_i) < rk(A_{i+1})$.

Given another $\mathcal{B} = B_1, \dots, B_q$, Say $\mathcal{A} < \mathcal{B}$ if either 1) $\exists\, k \le \min\{p, q\}$ s.t. $\forall\, i < k$, $rk(A_i) = rk(B_i)$ and $rk(A_k) < rk(B_k)$ or 2) $p > q$ and $\forall\, i \le q$, $rk(A_i) = rk(B_i)$. $\mathcal{A} \le \mathcal{B}$ if $\mathcal{A} < \mathcal{B}$ or $rk(\mathcal{A}) = rk(\mathcal{B})$.

- Any nonempty set of autoreduced sets in $K\{Y\}$ contains an autoreduced set of lowest rank.
- Diff reduction: (Theorem 2.1.12) Given $f \in K\{Y\}$, the δ-remainder of f, $R = \delta\text{-rem}(f, \mathcal{A})$, satisfies $\prod_{i=1}^{p} S_{A_i}^{t_i} I_{A_i}^{r_i} f \equiv r \mod [\mathcal{A}]$.

- characteristic set of a δ-ideal

Let $I \subseteq K\{y_1, \dots, y_n\}$ be a differential ideal and $\mathcal{A}$ be an autoreduced set contained in I w.r.t. a ranking R.

then $\mathcal{A}$ is a characteristic set of I

$\Longleftrightarrow$ $\mathcal{A}$ is of lowest rank among all autoreduced sets in I.

$\Longleftrightarrow$ $\forall f \in I$, $\delta\text{-rem}(f, \mathcal{A}) = 0$.

$\Longleftrightarrow$ I doesn't contain a nonzero δ-polynomial reduced w.r.t. $\mathcal{A}$.

We now come to the well-ordering of a (finite) differential polynomial set $\Sigma \subseteq K\{Y\}$. Fix a ranking $\mathcal{R}$ on $K\{Y\}$.

Def 6.1.1 An autoreduced set of lowest rank among all autoreduced sets belonging to Σ (i.e. each elt belongs to Σ) is called a basic set of Σ.

Lemma 6.1.2 Let Σ be a finite set of nonzero δ-polynomials in $K\{Y\}$. Then Σ necessarily has basic sets and there is a mechanical method in getting such a basic set in a finite number of steps.

Proof. As Σ is finite, the existence of basic sets is evident. So the problem reduces to a mechanical generation of such a set.

To show this, first choose $A_1 \in \Sigma$ of lowest rank. Let $\Sigma_1 = \{f \in \Sigma \mid f \text{ is reduced w.r.t. } A_1\}$. If $\Sigma_1 = \phi$, then output A_1. Otherwise, choose $A_2 \in \Sigma_1$ of lowest rank. Then A_1, A_2 is autoreduced. Let $\Sigma_2 = \{f \in \Sigma \mid f \text{ is reduced w.r.t. } A_1, A_2\}$. If $\Sigma_2 = \phi$, A_1, A_2 is a basic set of Σ. Otherwise, choose $A_3 \in \Sigma_2$ of lowest rank and proceed as before. As $A_1, A_2, A_3, \cdots$ constitute an autoreduced set, we have to stop in a finite number of steps and finally get a basic set in a mechanical manner. $\boxtimes$.

Lemma 6.1.3 Let Σ be a finite set of nonzero δ-polynomials with a basic set $\mathcal{A}: A_1, A_2, \cdots, A_p$ of which $A_1 \notin K$. Let B be a nonzero δ-polynomial reduced w.r.t. $\mathcal{A}$. Then the set $\Sigma_1 = \Sigma \cup \{B\}$ will have a basic set of rank lower than that of $\mathcal{A}$.

