

ImUp: A Maple Package for Uniformity-Improved Reparameterization of Plane Curves

Jing Yang

LMIB – Beihang University

Dongming Wang

CNRS – Université Pierre et Marie Curie

Hoon Hong

North Carolina State University

27 October, ASCM 2012

Outline

1 Problem

2 Methods

3 Implementation

4 Examples and Experiments

5 Summary

Outline

1 Problem

2 Methods

3 Implementation

4 Examples and Experiments

5 Summary

Angular Speed Uniformity

Angular Speed Uniformity

Given a parameterization $p = (x, y) : [0, 1] \rightarrow \mathbb{R}^2$, let

$$\begin{aligned}\theta_p &= \arctan \frac{y'}{x'}, & \omega_p &= |\theta'_p|, \\ \mu_p &= \int_0^1 \omega_p(t) dt, & \sigma_p^2 &= \int_0^1 (\omega_p(t) - \mu_p)^2 dt.\end{aligned}$$

Angular Speed Uniformity

Given a parameterization $p = (x, y) : [0, 1] \rightarrow \mathbb{R}^2$, let

$$\begin{aligned}\theta_p &= \arctan \frac{y'}{x'}, & \omega_p &= |\theta'_p|, \\ \mu_p &= \int_0^1 \omega_p(t) dt, & \sigma_p^2 &= \int_0^1 (\omega_p(t) - \mu_p)^2 dt.\end{aligned}$$

Definition (Angular Speed Uniformity)

The *angular speed uniformity* u_p of a parameterization p is defined as

$$u_p = \frac{1}{1 + \sigma_p^2 / \mu_p^2}$$

when $\mu_p \neq 0$. Otherwise, $u_p = 1$.

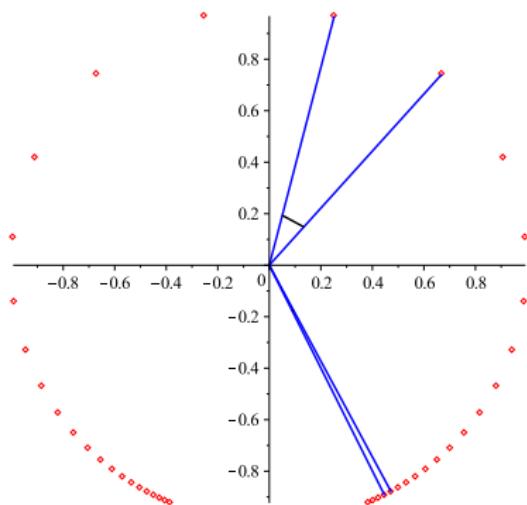
Angular Speed Uniformity

$$\omega_p = |\theta'_p| = \frac{|x'y'' - x''y'|}{x'^2 + y'^2}, \quad \sigma_p^2 = \int_0^1 (\omega_p(t) - \mu_p)^2 dt, \quad u_p = \frac{1}{1 + \sigma_p^2/\mu_p^2}$$

- $u_p \in (0, 1]$;
- When $u_p = 1$, ω_p is uniform;
- $\omega_p = \kappa \cdot \nu$, where κ is the curvature and ν is the speed at a point;
- u_p measures the goodness of a parameterization p .

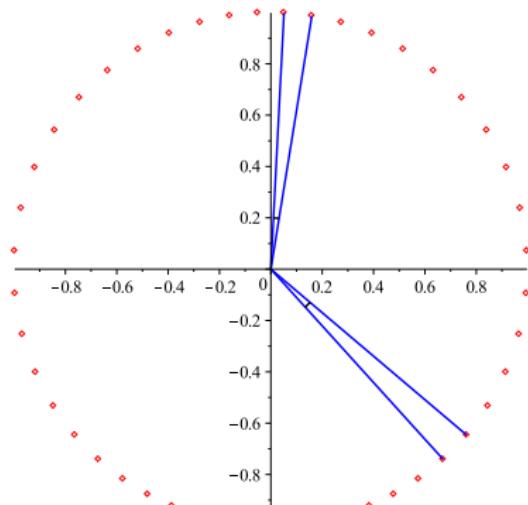
Examples: Angular Speed Uniformity

“Bad”



$$u_{p_1} \approx 0.482$$

“Good”



$$u_{p_2} \approx 0.977$$

Arc-angle Parameterization

Definition (Arc-angle Parameterization)

If $u_p = 1$, then p is called a *uniform parameterization* or an *arc-angle parameterization*.

