# Using Cayley Menger determinants

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**Abstract**

This article proposes an intrinsic formulation of geometric constraints, using classical or new Cayley Menger determinants. Sometimes this formulation has some advantages.

## 1 Introduction

### 1.1 The context

The problem of solving geometric constraints occurs in CAD, robotics, Computer Graphics, molecular biology [Doh95, BR98]. For instance, in CAD, today geometric modellers enable designers to describe geometric elements such as points, lines, circles, Bézier curves, etc in $2D$ and planes, quadrics, tori, Bézier patches, etc, in $3D$ by specifying constraints between them: distances, angles, incidence or tangency relations. Then the modeller has to solve the system of constraints, classically polynomial equations. It decomposes the system into irreducible subsystems, and solves them with symbolic methods or rather numerical ones: Newton-Raphson iterations from an initial guess interactively provided by the designer, or some kind of homotopy [LM95, Dur98, HD99]. Other numerical method can apply, for instance interval analysis, but the latter is less common in GCS (geometric constraint solving); however see [JAMSR01]. Numerical methods prevail because today symbolic packages are not powerfull enough to treat 3D real world geometric problems, like the Stewart platform problem discussed in this paper.

Most of the time cartesian coordinates are used to pose equations before solving geometric constraints. However it is not the only nor the best way. Section 2 shows the Menger Cayley determinant is much more convenient to solve the Stewart platform [Dur98, HD99, NW91]: the obtained system is much simpler, without spurious roots, and becomes tractable by symbolic methods. Classical Menger Cayley determinants [Ber90, Hav91, Blu53, dC87] are shortly presented in section 3; then new Menger Cayley determinants, for some asymmetric problems, are proposed for $2D$ and $3D$ examples in section 4. Section 1.3 presents the advantages of intrinsic formulations (independent from a particular coordinate system choice). Section 1.2 presents some related works.

Remark: this work has not been published so far, but was referenced several times since it is available on internet. Apart this introduction and the correction of some misprints and missings, this article is mainly the version available on internet. Main bibliographic missings in the web document are: the de Casteljau's book on Quaternions (in French) which contains a chapter on Menger-Cayley determinants [dC87], the Blumenthal's book [Blu53], and the paper by Zhang, Yang and Yang [ZYY94] (see also [Yan02]).

### 1.2 Related works

Recently, David Lesage, Philippe Serré and Jean-Claude Léon [LLS00, LLS02, Les02], in the wake of Serré's phD [Ser00], express all $2D$ constraints in a coordinate free way. They don't use the Cayley Menger formalism –which proves there are several intrinsic formulations; rather, they find independent angular and vectorial loops in some constraints graphs; then each loop gives a constraint, which is translated into equations. The unknowns are not the coordinates of points, lines, vectors, etc, but norms of vectors, and angles between vectors (which, again, are not represented by their coordinates); in other words, unknowns are scalar products between vectors. Their work proves that coordinate free approaches are indeed feasible, and can be realized in a systematic way. It also proves the advantages of an intrinsic approach (see section 1.3).

Researchers in the Laboratory for Geometric Modelling and Multimedia Algorithms [Pod02], and Lu Yang *et al* [Yan02, ZYY94] also propose non cartesian approaches.

In geometric theorem proving (an area close to, but distinct of GCS), intrinsic approaches are also investigated, by J. Richter-Gebert [RG95], by Neil L. White [Whi91], by Boy de la Tour and Fèvre and Wang [dlTFW99]. Up to now, these works seem unused in the GCS community – some symbolic methods propagated to GCS in the past, after some delay, for instance Wu-Ritt's method.

Finally, to prevent a very frequent confusion, note that cartesian coordinates, Grassman Plücker coordinates, pentaspheric coordinates, etc *are* coordinates: they depend on a particular coordinate system, thus they are not intrinsic formulations.