Proof. If $B \in K$, then B is a basic set of Σ_1 of rank lower than that of $\mathcal{A}$. Otherwise, there exists i s.t. $rk(B) < rk(A_i)$ and $rk(B) > rk(A_{i-1})$. Since B is reduced w.r.t. each A_j, $A_1, \cdots, A_{i-1}, B$ is an autoreduced set in Σ_1 of lower rank than $\mathcal{A}$. The basic set of Σ_1 will have therefore a fortiori a rank lower than that of $\mathcal{A}$. $\boxtimes$

Let Σ be a finite set of δ-polynomials in $K\{Y\}$. Set $\Sigma_1 = \Sigma$. By lemma 6.1.2, Σ_1 has a basic set, say $\mathcal{A}_1$.
Let $R_1 = \{\delta\text{-rem}(f, \mathcal{A}_1) \mid f \in \Sigma_1 \setminus \mathcal{A}_1\} \setminus \{0\}$. If $R_1 = \phi$, output $\mathcal{A}_1$. If $R_1 \neq \phi$, set $\Sigma_2 = \Sigma_1 \cup R_1$ and Σ_2 has a basic set, say $\mathcal{A}_2$. By Lemma 6.1.3, $\mathcal{A}_2$ is of lower rank than $\mathcal{A}_1$.
Let $R_2 = \{\delta\text{-rem}(f, \mathcal{A}_2) \mid f \in \Sigma_2 \setminus \mathcal{A}_2\} \setminus \{0\}$. If $R_2 = \phi$, output $\mathcal{A}_2$. Otherwise, we can proceed as before. In this way we shall get a sequence of sets of δ-polys $\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$ with corresponding basic sets $\mathcal{A}_1, \mathcal{A}_2, \cdots$ having decreasing ranks. Thus, such a sequence can have only a finite number of terms. In other words, if Σ_q is the last one of such a sequence with a basic set $\mathcal{A}_q$, then $R_q = \{0\}$, i.e., $\forall f \in \Sigma_q$, $\delta\text{-rem}(f, \mathcal{A}_q) = 0$. Output $\mathcal{A}_q$.

$$\begin{array}{l|ccccccc|}
 & \Sigma_1=\Sigma & \subseteq & \Sigma_2=\Sigma_1\cup R_1 & \subseteq \cdots \subseteq & \Sigma_q=\Sigma_{q-1}\cup R_{q-1} & \\
\text{basic sets} & \mathcal{A}_1 & > & \mathcal{A}_2 & > \cdots > & \mathcal{A}_q & \\
\delta\text{-reduction} & R_1\neq\phi & & R_2\neq\phi & \cdots & R_q=\phi. &
\end{array} \quad (*)$$

Def 6.1.4 The above $\mathcal{A}_q$ is called a characteristic set of the finite δ-polynomial set Σ.

Theorem 6.1.5 (Well-ordering principle).
Given a finite δ-polynomial set $\Sigma\subseteq K\{y_1,\dots,y_n\}$, there is an algorithm to obtain a characteristic set $\mathcal{A}$ of Σ after mechanically a finite number of steps.
Moreover, we have $\mathbb{V}(\mathcal{A}\backslash H_{\mathcal{A}})\subseteq\mathbb{V}(\Sigma)\subseteq\mathbb{V}(\mathcal{A})$,

$\shortparallel$

$\big(\{\eta\in \overline{K}^n \mid \mathcal{A}(\eta)=0 \text{ and } H_{\mathcal{A}}(\eta)\neq 0\}\big)$

and $\mathbb{V}(\Sigma)=\mathbb{V}(\mathcal{A}/H_{\mathcal{A}})\cup\bigcup_{A\in\mathcal{A}}\big(\mathbb{V}(\Sigma,I_A)\cup\mathbb{V}(\Sigma,S_A)\big)$.