Arc-angle Parameterization

Definition (Arc-angle Parameterization)

If $u_p = 1$, then p is called a *uniform parameterization* or an *arc-angle parameterization*.

Example

The parameterization $p = (\cos t, \sin t)$ is an arc-angle parameterization since $u_p = 1$.

Arc-angle Reparameterization

Question

How to compute an arc-angle reparameterization p^* if p is not an arc-angle one?

Arc-angle Reparameterization

Question

How to compute an arc-angle reparameterization p^* if p is not an arc-angle one?

Theorem

Let

$$\psi_p(t) = \frac{1}{\mu_p} \int_0^t \omega_p(s) ds$$

and $r_p = \psi_p^{-1}$, then $u_{p \circ r_p} = 1$, i.e. $p \circ r_p$ is an arc-angle reparameterization of p .

- Such r_p is called a *uniformizing parameter transformation*.

Rational Approximation of Arc-angle Reparameterization

Q: Is arc-angle parameterization **rational**?

A: The answer is “**No**” except for straight lines.

Rational Approximation of Arc-angle Reparameterization

Q: Is arc-angle parameterization **rational**?

A: The answer is “**No**” except for straight lines.

Problem

Given: $p \in \mathbb{Q}(t)^2$

Find: a rational p^* such that $u_{p^*} \approx 1$ or equivalently a rational r such that $u_{por} \approx 1$

Rational Approximation of Arc-angle Reparameterization

Q: Is arc-angle parameterization **rational**?

A: The answer is “**No**” except for straight lines.

Problem

Given: $p \in \mathbb{Q}(t)^2$

Find: a rational p^* such that $u_{p^*} \approx 1$ or equivalently a rational r such that $u_{por} \approx 1$

Two Approaches

- One-piece rational functions of high degree
 - e.g. Weierstrass approximation
- Piecewise rational functions of low degree
 - e.g. Piecewise Möbius transformation ✓

Piecewise Möbius Transformation

Notation

Let $T = (t_0, \dots, t_N)$, $S = (s_0, \dots, s_N)$, $\alpha = (\alpha_0, \dots, \alpha_{N-1})$ where $0 = t_0 < \dots < t_N = 1$, $0 = s_0 < \dots < s_N = 1$, and $0 < \alpha_0, \dots, \alpha_{N-1} < 1$.

Piecewise Möbius Transformation

Notation

Let $T = (t_0, \dots, t_N)$, $S = (s_0, \dots, s_N)$, $\alpha = (\alpha_0, \dots, \alpha_{N-1})$ where $0 = t_0 < \dots < t_N = 1$, $0 = s_0 < \dots < s_N = 1$, and $0 < \alpha_0, \dots, \alpha_{N-1} < 1$.

Definition (Piecewise Möbius Transformation)

A map m is called a *piecewise Möbius transformation* if

$$m(s) = \begin{cases} \vdots \\ m_i(s), & \text{if } s \in [s_i, s_{i+1}]; \\ \vdots \end{cases}$$

where

$$m_i(s) = t_i + \Delta t_i \frac{(1 - \alpha_i)\tilde{s}}{(1 - \alpha_i)\tilde{s} + (1 - \tilde{s})\alpha_i}$$

and $\Delta t_i = t_{i+1} - t_i$, $\Delta s_i = s_{i+1} - s_i$, $\tilde{s} = (s - s_i)/\Delta s_i$.

Remarks

- $m(s)$ is C^0 continuous and thus called C^0 piecewise Möbius transformation.
- When $N = 1$, it degenerates to an α -Möbius transformation.
- If m satisfies

$$m'_i(s_{i+1}) = m'_{i+1}(s_{i+1}),$$

it becomes a C^1 piecewise Möbius transformation.

- Different choices of T, S, α produce different $m(s)$. Thus m is represented as (T, S, α) .