### 1.3 Advantages of the intrinsic formulation

The intrinsic formulation [LLS00, LLS02, Les02] has several advantages. First, it takes naturally into account technologic unknowns and constraints. Thus it avoids the limitations of a lot of geometric decomposition methods. Second, the qualitative study of the resulting systems of equations is straightforward: the number of equations and unknowns are equal in correct (*ie* rigid) systems; the resulting system of equations is decomposed with bipartite graph matching methods [AAJM93], and structurally irreducible subsystems are studied with the probabilistic numerical method [LM97, LM98]. This contrasts with the classical cartesian formulation, where well-constrained (rigid) systems have less equations than unknowns (3 in 2D, 6 in 3D for "full dimensional" systems); this feature introduces a lot of complications. Moreover, with the non cartesian approach, the methods performing the qualitative analysis detect mistakes which are often hidden by the cartesian formulation, such as for example an angular over-constrainedness in 3D (mitigated by an elsewhere under-constrainedness). Last advantage, some problems become tractable with symbolic computations.

## 2 The Stewart platform problem

The lengths of the 12 edges of a $3D$ octahedron are given; the Stewart platform problem, also called the octahedron problem [Dur98, HD99, NW91], is then to find compatible coordinates for the 6 vertices $s_1, s_2 \dots s_6$. The 12 edges of the octahedron are:

$$s_2s_3, s_3s_4, s_4s_5, s_5s_2, s_1s_2, s_1s_3, s_1s_4, s_1s_5, s_6s_2, s_6s_3, s_6s_4, s_6s_5$$

This problem is met in CAD as a typical irreducible $3D$ problem in geometric modelling by constraints, and in robotic with the Stewart platform: the Stewart triangular platform: $s_1s_2s_3$ is driven with 6 jacks (with variable lengths) $s_1s_4, s_1s_5, s_2s_5, s_2s_6, s_3s_6, s_3s_4$ from a ground triangular base $s_4s_5s_6$. Edges of the triangular platform and of the base are rigid, *i.e.* their lengths is fixed once and for all. See figure 1.

It is possible to use Cartesian coordinates to pose the problem, of course. But which coordinates system turns out to be the more convenient for solving is not obvious, today computational algebra packages are not powerfull enough to solve the system, and a lot of work must be made by hand in order to reduce the system to an irreducible (or at least it seems so) system in 3 unknowns and 3 equations of degree 4: see [Dur98, HD99] for details. The resulting system has Bezout number $4 \times 4 \times 4 = 64$, and BKK bound (or mixed volume) 16.

Another possibility is to use Menger Cayley determinants (see [Ber90, Hav91] or section 3). It directly yields to 2 equations in 2 unknowns, each with degree 4. The Menger Cayley determinant

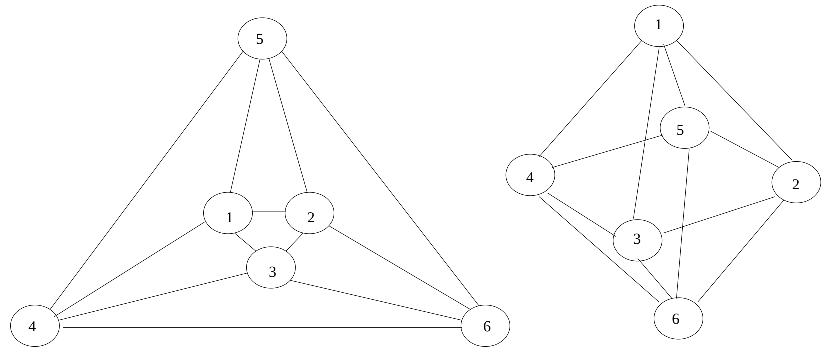


Figure 1: Two isomorphic graphs, or views, of the Stewart platform.

gives the relations between the distances between 5 points in $3D$ (between 4 points in $2D$, between $d+2$ points in $d$ dimensions). Here we can write the Menger Cayley determinant:

- first for points $s_1$ and $s_2$, $s_3$, $s_4$, $s_5$ (the equatorial square and the north vertex in the right part of figure 1). It gives an algebraic equation between squared distances

$$d_{12}, d_{13}, d_{14}, d_{15}, d_{23}, d_{24}, d_{25}, d_{34}, d_{35}, d_{45}$$

where $d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$. All these distances are known, except $d_{24}$ and $d_{35}$, the squared lengths of diagonals of the equatorial square in figure 1. The equation has degree 4 and involves 2 unknowns.

- second for points $s_6$ and $s_2$, $s_3$, $s_4$, $s_5$ (the equatorial square and the south vertex in the right part of figure 1). It gives another algebraic equation, with the same 2 unknowns $d_{24}$ and $d_{35}$, and degree 4. It is obvious this equation is generically independent of the previous one.