Proof. The first assertion has been shown above the scheme.
Note that $R_k\subseteq[\Sigma_k]$ for each k, so $\mathbb{V}(\Sigma_k)=\mathbb{V}(\Sigma_{k+1})$ and thus, $\mathbb{V}(\Sigma_1)=\mathbb{V}(\Sigma_2)=\cdots=\mathbb{V}(\Sigma_q)=\mathbb{V}(\Sigma)$.
On the other hand, since $\delta\text{-rem}(f,\mathcal{A}_q)=0$ for $\forall f\in\Sigma_q$, $\exists\, i_A, s_A\in\mathbb{N}$ s.t. $\prod_{A\in\mathcal{A}_q} I_A^{i_A}S_A^{s_A}f\in[\mathcal{A}_q]$. It follows that any δ-zero of $\mathcal{A}_q$, which doesn't annul $H_{\mathcal{A}}\triangleq\prod_{A\in\mathcal{A}}(I_AS_A)$, is necessarily also a δ-zero of Σ_q and thus a δ-zero of Σ. ∎

Remark: Each newly obtained δ-poly set $\Sigma\cup\{I_A\}$ or $\Sigma\cup\{S_A\}$ has basic sets of lower rank than that of Σ.

Example Let $f = y_1' + 1$ and $g = y_1 + y_2'$ in $\mathbb{Q}(t)\{y_1, y_2\}$.

(1) Consider the elimination ranking R_1 with $y_1 > y_2$. We compute a characteristic set of the set $\Sigma = \{f, g\}$ following the scheme (*).

Let $\Sigma_1 = \Sigma$. A basic set of Σ_1 is $\mathcal{A}_1 := g$. Compute $r_1 \triangleq \delta\text{-Rem}(f, \mathcal{A}_1) = f - g' = 1 - y_2''$. So $R_1 = \{r_1\}$.

Let $\Sigma_2 = \Sigma \cup \{r_1\} = \{f, g, r_1\}$. A basic set of Σ_2 is $\mathcal{A}_2 := r_1, g$. Compute $r_2 = \delta\text{-Rem}(f, \mathcal{A}_2) = \delta\text{-Rem}(r_1, r_1) = 0$. So $R_2 = \phi$ and a characteristic set of Σ is $\mathcal{A}_2 = r_1, g$.

(2) Consider the orderly ranking R_2 with $y_1 > y_2$.

Let $\Sigma_1 = \Sigma$. A basic set of Σ_1 is $\mathcal{A}_1 = g, f$. So $R_1 = \phi$ and a characteristic set of Σ w.r.t. R_2 is $\mathcal{A} = g, f$.

Review:

Chapter 1. Basic notions: differential ring (R, δ) / differential ideal

Notation: given $S \subseteq R$, $\{S\}, [S] \subseteq R$: the radical diff ideal generated by S

Results: 1) In general, $\{S\} \neq \sqrt{[S]}$ and a maximal δ-ideal might not be prime.

2) Each radical δ-ideal $I \neq R$ is the intersection of all prime δ-ideals containing I (the intersection could be an infinite one).

3) If R is a Ritt algebra (i.e., $\mathbb{Q} \subseteq R$), then $\{S\} = \sqrt{[S]}$ and a maximal δ-ideal is prime.

Chapter 2. Notions: differential indeterminates (differentially dependent/independent); differential polynomial ring $K\{y_1, \ldots, y_n\}$ ((K, δ): a δ-field of char 0); differential homomorphism; differential zero; differential variety differential characteristic set (ranking, autoreduced set, diff reduction)

Ritt-Raudenbush basis theorem: If char(K)=0, $K\{y_1,\ldots,y_n\}$ is Ritt-Noetherian.

Minimal prime decomposition for radical δ-ideals: $\sqrt{I} = \bigcap_{i=1}^{l} p_i$.

Chapter 3. Two inclusion-reversing maps:

$\mathbb{I}$: { δ-varieties $\subseteq \mathbb{A}^n(\bar{K})$ } $\longrightarrow$ { radical δ-ideals in $K\{y_1,\ldots,y_n\}$ }

$V \qquad\qquad \mathbb{I}(V)$

and $\mathbb{V}$: { radical δ-ideals in $K\{y_1,\ldots,y_n\}$ } $\longrightarrow$ { δ-varieties in $\mathbb{A}^n(\bar{K})$ }

$I \qquad\qquad \mathbb{V}(I)$

Fact: $\mathbb{V}(\mathbb{I}(V)) = V$.

