Recall: Let $(K, \delta)$ be a differential field of char 0.

Ritt's Component theorem Let $A \in K\{y_1, \dots, y_n\} \setminus K$ be irreducible. Then $\exists\, B_1, \dots, B_{\ell-1} \in K\{y_1, \dots, y_n\} \setminus K$ w/ for each $i$, $\mathrm{ord}(B_i) < \mathrm{ord}(A)$ and $A$ involving a proper derivative of the leader of $B_i$ under any ranking such that

$$\{A\} = \mathrm{Sat}(A) \cap \mathrm{Sat}(B_1) \cap \dots \cap \mathrm{Sat}(B_{\ell-1})$$

is the minimal prime decomposition of $\{A\}$.

Extensions of differential fields

Theorem 4.1 Let $K \subseteq L$ be fields of char 0. Then any derivation on $K$ could be extended to a derivation on $L$. This extension is unique iff $L$ is algebraic over $K$.

The proof shows that if $\gamma \in L$ is algebraic over $(K, \delta)$, then $\delta(\gamma) \in K(\gamma)$.

(Recall $\delta(\gamma) = -\dfrac{\sum_{i=0}^{d} \delta(p_i)\gamma^i}{\left.\frac{\partial p(x)}{\partial x}\right|_{x=\gamma}}$ if $p(x) = \sum_{i=0}^{d} p_i x^i$ is the minimal poly of $\gamma$ over $K$)

And if $\gamma$ is alg over the constant field of $(K, \delta)$, then $\delta(\gamma) = 0$.

Def Let $(K, \delta) \subseteq (L, \delta)$ and $\gamma \in L$. $\gamma$ is called diff-algebraic over $K$ if $\exists\, 0 \neq p(y) \in K\{y\}$ s.t. $p(\gamma) = 0$. Otherwise, $\gamma$ is said to be differentially transcendental over $K$.

Lemma 4.4 Let $K \subseteq L$ be differential fields of char 0 and $\gamma \in L$. Then $\gamma$ is diff algebraic over $K \Longleftrightarrow \mathrm{tr.deg}\, K\langle\gamma\rangle / K < \infty$.

Remark: 1) If $\gamma$ is diff alg over $K$ and $f(y) \neq 0$ is a diff poly of minimal order which vanishes at $\gamma$, then $\mathrm{tr.deg}\, K\langle\gamma\rangle / K = \mathrm{ord}(f)$.

Example: $K = (\mathbb{R}(x), \frac{d}{dx})$, $L = (K\langle e^x, \sin(x)\rangle, \frac{d}{dx})$. Since $\frac{d}{dx}(e^x) = e^x$ and $(\frac{d}{dx})^2(\sin(x)) = -\sin(x)$, $e^x$ and $\sin(x)$ are $\delta$-alg over $K$ w/ $\mathrm{tr.deg}\, K\langle e^x\rangle / K = 1$, $\mathrm{tr.deg}\, K\langle \sin(x)\rangle / K = 1$.

2) The result "$\Rightarrow$" is false in the partial differential case $(K, \{\delta_1, \cdots, \delta_m\})$, where tr.deg $K\langle \bar{a}\rangle/K$ might be infinity but the differential type of $K\langle\bar{a}\rangle$ is $\leq m-1$. (differential type is the degree of differential dimension poly of $\mathbb{I}(\bar{a})$).

We say $L \supseteq K$ is diff algebraic over $K$, if each elt $a \in L$ is diff algebraic over $K$. Note that every diff field extension with finite transcendence degree is diff algebraic over $K$. But the converse doesn't hold.

Lemma 4.5 Let $L \supseteq K$ be a diff field extension and $a, b \in L$ which are diff algebraic over $K$. Then $a+b$, $ab$, $\delta(a)$ and $a^{-1}$ $(a \neq 0)$ are $\delta$-algebraic over $K$. In particular, a $\delta$-field extension generated by $\delta$-algebraic elements is $\delta$-algebraic over $K$ and the set of all elements in $L$ which are $\delta$-algebraic over $K$ is a $\delta$-algebraic $\delta$-field extension of $K$.