Rational Approximation of Arc-angle Reparameterization

Sub-Problem A

Given: $p \in \mathbb{Q}(t)^2$ which is not a straight line,

N the number of pieces

Find: a C^0 N -piecewise Möbius transformation m such that u_{pom} is optimal

Rational Approximation of Arc-angle Reparameterization

Sub-Problem A

Given: $p \in \mathbb{Q}(t)^2$ which is not a straight line,

N the number of pieces

Find: a C^0 N -piecewise Möbius transformation m such that $u_{p \circ m}$ is optimal

Sub-Problem B

Given: $p \in \mathbb{Q}(t)^2$ which is not a straight line,

\bar{u} an object uniformity

Find: a C^1 piecewise Möbius transformation m such that $u_{p \circ m}$ is close to \bar{u}

Rational Approximation of Arc-angle Reparameterization

Sub-Problem A

Given: $p \in \mathbb{Q}(t)^2$ which is not a straight line,

N the number of pieces

Find: a C^0 N -piecewise Möbius transformation m such that $u_{p \circ m}$ is optimal

Sub-Problem B

Given: $p \in \mathbb{Q}(t)^2$ which is not a straight line,

\bar{u} an object uniformity

Find: a C^1 piecewise Möbius transformation m such that $u_{p \circ m}$ is close to \bar{u}

Assumption: the angular speed ω_p is nonzero over $[0, 1]$

Outline

1 Problem

2 Methods

3 Implementation

4 Examples and Experiments

5 Summary

State of Art

- Yang, Wang, and Hong: Improving angular speed uniformity by reparameterization (revised version under review)
- Yang, Wang, and Hong: Improving angular speed uniformity by optimal C^0 piecewise reparameterization, CASC 2012
- Yang, Wang, and Hong: Improving angular speed uniformity by C^1 piecewise reparameterization, ADG 2012

C^0 Piecewise Reparameterization

- Let T be an arbitrary but fixed sequence. Then the globally optimal α and S are computed by

$$\alpha_i = (\alpha_i)_T = \frac{1}{1 + \sqrt{C_i/A_i}} \quad \text{and} \quad s_i = (s_i)_T = \frac{\sum_{k=0}^{i-1} \sqrt{M_k}}{\sum_{k=0}^{N-1} \sqrt{M_k}},$$

where

$$\begin{aligned} A_i &= \int_{t_i}^{t_{i+1}} \omega_p^2(t) \cdot (1 - \tilde{t})^2 dt, & B_i &= \int_{t_i}^{t_{i+1}} \omega_p^2(t) \cdot 2\tilde{t}(1 - \tilde{t}) dt, \\ C_i &= \int_{t_i}^{t_{i+1}} \omega_p^2(t) \cdot \tilde{t}^2 dt, & M_k &= \Delta t_k \left(2\sqrt{A_k C_k} + B_k \right), \\ \tilde{t} &= (t - t_i)/\Delta t_i. \end{aligned}$$

C^0 Piecewise Reparameterization

- Let T be an arbitrary but fixed sequence and m_T denote the optimal transformation. Then

$$u_{p \circ m_T} = \frac{\mu_p^2}{\phi_p^2}, \quad \text{where} \quad \phi_p = \sum_{i=0}^{N-1} \sqrt{\Delta t_i (2\sqrt{A_i C_i} + B_i)}.$$

C^0 Piecewise Reparameterization

- Let T be an arbitrary but fixed sequence and m_T denote the optimal transformation. Then

$$u_{p \circ m_T} = \frac{\mu_p^2}{\phi_p^2}, \quad \text{where} \quad \phi_p = \sum_{i=0}^{N-1} \sqrt{\Delta t_i (2\sqrt{A_i C_i} + B_i)}.$$

- $\max u_{p \circ m_T} \Leftrightarrow \min \phi_p$
s.t. $0 < t_1 < \dots < t_{N-1} < 1$ s.t. $0 < t_1 < \dots < t_{N-1} < 1$

C^0 Piecewise Reparameterization

- Let T be an arbitrary but fixed sequence and m_T denote the optimal transformation. Then

$$u_{p \circ m_T} = \frac{\mu_p^2}{\phi_p^2}, \quad \text{where} \quad \phi_p = \sum_{i=0}^{N-1} \sqrt{\Delta t_i (2\sqrt{A_i C_i} + B_i)}.$$

- $\max u_{p \circ m_T} \Leftrightarrow \min \phi_p$
s.t. $0 < t_1 < \dots < t_{N-1} < 1$ s.t. $0 < t_1 < \dots < t_{N-1} < 1$



Zoutendijk's method of
feasible directions

C^0 Piecewise Reparameterization

- Let T be an arbitrary but fixed sequence and m_T denote the optimal transformation. Then

$$u_{p \circ m_T} = \frac{\mu_p^2}{\phi_p^2}, \quad \text{where} \quad \phi_p = \sum_{i=0}^{N-1} \sqrt{\Delta t_i (2\sqrt{A_i C_i} + B_i)}.$$

- $\max u_{p \circ m_T} \Leftrightarrow \min \phi_p$
s.t. $0 < t_1 < \dots < t_{N-1} < 1$ s.t. $0 < t_1 < \dots < t_{N-1} < 1$



Zoutendijk's method of
feasible directions

- Procedure:** (1) construct ϕ_p ; (2) locally optimize T ; (3) globally optimize α and S ; (4) construct optimal m and $p^* = p \circ m$.

