10.

Recall: A differential ring (R, δ):

- R is a commutative ring with unity 1;
- $\delta: R \to R$ is a derivation satisfying $\begin{cases} \delta(a+b) = \delta(a) + \delta(b) \\ \delta(ab) = \delta(a)b + a\delta(b). \end{cases}$

(If R is a domain, then δ could be extended to $\mathrm{Frac}(R)$. In this case, $(\mathrm{Frac}(R), \delta)$ is a differential field.)

A differential ideal $I \subseteq (R, \delta)$ $\begin{cases} I \trianglelefteq R \\ \delta(I) \subseteq I. \end{cases}$

Notation: $S \subseteq R$, (S), $[S] = (\Theta(S))$ w/ $\Theta = \{\delta^i : i \in \mathbb{N}\}$.

$\{S\}$: the radical diff ideal generated by S.

$\sqrt{I} = \{f \in R \mid \exists n \in \mathbb{N},\ f^n \in I\}$.

In general, $\{S\} \neq \sqrt{[S]}$ and a maximal δ-ideal may not be prime.

Lemma: If $\mathbb{Q} \subseteq R$, then $\{S\} = \sqrt{[S]}$.

1.3. Decomposition of radical differential ideals.

In computational algebraic geometry, we have studied decompositions of radical ideals. In differential algebra, we have analogous arguments.

Let (R, δ) be a differential ring and I be a radical differential ideal of R.

Lemma 1.10. If $ab \in I$, then $a\delta(b) \in I$ and $\delta(a)b \in I$.

Proof. $ab \in I \Rightarrow \delta(ab) = \delta(a)b + a\delta(b) \in I$

$\Rightarrow a\delta(b) \cdot \delta(ab) = (a\delta(b))^2 + ab\,\delta(a)\delta(b) \in I \Rightarrow (a\delta(b))^2 \in I$

$\Rightarrow a\delta(b) \in I$ and $\delta(a)b \in I$.

Lemma 1.11. Let $S \subseteq R$ be any subset. Then

$I : S = \{a \in R \mid aS \subseteq I\}$ is a radical differential ideal.

proof. 1) $\forall\, a,b\in I:S,\ r\in R,\quad aS\subseteq I$ & $bS\subseteq I \Rightarrow (a+b)S\subseteq I$ & $raS\subseteq I$
$\Rightarrow a+b\in I:S,\ ra\in I:S$. So $I:S$ is an ideal.

2) $\forall\, a\in I:S,\ aS\subseteq I$. By Lemma 1.10, $\delta(a)S\subseteq I \Rightarrow \delta(a)\in I:S$.
So $I:S$ is a differential ideal.

3). $\forall\, a\in R$, suppose $\exists\, n\in\mathbb{N},\ a^n\in I:S$. Then $a^nS\subseteq I$. So for $\forall\, s\in S$,
$a^n s\in I \xRightarrow{*s^{n-1}} a^n s^n\in I \Rightarrow as\in I$ for $\forall\, s\in S \Rightarrow a\in I:S$.

Thus, $I:S$ is a radical differential ideal. $\square$.

Lemma 1.12. Let S be any subset. Let $a\in R$. Then $a\{S\}\subseteq\{aS\}$.
proof. Consider $J=\{aS\}:a$. By Lemma 1.11, J is a radical differential ideal.
Since $S\subseteq J$, $\{S\}\subseteq J$. Thus, $a\{S\}\subseteq\{aS\}$. $\square$.

Lemma 1.13. For all subsets $S,T\subseteq R$, we have $\{S\}\{T\}\subseteq\{ST\}$.
Furthermore, $\{S\}\cap\{T\}=\{ST\}$.

proof. Since for each $a\in S$, by Lemma 1.12, $a\{T\}\subseteq\{aT\}\subseteq\{ST\}$.
By Lemma 1.11, $\{ST\}:\{T\}$ is a radical differential ideal containing S.
Thus, $\{S\}\{T\}\subseteq\{ST\}$.

The assertion $\{S\}\cap\{T\}=\{ST\}$ follows from i) and ii):
i) $ST\subseteq\{S\},\{T\}\Rightarrow\{ST\}\subseteq\{S\}\cap\{T\}$;
ii) $\forall\, a\in\{S\}\cap\{T\},\ a^2\in\{S\}\cdot\{T\}\subseteq\{ST\}$. So $a\in\{ST\}$.