Proof. Since tr.deg $K\langle a\rangle/K < \infty$ and tr.deg $K\langle b\rangle/K < \infty$, we have

$$\text{tr.deg } K\langle a,b\rangle/K = \text{tr.deg } K\langle a\rangle/K + \text{tr.deg } K\langle a\rangle\langle b\rangle/K\langle a\rangle < \infty. \quad \square$$

Lemma 4.6. Let $K \subseteq L \subseteq M$ be $\delta$-fields. Then $M$ is $\delta$-algebraic over $K \iff M$ is $\delta$-algebraic over $L$ and $L$ is $\delta$-algebraic over $K$.

Proof. "$\Rightarrow$" valid by definition.

"$\Leftarrow$" For any $a \in M$, $a$ is $\delta$-algebraic over $L$, so $\exists\, 0 \neq p(y) \in L\{y\}$ s.t. $p(a) = 0$. Denote the coefficient set of $p(y)$ to be $\{b_1, \cdots, b_t\} \subseteq L$.

Then $\text{tr.deg } K\langle b_1, \cdots, b_t, a\rangle/K = \text{tr.deg } \underbrace{K\langle b_1, \cdots, b_t\rangle/K}_{<\infty} + \text{tr.deg } \underbrace{K\langle b_1, \cdots, b_t\rangle\langle a\rangle/K\langle b_1, \cdots, b_t\rangle}_{<\infty} < \infty$.

for $a$ $\delta$-alg / $K\langle b_1, \cdots, b_t\rangle$

Thus, tr.deg $K\langle a\rangle/K < \infty$ and $a$ is $\delta$-algebraic over $K$. $\square$

## §4.1 Differential Primitive Theorem

It is a well-known theorem of algebra that a finite algebraic extension of a field K of characteristic 0 has a primitive element $w$:

$$K(a_1, \cdots, a_n) = K(w).$$

In this section, we treat analogous problem for ordinary differential field of characteristic 0.

Note that $\mathbb{Q}\langle \pi, e\rangle$ is a finitely generated differential extension field of $\mathbb{Q}$ ($\delta(\pi)=\delta(e)=0$). Clearly, $\mathbb{Q}\langle \pi, e\rangle \neq \mathbb{Q}\langle w\rangle$ for any $w \in \mathbb{Q}\langle \pi, e\rangle$. So to derive an analog of primitive element theorem in differential algebra, we need some restrictions. For the ordinary differential fields, the mild condition is that $(K, \delta)$ contains a non-constant element (i.e., $\exists\, y \in K$ s.t. $y' \neq 0$)

We need two lemmas for preparation to state the main theorem. Throughout this section, $(K, \delta)$ is a fixed $\delta$-field of char 0 containing a non-constant.

A set of elements $y_1, \cdots, y_s$ of K is called linearly dependent if there exists a relation

$$c_1 y_1 + \cdots + c_s y_s = 0$$

where the $c$'s are constant elements in K, not all zero.

The Wronskian determinant of $y_1, \cdots, y_s$ is defined as

$$\mathrm{Wr}(y_1, \cdots, y_s) = \begin{vmatrix} y_1 & \cdots & y_s \\ y_1' & \cdots & y_s' \\ \cdots & \cdots & \cdots \\ y_1^{(s-1)} & \cdots & y_s^{(s-1)} \end{vmatrix}.$$

**Lemma 4.1.1** Let $y_1, \cdots, y_s$ be elements in K. Then

$y_1, \cdots, y_s$ are linearly dependent $\Longleftrightarrow$ $\mathrm{Wr}(y_1, \cdots, y_s) = 0$, i.e.,

$$\begin{vmatrix} y_1 & \cdots & y_s \\ y_1' & \cdots & y_s' \\ \cdots & \cdots & \cdots \\ y_1^{(s-1)} & \cdots & y_s^{(s-1)} \end{vmatrix} = 0. \quad (*)$$

Proof. "$\Rightarrow$" Sps $y_1, \cdots, y_s$ are linearly dependent. Then $\exists\, c_1, \cdots, c_s$, constants of $K$, not all zero s.t. $c_1 y_1 + \cdots + c_s y_s = 0$.