C^1 Piecewise Reparameterization

1. Construct a near optimal C^1 piecewise Möbius transformation as follows.

- 1.1. Compute a partition T of the unit interval $[0, 1]$ by solving

$$\omega'_p(t) = 0$$

for t over $(0, 1)$.

- 1.2. Choose

$$s_i = \int_0^{t_i} \omega_p dt / \mu_p \quad (1 \leq i \leq N - 1)$$

to obtain S .

- 1.3. Compute the exact optimal α using T and S .

- 1.4. Construct a C^1 piecewise Möbius transformation from (T, S, α) .

C^1 Piecewise Reparameterization

2. Iterative process

- Traditional approach: construct

$$p_1 = p \circ m_1 \implies p_2 = p_1 \circ m_2 \implies \dots \implies$$

$$p_n = p_{n-1} \circ m_n = p \circ m_1 \circ \dots \circ m_n$$

such that $u_{p_1} < u_{p_2} < \dots < u_{p_n}$ until u_{p_n} is a desirable uniformity.

- Alternative approach: refine T at each iteration by solving

$$\omega_p' \cdot \Delta t_i \cdot [\alpha_i \tilde{t} + (1 - \alpha_i)(1 - \tilde{t})] - 2 \omega_p \cdot (1 - 2 \alpha_i) = 0$$

for t , where T and α are sequences computed before.