Differential Nullstellensatz: $\mathbb{I}(\mathbb{V}(S)) = \{S\}$. } ⇒ 1-1 correspondence b/w Alg and Geometry

Irreducible decomposition of diff varieties: $V = V_1 \cup \cdots \cup V_l$, V_i irr.

Ritt's Component theorem for a single differential poly:

Let $A \in K\{y_1,\ldots,y_n\}\setminus K$ be irreducible. Then the minimal prime decomposition is of the form $\{A\} (= \mathrm{sat}(A) \cap \{A, S_A\}) = \mathrm{sat}(A) \cap p_2 \cap \cdots \cap p_l$, where $\mathrm{sat}(A) = \{A\} : S_A$ is prime and A is a characteristic set of $\mathrm{sat}(A)$ under any ranking. $\mathrm{Sat}(A)$ is called the general component of A and $p_2, \ldots, p_l$ are singular components with $S_A \in p_i$ for each. Moreover, $p_i = \mathrm{sat}(B_i)$ for $B_i \in K\{y_1,\ldots,y_n\}$ with $\mathrm{ord}(B_i) < \mathrm{ord}(A)$.

Chapter 4. Notions: differential algebraic/transcendental

differential transcendence basis/degree, differential dimension

parametric set of a prime δ-ideal; diff dimension polynomial

Main results on extensions of diff fields:

1) (K, δ): a δ-field of char 0. $K \subseteq L \Rightarrow \delta$ could be extended to L.

This extension is unique ⟺ L is algebraic over K.

2) Primitive element theorem: (K,δ): char 0 and $C_K \neq K$.
Each η_i is δ-algebraic$/K \Rightarrow K\langle \eta_1,\cdots,\eta_n\rangle = K\langle \sum_{i=1}^{n} c_i\eta_i \rangle$ for some $c_i \in K$.
Key lemmas: • $\xi_1,\cdots,\xi_n$ are linearly dependent over C_K
$\Longleftrightarrow$ the wronskian determinant $Wr(\xi_1,\cdots,\xi_n)=0$.
• Non-vanishing of nonzero diff polynomials.
3) properties of diff trancedence basis/ degree
4) δ-$\dim(V) = \delta.\mathrm{tr.deg}\, K\langle V\rangle/K$. Not fine enough: $V \subsetneq W \not\Rightarrow \delta\text{-}\dim(V) < \delta\text{-}\dim(W)$
$= \delta.\mathrm{tr.deg}\, K\langle \eta_1,\cdots,\eta_n\rangle/K$. ($(\eta_1,\cdots,\eta_n)$: a generic point of V)
$\omega_V(t) = \mathrm{tr.deg}\, K(\eta_1^{[t]},\cdots,\eta_n^{[t]})/K$ (for $t \gg 0$)
$= \delta\text{-}\dim(V)(t+1) + \mathrm{ord}(V)$.
Chapter 5. Notions: elementary extension (exponential, logarithmic)
elementary function, elementary integrable ($y'=f$ has an elementary function solution)
special/normal polynomial, order function and its properties
Liouville's theorem: Given (F,δ) and $f\in F$, if f is elementary integrable in some elementary extension E of F with $C_E = C_F$,
then $\exists\, v\in F$, $c_1,\cdots,c_m \in C_F$ and $u_1,\cdots,u_m \in F^*$ s.t. $f = v' + \sum_{i=1}^{m} c_i v_i$.
Chapter 6 Basic set, characteristic set of a δ-poly set
Well-ordering principle
Next time: Zero-decomposition and its application
(Example: Kepler's law $\Longrightarrow$ Newton's Gravitational law)