Differentiate the relation $s-1$ times, we get a system of linear equations for $c$'s:

$$\begin{cases} c_1 y_1 + \cdots + c_s y_s = 0 \\ c_1 y_1' + \cdots + c_s y_s' = 0 \\ \cdots\cdots \\ c_1 y_1^{(s-1)} + \cdots + c_s y_s^{(s-1)} = 0 \end{cases}$$ has a nonzero solution.

So $(*)$ holds.

"$\Leftarrow$" Suppose we have $(*)$. We now show $y_1, \cdots, y_s$ are linearly dependent by induction on $s$.

If $s=1$, $y_1 = 0 \Rightarrow y_1$ is linearly dependent.

Suppose it is valid for the case $\leq s-1$ and we treat for the case $s$.

By $(*)$, $\exists\, c_1, \cdots, c_m \in K$, not all zero s.t.

$$c_1 y_1^{(j)} + \cdots + c_s y_s^{(j)} = 0 \quad (**) \text{ for } j = 0, \cdots, s-1.$$

If $wr(y_1, \cdots, y_{s-1}) = \begin{vmatrix} y_1 & \cdots & y_{s-1} \\ y_1' & \cdots & y_{s-1}' \\ y_1^{(s-2)} & \cdots & y_s^{(s-2)} \end{vmatrix} = 0$, by the induction hypothesis,

$y_1, \cdots, y_{s-1}$ are linearly dependent, so $y_1, \cdots, y_s$ are linearly dependent too.

So it suffices to consider the case $wr(y_1, \cdots, y_{s-1}) \neq 0$.

Then in this case $c_s \neq 0$. By dividing $c_s$ on both sides when necessary, we can take $c_s = 1$. For $j = 0, \cdots, s-2$, differentiate $(**)$ and then subtract the equation $(**)$ corresponding to $j+1$, then we have

$$c_1' y_1^{(j)} + \cdots + c_{s-1}' y_{s-1}^{(j)} = 0 \quad \text{for } j = 0, \cdots, s-2.$$

Since $wr(y_1, \cdots, y_{s-1}) \neq 0$, we have $c_i' = 0$ for $i = 1, \cdots, s-1$. Thus $y_1, \cdots, y_s$ are linearly dependent. ∎

Lemma 4.1.2. If $G$ is a nonzero differential polynomial in $K\{y_1,\cdots,y_n\}$, there exist elements $\eta_1,\cdots,\eta_n$ in $K$ such that $G(\eta_1,\cdots,\eta_n)\neq 0$.

Proof. It suffices to treat a differential polynomial in a single indeterminate $y$ (the case $n=1$). Take a nonconstant $\xi\in K$. Fix any $r\in\mathbb{N}$.

<u>Claim</u>: If $G\in K\{y\}$ is a nonzero diff poly of order $\leq r$, there exists

$$\eta = c_0 + c_1\xi + \cdots + c_r\xi^r$$

where all the $c_i$'s are constants in $K$, satisfying $G(\eta)\neq 0$.

Suppose the claim is false and let $H\neq 0$ be a nonzero diff poly of lowest rank which vanishes for every elt $c_0+c_1\xi+\cdots+c_r\xi^r$ ($c_i$ are constants from $K$). Let $\operatorname{ord}(H,y)=s$. Then $0<s\leq r$. Introduce algebraic indeterminates $z_0,\cdots,z_r$ with $z_i'=0$. Then $\overline{H}=H(z_0+z_1\xi+\cdots+z_r\xi^r)\in K[z_0,\cdots,z_r]$ is the zero polynomial. Take the partial derivative of $\overline{H}$ w.r.t. $z_0,\cdots,z_s$ then