We use Lemmas 1.10 – 1.13 to show the following:

Lemma 1.14. Let $T\subseteq R$ be a subset closed under multiplication and let p be maximal among radical differential ideals that do not intersect T. Then p is prime.

Proof. Suppose the contrary, i.e., P is not prime. Let $a, b \in R$ be such that $ab \in P$ but $a \notin P$ and $b \notin P$. Hence $P \subsetneqq \{P, a\}$, $P \subsetneqq \{P, b\}$. Thus, $\exists\, t_1 \in \{P, a\} \cap T$, $\exists\, t_2 \in \{P, b\} \cap T$. So $t_1 t_2 \in T$ but $t_1 t_2 \in \{P, a\} \cdot \{P, b\} \subseteq \{P, ab\} = P$, a contradiction to $P \cap T = \phi$. ▨

In a commutative ring R, the nilradical $\sqrt{(0)}$ of R is the intersection of all the prime ideals of R and every radical ideal of R is the intersection of all prime ideals containing it. In differential algebra, we have a similar result.

Now, we are ready to state our main theorem of this section:

Theorem 1.15. Let $I \subsetneqq R$ be a radical differential ideal. Then I can be represented as an intersection of prime differential ideals.

Proof. We first construct for each $x \notin I$ a prime differential ideal P_x such that $P_x \supseteq I$ and $x \notin P_x$. Let $T = \{x^n \mid n \in \mathbb{N}\}$. The set

$$U = \{P \subseteq R \mid P \text{ is a radical differential ideal of } R,\ I \subseteq P,\ P \cap T = \phi\}$$

is nonempty since $I \in U$. By Zorn's Lemma, $\exists$ a maximal element P_x in U. P_x is prime by Lemma 1.14, and since $P_x \cap T = \phi$, $x \notin P_x$.

Clearly, $I = \bigcap_{x \notin I} P_x$ is an intersection of prime differential ideals. ▨.

In section 1.2, we gave an example showing a maximal differential ideal might be not prime. But if $\mathbb{Q} \subseteq R$, then a maximal differential ideal in R is always prime.

Cor 1.16. Let $\mathbb{Q} \subseteq (R, \delta)$ and M be maximal among proper differential ideals. Then M is prime.

Proof. Consider $\{M\} = \sqrt{[M]} = \sqrt{M}$. If $\sqrt{M} = R$, then $1 \in \sqrt{M} \Rightarrow 1 \in M$, which contradicts M being proper. Therefore, $\sqrt{M} = M$, M is a radical diff ideal.

By Theorem 1.15, $M = \bigcap_{\gamma \in J} P_\gamma$ where P_γ is a prime differential ideal.

Therefore, for all $\gamma \in J$, $M = P_\gamma$ and thus, M is prime. ∎.

Remark: A differential ring R with $\mathbb{Q} \subseteq R$ is called a Ritt Algebra.

We have shown in section 1.2 and section 1.3, in a Ritt Algebra

1) The radical differential ideal $\{S\} = \sqrt{[S]}$;
2) A maximal differential ideal is a prime differential ideal.
3) Even in a Ritt algebra R, the quotient R/M (M is a maximal differential ideal) might not be a differential field.

Example: Let $R = \mathbb{Q}[x]$ w/ $\delta(x) = 1$. Then $[0]$ is the unique maximal differential ideal. $R/[0] = R$ is not a δ-field.

Chapter 2. Differential polynomial Rings and Differential Varieties

Let (K, δ) be a differential field of characteristic 0. We hope to develop an algebraic structure and algebraic theory for ordinary differential equations.

Def 2.1. Let (L, δ) be a differential field extension of (K, δ). A subset S of L is said to be differentially dependent over K if the set $(\delta^k(s))_{k\in\mathbb{N}, s\in S}$ is algebraically dependent over K. In the contrary case, S is said to be δ-independent over K, or a family of differential indeterminates over K. In the case $S=\{\alpha\}$, we say that α is differentially algebraic over K or differentially transcendental over K respectively.

Example: Let $(K, \delta) = (\mathbb{Q}(x), \frac{d}{dx})$ and $(L, \delta) = (\mathbb{C}(x, e^x), \frac{d}{dx})$. Clearly each $c \in \mathbb{C}$ and $\alpha = e^x$ are differentially algebraic over K.