Thus we obtain an algebraic system in 2 unknowns and 2 equations, each of degree 4. If we want, we can draw the corresponding curves in the plane, with any standard method of Computer Graphics. From Bezout's theorem, this system cannot have more than 16 solutions in complex projective space (*i.e.* taking into account multiple solutions, real and complex solutions, solutions at infinity). Other methods yield to systems with greater Bezout number (typically 64), and in such a case it is not obvious at all to prove there are only 16 solutions.

The system can be solved by any standard numerical method, say homotopy. But since there is only 2 equations in 2 unknowns, it becomes tractable with symbolic methods. For instance the Sylvester resultant, gives a degree 16 equation in only one of the unknowns Maybe it also becomes possible to discuss degeneracies, but this question has not been investigated at this moment.

Once we have the length of diagonals $d_{24}$ and $d_{35}$, it is trivial to find consistent coordinates for the six vertices.

The trick here was to not use coordinates, but to compute distances, which are independent of the coordinates system (once the scale, say meters or millimeters, has been chosen, of course). Other parameters independent of coordinates system are angles and cross ratios, and they may be more convenient in other cases.

# 3 Classical Menger Cayley determinants

Here is an introduction to classical Menger Cayley determinants. See [Ber90, Hav91] for more.

### 3.1 Distances between points

Let 5 points in $3D$ (our usual euclidean $3D$ space). Then the following relation holds, between all their squared distances:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12} & d_{13} & d_{14} & d_{15} \\ 1 & d_{21} & 0 & d_{23} & d_{24} & d_{25} \\ 1 & d_{31} & d_{32} & 0 & d_{34} & d_{35} \\ 1 & d_{41} & d_{42} & d_{43} & 0 & d_{45} \\ 1 & d_{51} & d_{52} & d_{53} & d_{54} & 0 \end{vmatrix} = 0$$

where $d_{ij} = (p_i - p_j).(p_i - p_j)$ is the square of the distance $ij$ between point $i$ and $j$.

$|M|$ is the so called Menger Cayley determinant. For the proof, note that $M = AB^t$ where $A$ and $B^t$ are matrices with rank at most 5 ($A$ and $B$ have 6 rows but only 5 columns):

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x_1^2 + y_1^2 + z_1^2 & 2x_1 & 2y_1 & 2z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & 2x_3 & 2y_3 & 2z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & 2x_4 & 2y_4 & 2z_4 & 1 \\ x_5^2 + y_5^2 + z_5^2 & 2x_5 & 2y_5 & 2z_5 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -x_1 & -y_1 & -z_1 & x_1^2 + y_1^2 + z_1^2 \\ 1 & -x_2 & -y_2 & -z_2 & x_2^2 + y_2^2 + z_2^2 \\ 1 & -x_3 & -y_3 & -z_3 & x_3^2 + y_3^2 + z_3^2 \\ 1 & -x_4 & -y_4 & -z_4 & x_4^2 + y_4^2 + z_4^2 \\ 1 & -x_5 & -y_5 & -z_5 & x_5^2 + y_5^2 + z_5^2 \end{pmatrix}$$

Actually, $|M|$ still vanishes when points in $A$ are not the same than in $B$. It gives another non trivial relation for distances between points $P_i, i = 1 \ldots 5$ and points $Q_j, j = 1 \ldots 5$ (the diagonal entries in $M$ are no more zeros, but squared distances between $P_i$ and $Q_i$).

The previous determinants extends in $2D$, 4D, etc. In $2D$, it gives Ptolemee theorem. Finally let us mention that the Menger Cayley determinant is equal to a signed volume, up to some multiplicative constant.

### 3.2 Distances between spheres

In $3D$, define the signed distance (or power) of 2 spheres $S_i = (x_i \quad y_i \quad z_i)$ with radius $R_i$ and $S_j = (x_j \quad y_j \quad z_j)$ with radius $R_j$ as

$$K_{ij} = K_{ji} = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (R_i^2 + R_j^2)$$

Actually, this signed distance does not depend on the used coordinate system (once the scale is chosen, say meter or millimeter), as easily seen.