$$\begin{cases}\overline{\frac{\partial H}{\partial y}} = 0\\ \overline{\frac{\partial H}{\partial y}}\xi + \overline{\frac{\partial H}{\partial y'}}\xi' + \cdots + \overline{\frac{\partial H}{\partial y^{(s)}}}\xi^{(s)} = 0\\ \cdots\quad\cdots\\ \overline{\frac{\partial H}{\partial y}}\xi^s + \overline{\frac{\partial H}{\partial y'}}(\xi^s)' + \cdots + \overline{\frac{\partial H}{\partial y^{(s)}}}(\xi^s)^{(s)} = 0\end{cases} \qquad \left(\overline{\frac{\partial H}{\partial y^{(j)}}} = \frac{\partial H}{\partial y^{(j)}}(z_0+\cdots+z_r\xi^r)\right)$$

So
$$\begin{pmatrix}1 & 0 & \cdots & 0\\ \xi & \xi' & \cdots & \xi^{(s)}\\ \cdots & & & \\ \xi^s & (\xi^s)' & \cdots & (\xi^s)^{(s)}\end{pmatrix}\begin{pmatrix}\overline{\frac{\partial H}{\partial y}}\\ \overline{\frac{\partial H}{\partial y'}}\\ \vdots\\ \overline{\frac{\partial H}{\partial y^{(s)}}}\end{pmatrix} = 0.$$
Since $\frac{\partial H}{\partial y^{(s)}}$ is of lower rank than $H$, $\overline{\frac{\partial H}{\partial y^{(s)}}}\neq 0$.

Thus,
$$\begin{vmatrix}\xi' & (\xi^2)' & \cdots & (\xi^s)'\\ \xi'' & (\xi^2)'' & \cdots & (\xi^s)''\\ \cdots & & & \\ \xi^{(s)} & (\xi^2)^{(s)} & \cdots & (\xi^s)^{(s)}\end{vmatrix} = W r(\xi', (\xi^2)', \cdots, (\xi^s)') = 0.$$

So $\exists\, c_1,\cdots,c_s$ constants of $K$, not all zero s.t. $c_1\xi'+c_2(\xi^2)'+\cdots+c_s(\xi^s)'=0$.

Then $c_1\xi+c_2\xi^2+\cdots+c_s\xi^s=c_0$ with $c_0$ a constant.

Thus, $\xi$ is algebraic over the constant field of $K$.
By Corollary 4.2, $\xi'=0$, a contradiction to the hypothesis $\xi'\neq 0$.
So we can find some $y=c_0+c_1\xi+\cdots+c_r\xi^r$ w/ $c_i$ constants s.t. $G(y)\neq 0$. $\boxed{\xi}$.

Remark: 1) Lemma 4.1.2 is false without the restriction that $(K,\delta)$ contains at least a nonconstant element. A non-example: $K=\mathbb{Q}$, $G(y)=y'$.

2) For the partial differential case $(K,\{\delta_1,\cdots,\delta_m\})$, the condition that "$\exists\ \xi\in K$ s.t. $\xi'\neq 0$" should be replaced by
"$\exists\ \xi_1,\cdots,\xi_m\in K$ s.t. $\begin{vmatrix} \delta_1(\xi_1) & \cdots & \delta_1(\xi_m) \\ \delta_2(\xi_1) & \cdots & \delta_2(\xi_m) \\ \vdots & & \vdots \\ \delta_m(\xi_1) & \cdots & \delta_m(\xi_m) \end{vmatrix}\neq 0$."

The lemma is called "non-vanishing of diff polynomials".

3) This is the differential analog of the following result in Algebra:
"Let $K$ be an infinite field. Then for any nonzero poly $f\in K[y_1,\cdots,y_n]$, $\exists\ (a_1,\cdots,a_n)\in K^n$ s.t. $f(a_1,\cdots,a_n)\neq 0$."