Def 2.1. The ring of differential polynomials with coefficients in K in the differential indeterminates $y_1, \cdots, y_n$ is the ring of polynomials

$$K[\delta^k y_j \mid k\in\mathbb{N},\ j=1,\cdots,n], \text{ denoted by } K\{y_1,\cdots,y_n\}.$$

Its elements are called differential polynomials. $K\{y_1,\cdots,y_n\}$ is a differential ring with the derivation operator δ extending $\delta|_K$ and $\delta(\delta^k y_j) = \delta^{k+1}(y_j)$.

Example: 1) $u_{xx} = v_x \longleftrightarrow \delta^2 y_1 - \delta y_2 = 0$

2) $\left(\frac{du}{dt}\right)^2 = 4u\frac{d^2u}{dt^2} \longleftrightarrow (\delta y_1)^2 - 4y_1\delta^2(y_1) = 0.$

Definition 2.2. Let (R_1, δ_1) and (R_2, δ_2) be two differential rings. A differential homomorphism of (R_1, δ_1) to (R_2, δ_2) is a ring homomorphism $\varphi: R_1 \to R_2$ such that $\varphi \circ \delta_1 = \delta_2 \circ \varphi$. If R_0 is a common differential subring of R_1 and R_2, and $\varphi|_{R_0} = id_{R_0}$, φ is called a differential homomorphism over R_0.

$$\left(\begin{tikzcd} a \arrow[r, "\varphi"] \arrow[d, "\delta_1"] & \varphi(a) \arrow[d, "\delta_2"] \\ \delta_1(a) \arrow[r, "\varphi"'] & \varphi(\delta_1(a)) \end{tikzcd}\right)$$

We give two examples of differential homomorphisms:

1) Let $(K, \delta) \subseteq (L, \delta)$ be two differential fields. Then

$$id_K : (K, \delta) \to (L, \delta) \text{ is a differential homomorphism.}$$

2) Take an elt $\vec{a} = (a_1, \cdots, a_n) \in L^n$, then the map

$$\varphi_{\vec{a}} : K\{y_1, \cdots, y_n\} \longrightarrow L \text{ defined by}$$

$$f(y_1, \cdots, y_n) \mapsto f(a_1, \cdots, a_n)$$

$$\delta^k(y_i) \qquad\quad \delta^k(a_i)$$

is a differential homomorphism over K. (uniquely determined by the value $\varphi(y_i)$.)

Here, $f(a_1, \cdots, a_n)$ means replacing $\delta^k y_i$ by $\delta^k(a_i)$ in $f(y_1, \cdots, y_n)$.

Prop 2.3. Let (R_1, δ) and (R_2, δ) be two differential rings and $\varphi: R_1 \to R_2$ be a differential homomorphism. Then $Ker(\varphi)$ is a differential ideal.

Proof. $Ker(\varphi)$ is an ideal of R_1 since φ is a homomorphism of rings.

For each $r \in Ker(\varphi)$, $\varphi(r) = 0$, so $\delta(\varphi(r)) = 0 = \varphi(\delta(r)) \Rightarrow \delta(r) \in Ker(\varphi)$. ☒.

Cor 2.4. Let (R, δ) be a differential ring and I be an ideal of R.

Then I is a differential ideal of $R \iff (R/I, \delta)$ is a differential ring.

proof. "$\Rightarrow$" Let $r + I \in R/I$. Define

$$\delta(r + I) = \delta(r) + I \quad (*).$$

To show $(*)$ is well-defined, let $r_1 + I = r_2 + I$, we need to show $\delta(r_1) + I = \delta(r_2) + I$.

Since $r_1 - r_2 \in I$ and I is a differential ideal, $\delta(r_1 - r_2) = \delta(r_1) - \delta(r_2) \in I$.

So $\delta(r_1) + I = \delta(r_2) + I$. To show $(*)$ is a derivation on R/I.

Let $r_1+I,\ r_2+I \in R/I$, then $\delta(r_1+I+r_2+I) = \delta(r_1+r_2+I) = \delta(r_1)+\delta(r_2)+I = \delta(r_1+I)+\delta(r_2+I)$ and $\delta((r_1+I)(r_2+I)) = \delta(r_1r_2+I) = \delta(r_1)r_2 + r_1\delta(r_2)+I = \delta(r_1+I)\cdot(r_2+I)+(r_1+I)\delta(r_2+I)$.