Then the distances between any six spheres in $3D$ fullfill:

$$|K| = \begin{vmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{vmatrix} = 0$$

Proof: $K = AB^t$ and $A$ and $B$ have rank at most 5 (6 rows, 5 columns):

$$A = \begin{pmatrix} x_1^2+y_1^2+z_1^2-R_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2-R_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2-R_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2-R_4^2 & x_4 & y_4 & z_4 & 1 \\ x_5^2+y_5^2+z_5^2-R_5^2 & x_5 & y_5 & z_5 & 1 \\ x_6^2+y_6^2+z_6^2-R_6^2 & x_6 & y_6 & z_6 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -2x_1 & -2y_1 & -2z_1 & x_1^2+y_1^2+z_1^2-R_1^2 \\ 1 & -2x_2 & -2y_2 & -2z_2 & x_2^2+y_2^2+z_2^2-R_2^2 \\ 1 & -2x_3 & -2y_3 & -2z_3 & x_3^2+y_3^2+z_3^2-R_3^2 \\ 1 & -2x_4 & -2y_4 & -2z_4 & x_4^2+y_4^2+z_4^2-R_4^2 \\ 1 & -2x_5 & -2y_5 & -2z_5 & x_5^2+y_5^2+z_5^2-R_5^2 \\ 1 & -2x_6 & -2y_6 & -2z_6 & x_6^2+y_6^2+z_6^2-R_6^2 \end{pmatrix}$$

This relation also holds when some radii are 0. This way it is possible to compute, say, the relation between any point and any 5 spheres in $\mathbb{R}^3$.

Remark: this other definition for the distance between 2 spheres:

$$K = (x-x')^2 + (y-y')^2 + (z-z')^2 - (R-R')^2$$

yields to the same kind of relations.

## 3.3 Cocyclicity or cosphericity of points

It is possible to express the cocyclicity of 4 points in $2D$, or the cosphericity of 5 points in $3D$, of $d+1$ points in $\mathbb{R}^d$ without coordinates, just by using squared distances between points.

In $2D$, 4 points are cocyclic (belong to the same circle) iff

$$|C| = \begin{vmatrix} 0 & d_{12} & d_{13} & d_{14} \\ d_{21} & 0 & d_{23} & d_{24} \\ d_{31} & d_{32} & 0 & d_{34} \\ d_{41} & d_{42} & d_{43} & 0 \end{vmatrix} = 0$$

where as usual $d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2$ is the squared distance between point $i$ and $j$, and thus is independent of cartesian systems.

Proof: let $(x_0, y_0)$ be the center and $R_0$ be the radius of the (unknown) circle. We have, in some cartesian frame (we will remove this dependency later): $(x_i - x_0)^2 + (y_i - y_0)^2 - R_0^2 = 0$ for $i = 1, 2, 3, 4$. We can express these conditions this way:

$$\begin{pmatrix} x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2x_0 \\ -2y_0 \\ x_0^2+y_0^2-R_0^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

It can be seen as a linear homogenous system with unknows in the column vector. There is a non zero solution iff the determinant of the matrice (call it $C_1$) is zero. We have a condition for cocyclicity, but it depends on the cartesian frame.

We also can express the system this way:

$$\begin{pmatrix} 1 & -2x_1 & -2y_1 & x_1^2+y_1^2 \\ 1 & -2x_2 & -2y_2 & x_2^2+y_2^2 \\ 1 & -2x_3 & -2y_3 & x_3^2+y_3^2 \\ 1 & -2x_4 & -2y_4 & x_4^2+y_4^2 \end{pmatrix} \begin{pmatrix} x_0^2+y_0^2-R_0^2 \\ x_0 \\ y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Here again, the determinant of the matrice (call it $C_2$) must vanish.

Now remark that $C = C_1 C_2^t$. Thus the determinant of $C = C_1 C_2^t$ must also vanish. We have proved the cocyclicity condition. This relation is trivially extended to $\mathbb{R}^3$ and beyond.

## 4 New Menger Cayley determinants

Classical Menger Cayley determinants apply to very symmetric problems. But typical problems, in CAD and modelling by geometric constraints, are not as symmetrical as the Stewart platform problem: constraints involve heterogeneous data: points, planes, lines, spheres... Here are simple examples of heterogeneous Menger Cayley determinants.