<u>Theorem 4.1.3</u> (Differential Primitive Element Theorem)
Let $(K,\delta)$ be a <u>non-constant</u> differential field of char 0 (i.e., $\exists\ b\in K$, $\delta(b)\neq 0$).
Assume $K\langle\partial_1,\cdots,\partial_n\rangle$ is differential algebraic over $K$.
Then $\exists\ \xi\in K\langle\partial_1,\cdots,\partial_n\rangle$ s.t. $K\langle\partial_1,\cdots,\partial_n\rangle=K\langle\xi\rangle$.
(In other words, every finitely generated differential algebraic extension field of $(K,\delta)$ is generated by a single element.)

Proof. It suffices to show that if $\gamma,\beta$ are differential algebraic over $K$, then $\exists\ e\in K$ s.t.
$$K\langle\gamma,\beta\rangle=K\langle\gamma+e\beta\rangle.$$
Introduce a new differential indeterminate $t$ over $K\langle\gamma,\beta\rangle$ and consider $\gamma+t\beta\in K\langle t\rangle\langle\gamma,\beta\rangle$.

By lemma 4.5, $\gamma+t\beta$ is differential algebraic over $K\langle t\rangle$.
Consider the prime differential ideal $\mathbb{I}(\gamma+t\beta)\subseteq K\langle t\rangle\{y\}$
and suppose $A(y)\in K\langle t\rangle\{y\}$ is a characteristic set of $\mathbb{I}(\gamma+t\beta)$.
Then $A(\gamma+t\beta)=0$ but $S_A(\gamma+t\beta)\neq 0$. Assume $\mathrm{ord}(A)=s$.
Clearing denominators when necessary, we can take $A\in K\{t,y\}$
and write $A(t,y)$ for convenience.
Now, we have $A(t,\gamma+t\beta)=0$ but $\frac{\partial A}{\partial y^{(s)}}(t,\gamma+t\beta)\neq 0$.
Note that $\dfrac{\partial(\gamma+t\beta)^{(k)}}{\partial t^{(s)}}=\begin{cases}0 & k<s\\ \beta & k=s\end{cases}$ for $k\leq s$.

Take the partial derivatives of $A(t,\gamma+t\beta)=0$ w.r.t. $t^{(s)}$, we have

$$\frac{\partial A}{\partial t^{(s)}}(t,\gamma+t\beta)+\beta\cdot\frac{\partial A}{\partial y^{(s)}}(t,\gamma+t\beta)=0.$$

Since $\frac{\partial A}{\partial y^{(s)}}(t,\gamma+t\beta)\neq 0$ belongs to $K\langle\gamma,\beta\rangle\{t\}$, by lemma 4.1.2,
$\exists\, e\in K$ s.t. $\frac{\partial A}{\partial y^{(s)}}(e,\gamma+e\beta)\neq 0$.
Thus,

$$\beta=-\frac{\frac{\partial A}{\partial t^{(s)}}(e,\gamma+e\beta)}{\frac{\partial A}{\partial y^{(s)}}(e,\gamma+e\beta)}\in K\langle\gamma+e\beta\rangle$$

and $K\langle\gamma,\beta\rangle=K\langle\gamma+e\beta\rangle$ follows. $\boxtimes$

<u>Corollary</u>: Let $(K,\delta)$ be a nonconstant differential field. Let
$K\langle\eta_1,\cdots,\eta_n\rangle$ be a differential algebraic extension field of $K$.
Then $\exists\, e_1,\cdots,e_n\in K$ s.t. $K\langle\eta_1,\cdots,\eta_n\rangle=K\langle e_1\eta_1+\cdots+e_n\eta_n\rangle$.

Remark: G. Pogudin proved the diff primitive theorem for the case ① $K'=\{0\}$; ② $K\langle\eta_1,\cdots,\eta_n\rangle$ has a nonconstant.
("The primitive element theorem for differential fields with zero derivation on the base field. J. Pure Appl. Algebra, 4035-4041, 2015.")