"$\Leftarrow$" Let $\varphi: R \to R/I$ be defined by $\varphi(r) = r+I$ for each $f \in R$.

Then $\forall f \in R$, $\varphi(\delta(f)) = \delta(f)+I = \delta(f+I) = \delta(\varphi(f))$, so φ is a differential homomorphism. By Prop 2.3, $I = \mathrm{Ker}(\varphi)$ is a differential ideal of R.

Def. Let $\Sigma \subseteq K\{Y_1,\dots,Y_n\}$ and $\eta = (\eta_1,\dots,\eta_n)$ be a point from $(L,\delta) \supseteq (K,\delta)$. We call η a differential zero of Σ if for each $f \in \Sigma$, $f(\eta)=0$. (that is, $\Sigma \subseteq \mathrm{Ker}(\varphi_\eta : K\{Y_1,\dots,Y_n\} \to L^n)$).

In Algebraic Geometry, we consider algebraic varieties in an algebraic closed field. In differential algebra, we have similar concepts to define differential varieties.

Def. (K,δ) is called differentially closed if for all $F \subseteq K\{Y_1,\dots,Y_n\}$, if $\exists (L,\delta) \supseteq (K,\delta)$ and $\eta \in L^n$ s.t. $F(\eta)=0$, then $\exists \xi \in K^n$ s.t. $F(\xi)=0$.

Let $(L,\delta) \supseteq (K,\delta)$. (L,δ) is called a differential closure of (K,δ) if 1) (L,δ) is differentially closed and 2) for every differential closed field $(M,\delta) \supseteq (K,\delta)$, there is a differential embedding $\varphi: L \hookrightarrow M$ w/ $\varphi|_K = id_K$.

Def. Let (E,δ) be a fixed differential closure of (K,δ). The set of differential zeros of $\Sigma \subseteq K\{Y_1,\dots,Y_n\}$ is called a differential variety over K, denoted by $\mathbb{W}_E(\Sigma)$ or $\mathbb{W}(\Sigma)$. For a subset $V \subseteq E^n$, we denote $\mathbb{I}(V) = \{f \in K\{Y_1,\dots,Y_n\} \mid \forall \xi \in V,\ f(\xi)=0\}$ to be the set of all differential polynomials in $K\{Y_1,\dots,Y_n\}$ which vanish at each point of V. Clearly, $\mathbb{I}(V)$ is a radical differential ideal.

Let $\eta=(\eta_1,\cdots,\eta_n)$ be a point from a differential extension field of (K,δ). η is called a generic point of a differential ideal $I\subseteq K\{Y_1,\cdots,Y_n\}$ if $\forall f\in K\{Y_1,\cdots,Y_n\}$, $f(\eta_1,\cdots,\eta_n)=0 \Leftrightarrow f\in I$.

Example. In the algebraic case, $I=(x^2+y^2-1)\subseteq \mathbb{Q}[x,y]$ has a generic point $\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)$. Also, $(\cos(\theta),\sin(\theta))$ is another generic point. So generic points are not unique.

<u>Lemma</u> Let $P\subseteq K\{Y_1,\cdots,Y_n\}$ be a differential ideal. Then P has a generic point $\Leftrightarrow$ P is prime. (A prime δ-ideal is assumed not to be the unit.)

Proof. "$\Rightarrow$" Suppose η is a generic point of P. Then $P=\mathbb{I}(\eta)$ is a prime differential ideal.

"$\Leftarrow$" Suppose P is a prime differential ideal. Then $K\{Y_1,\cdots,Y_n\}/P$ is a differential domain. Let $L=\mathrm{Frac}(K\{Y_1,\cdots,Y_n\}/P)$ and $\bar{Y}_i=Y_i+P$. Then $(\bar{Y}_1,\cdots,\bar{Y}_n)\in L^n$ is a generic point of P. Indeed, $\forall f\in P$, $f(\bar{Y}_1,\cdots,\bar{Y}_n)=f(Y_1,\cdots,Y_n)+P=\bar{0}\in L$ and $\forall f\in K\{Y_1,\cdots,Y_n\}$, if $f(\bar{Y}_1,\cdots,\bar{Y}_n)=0$, then $f(Y_1,\cdots,Y_n)\in P$. $\square$.

Next class: $\begin{cases}\text{Differential characteristic sets}\\ \text{Ritt-Raudenbush basis Theorem.}\end{cases}$