### 4.1 3 points and 1 line in $2D$

In $2D$, consider 3 points $P_1$, $P_2$, $P_3$ and a line $L$. Let $d_{ij}$ for $i, j = 1, 2, 3$ be as usual the squared distances between point $P_i$ and $P_j$, and let $d_i$ be the signed (non squared) signed distance between point $P_i$ and line $L$. $d_i = ax_i + by_i + c$ assuming $L$ has equation: $ax + by + c = 0$ and $a^2 + b^2 = 1$, in the cartesian frame we want to get rid of. Of course, the $d_i$ do not depend on a particular cartesian frame, once scale is chosen.

Due to coplanarity, there is a relation between the $d_{ij}$ and the $d_i$ (in passing, there is only one equality: the configuration involves 6 distances but has only five "degrees of freedom"; there are other constraints, like triangular inequalities for the triangle to be realizable; we ignore them). This relation may seem a bit strange at first glance:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & d_{12} & d_{13} & d_1 \\ 1 & d_{21} & 0 & d_{23} & d_2 \\ 1 & d_{31} & d_{32} & 0 & d_3 \\ 0 & d_1 & d_2 & d_3 & \frac{-1}{2} \end{vmatrix} = 0$$

where diagonal zeros stand for $d_{ii}$ and $d_{ij} = d_{ji}$ of course. Note that $M$ is symmetric though the dissymmetry of the problem.

For proof, just check below that $M$ is the product of the 2 matrices $5 \times 4$ and $4 \times 5$ (thus with rank 4, generically):

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1^2 + y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2 + y_2^2 & 2x_2 & 2y_2 & 1 \\ x_3^2 + y_3^2 & 2x_3 & 2y_3 & 1 \\ c & -a & -b & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & -x_1 & -x_2 & -x_3 & \frac{a}{2} \\ 0 & -y_1 & -y_2 & -y_3 & \frac{b}{2} \\ 1 & x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 & c \end{pmatrix}$$

### 4.2 2 points and 2 lines in $2D$

In $2D$, consider 2 points $s_1$ and $s_2$ and 2 lines $l_1$ and $l_2$. Call $s_i l_j$ with $i, j = 1, 2$ the distance between point $s_i$ and line $l_j$. Lines $l_j$ have equation $a_j x + b_j y + c_j = 0$ in the cartesian frame we want to get rid of and we suppose for simplicity that $a_j^2 + b_j^2 = 1$. Thus $s_i l_j = a_j x_i + b_j y_i + c_j$. Call $l_1 l_2$ the "distance", actually the cosinus, between the 2 line directions: $l_1 l_2 = a_1 a_2 + b_1 b_2$. Call $s_1 s_2$ the squared distance between points $s_1$ and $s_2$. The relation between these distances is given by the nullity of the non symmetric determinant:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & s_1 s_2 & 2s_1 l_1 & 2s_1 l_2 \\ 1 & s_1 s_2 & 0 & 2s_2 l_1 & 2s_2 l_2 \\ 0 & -s_1 l_1 & -s_2 l_1 & 1 & l_1 l_2 \\ 0 & -s_1 l_2 & -s_2 l_2 & l_1 l_2 & 1 \end{vmatrix} = 0$$

Proof: the $5 \times 5$ matrice $M$ is the product of the following $5 \times 4$ and $4 \times 5$ matrices (thus these matrices have rank at most 4, and idem for their product):

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1^2+y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2+y_2^2 & 2x_2 & 2y_2 & 1 \\ -c_1 & a_1 & b_1 & 0 \\ -c_2 & a_2 & b_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & -x_1 & -x_2 & a_1 & a_2 \\ 0 & -y_1 & -y_2 & b_1 & b_2 \\ 1 & x_1^2+y_1^2 & x_2^2+y_2^2 & 2c_1 & 2c_2 \end{pmatrix}$$

Note that $|M|$ is not identically zero (*i.e.* we can find entries such that $|M|$ does not vanish), since we can find in it a perfect matching (*i.e.* one generically non zero element in each and every row and column).

## 4.3 3 lines in $2D$

Let $l_i, i = 1, 2, 3$ be any 3 lines in $2D$, with equation: $a_i x + b_i y + c_i = 0$. Assume w.l.o.g. that $a_i^2 + b_i^2 = 1$. Let $c_{ij} = c_{ji} = a_i a_j + b_i b_j$ be the cosinus of the angle between $l_i$ and $l_j$. As well known, they fullfill:

$$\begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{21} & 1 & c_{23} \\ c_{31} & c_{32} & 1 \end{vmatrix} = 0 \text{ since } \begin{pmatrix} 1 & c_{12} & c_{13} \\ c_{21} & 1 & c_{23} \\ c_{31} & c_{32} & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

these 2 last matrices having rank 2. For $d+1$ vectors to belong to the same $d$ flat, their Gram matrix (the matrix of their scalar product) must have rank $d$, thus the determinant must vanish.

## 4.4 4 points and 1 plane in $3D$

In $3D$, consider 4 points $S_i = S_1 \dots S_4$ and 1 plane $P$ with equation: $ax + by + cz + d = 0$, where w.l.o.g. $a^2 + b^2 + c^2 = 1$. The squared distance between $S_i$ and $S_j$ is $d_{ij}$ and the signed distance between $S_i$ and $P$ is $d_i = ax_i + by_i + cz_i + d$. The relation between all these distances is:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & d_{12} & d_{13} & d_{14} & d_1 \\ 1 & d_{21} & 0 & d_{23} & d_{24} & d_2 \\ 1 & d_{31} & d_{32} & 0 & d_{34} & d_3 \\ 1 & d_{41} & d_{42} & d_{43} & 0 & d_4 \\ 0 & d_1 & d_2 & d_3 & d_4 & \frac{-1}{2} \end{vmatrix} = 0$$

Proof:

$$\begin{aligned} M \quad &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x_1^2+y_1^2+z_1^2 & 2x_1 & 2y_1 & 2z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & 2x_3 & 2y_3 & 2z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & 2x_4 & 2y_4 & 2z_4 & 1 \\ d & -a & -b & -c & 0 \end{pmatrix} \\ &\times \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -x_1 & -x_2 & -x_3 & -x_4 & a/2 \\ 0 & -y_1 & -y_2 & -y_3 & -y_4 & b/2 \\ 0 & -z_1 & -z_2 & -z_3 & -z_4 & c/2 \\ 1 & x_1^2+y_1^2+z_1^2 & x_2^2+y_2^2+z_2^2 & x_3^2+y_3^2+z_3^2 & x_4^2+y_4^2+z_4^2 & d \end{pmatrix} \end{aligned}$$

## 4.5 3 points and 2 planes in $3D$

Consider 3 points $s_1$, $s_2$, $s_3$ and 2 planes $p_1$ and $p_2$ in $3D$. Assume as usual that $p_i$ has equation: $a_i x + b_i y + c_i z + d_i = 0$ in some coordinate frame we want to get rid of, with $a_i^2 + b_i^2 + c_i^2 = 1$. Note

$s_ip_j$ the signed distance between point $s_i$ and plane $p_j$: $s_ip_j = a_jx_i + b_jy_i + c_jz_i + d_j$, and note $p_ip_j$ the cosinus of the angle between $p_i$ and $p_j$: $p_ip_j = a_ia_j + b_ib_j + c_ic_j$. The relation between all these distances is:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & s_1s_2 & s_1s_3 & 2s_1p_1 & 2s_1p_2 \\ 1 & s_1s_2 & 0 & s_2s_3 & 2s_2p_1 & 2s_2p_2 \\ 1 & s_1s_3 & s_2s_3 & 0 & 2s_3p_1 & 2s_3p_2 \\ 0 & -s_1p_1 & -s_2p_1 & -s_3p_1 & 1 & p_1p_2 \\ 0 & -s_1p_2 & -s_2p_2 & -s_3p_2 & p_1p_2 & 1 \end{vmatrix} = 0$$

Proof:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x_1^2+y_1^2+z_1^2 & 2x_1 & 2y_1 & 2z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & 2x_3 & 2y_3 & 2z_3 & 1 \\ -d_1 & a_1 & b_1 & c_1 & 0 \\ -d_2 & a_2 & b_2 & c_2 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -x_1 & -x_2 & -x_3 & a_1 & a_2 \\ 0 & -y_1 & -y_2 & -y_3 & b_1 & b_2 \\ 0 & -z_1 & -z_2 & -z_3 & c_1 & c_2 \\ 1 & x_1^2+y_1^2+z_1^2 & x_2^2+y_2^2+z_2^2 & x_3^2+y_3^2+z_3^2 & 2d_1 & 2d_2 \end{pmatrix}$$

### 4.6 2 points and 3 planes in $3D$

Without comment, the "distances" between 2 points $s_1, s_2$ and 3 planes $p_1, p_2, p_3$ in $3D$ fullfill:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & s_1s_2 & 2s_1p_1 & 2s_1p_2 & 2s_1p_3 \\ 1 & s_1s_2 & 0 & 2s_2p_1 & 2s_2p_2 & 2s_2p_3 \\ 0 & -s_1p_1 & -s_2p_1 & 1 & p_1p_2 & p_1p_3 \\ 0 & -s_1p_2 & -s_2p_2 & p_1p_2 & 1 & p_2p_3 \\ 0 & -s_1p_3 & -s_2p_3 & p_1p_3 & p_2p_3 & 1 \end{vmatrix} = 0$$

Proof:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x_1^2+y_1^2+z_1^2 & 2x_1 & 2y_1 & 2z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ -d_1 & a_1 & b_1 & c_1 & 0 \\ -d_2 & a_2 & b_2 & c_2 & 0 \\ -d_3 & a_3 & b_3 & c_3 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -x_1 & -x_2 & a_1 & a_2 & a_3 \\ 0 & -y_1 & -y_2 & b_1 & b_2 & b_3 \\ 0 & -z_1 & -z_2 & c_1 & c_2 & c_3 \\ 1 & x_1^2+y_1^2+z_1^2 & x_2^2+y_2^2+z_2^2 & 2d_1 & 2d_2 & 2d_3 \end{pmatrix}$$

### 4.7 4 planes in $3D$

Idem 4.3. The determinant of the Gram matrice of 4 planes in $3D$ vanishes.

### 4.8 Which method ?

A first open problem is to find relations involving also lines in $3D$, and not only points and planes. Probably Grassman Plücker coordinates for lines in some cartesian frame must be used, before eliminating it. One such relation, due to Neil White, is given in Sturmfels's book [Stu93], th. 3.4.7: it is the condition for five lines in $3D$ space for having a common transversal line. Philippe Serré, in his PhD [Ser00], also gives some relations, for example for the distance between two lines $AB$ and $CD$ and the distances between $A$, $B$, $C$, $D$.

A second question is how to build such determinants. From a theoretical point of view, it suffices to use a Grobner package to eliminate variables representing coordinates in some set of equations (for instance equations: $(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2-d_{ij}^2=0, i=1\dots 4, j=i+1\dots 5$, to find the Cayley Menger equation relating distances between 5 points in $3D$). In practice, Grobner packages are not powerfull enough.

The polynomial condition can be computed by interpolation: for instance, to guess the Cayley Menger equation in $3D$, generate random configurations of 5 points $(x_i, y_i, z_i) \in \mathbb{Z}^3$, compute (trivially) square distances $d_{ij}$, then reconstruct the interpolating polynomial in the variables $d_{ij}$, with increasing degree, until the correct degree is found. To be practicable, more precisely to decrease the number of unknown coefficients, this approach must exploit the symmetry, *ie* the fact that (for the same example) the searched polynomial must have equal coefficients for monomials in the same "class", like $d_{12}^2 d_{34}^2$, $d_{13}^2 d_{24}^2$, $d_{12}^2 d_{35}^2$, etc; these monomials define isomorphic weighted subgraphs in $K_5$. In passing, the fast generation of these classes (and of one instance per class) is an interesting and non trivial combinatorial problem by itself, related to the Polya's counting theory (the latter computes the number of such classes).

I implemented a naive interpolation method (naive since it lists all monomials before merging them into classes), to compute Cayley-Menger relations (to check the correction of the method) and the distances relation for 6 points (10 points) to lie on the same conic (the same quadric). Resulting polynomials are lengthy and not convenient: a method outputing determinant polynomials, like Menger Cayley determinants, would be more attractive. These questions deserve further studies.

## 5 Conclusion

This paper has shown that Menger Cayley determinants may give simpler algebraic systems, with less spurious roots, and more tractable with today symbolic algebra packages. Unfortunately, classical Menger Cayley determinants involve geometric elements all of the same kind, for instance only points. Thus this paper also proposed new Menger Cayley determinants for some $2D$ and $3D$ heterogeneous configurations. It remains to propose an automatic method to generate such new Menger Cayley relations, especially to handle lines in 3D space: a new challenging problem for computational algebra.

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