Outline

1 Problem

2 Methods

3 Implementation

4 Examples and Experiments

5 Summary

Architecture of ImUp

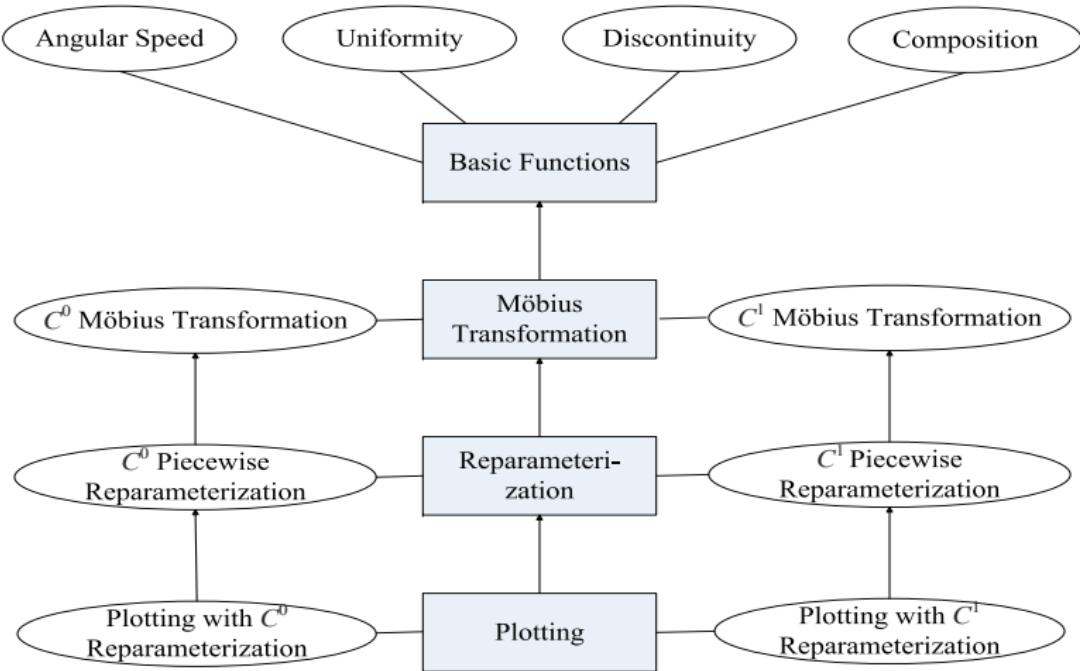


Figure: ImUp structure

Architecture of ImUp

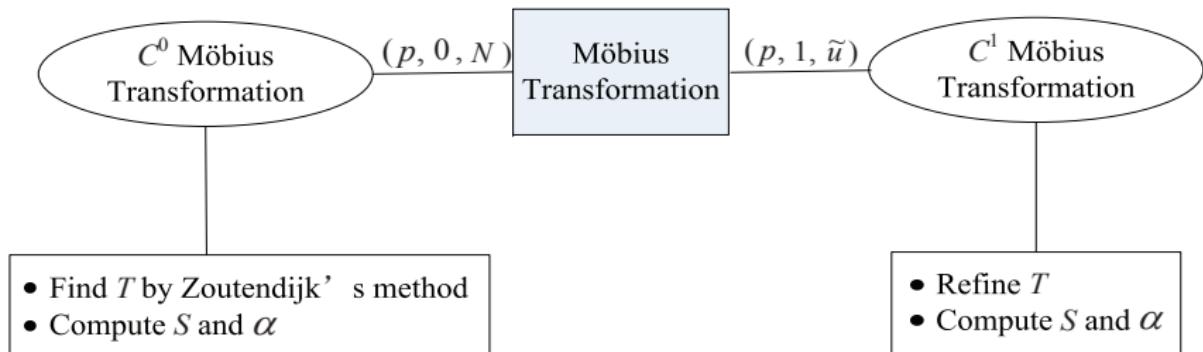


Figure: Module of Möbius transformation

Public Interface

Function	Calling Sequence
AngularSpeed	$\text{AngularSpeed}(p, m)$
Uniformity	$\text{Uniformity}(p, m)$
MoebiusTransformation	$\text{MoebiusTransformation}(p, \text{opt}, N \bar{u})$
ReparameterizationN	$\text{ReparameterizationN}(p, N)$
ReparameterizationU	$\text{ReparameterizationU}(p, \bar{u})$
ImUpPlotN	$\text{ImUpPlotN}(p, N, N_{\text{pt}})$
ImUpPlotU	$\text{ImUpPlotU}(p, \bar{u}, N_{\text{pt}})$

Outline

1 Problem

2 Methods

3 Implementation

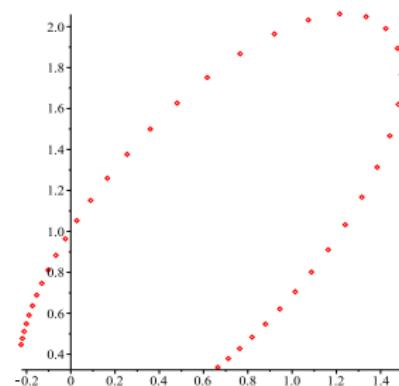
4 Examples and Experiments

5 Summary

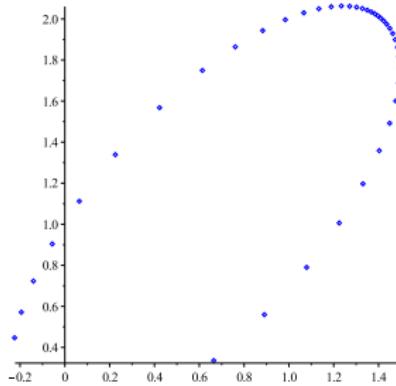
Plot with C^0 Piecewise Reparameterization

Let $p = \left(\frac{t^3 - 6t^2 + 9t - 2}{2t^4 - 16t^3 + 40t^2 - 32t + 9}, \frac{t^2 - 4t + 4}{2t^4 - 16t^3 + 40t^2 - 32t + 9} \right)$,

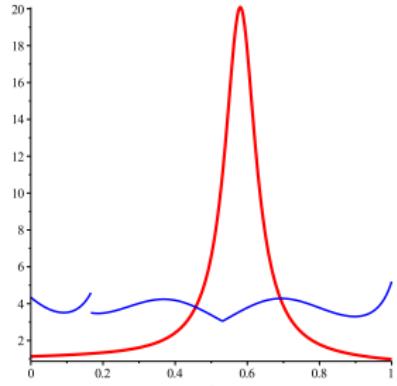
$N = 3$ and $N_{\text{pt}} = 40$.



Original p



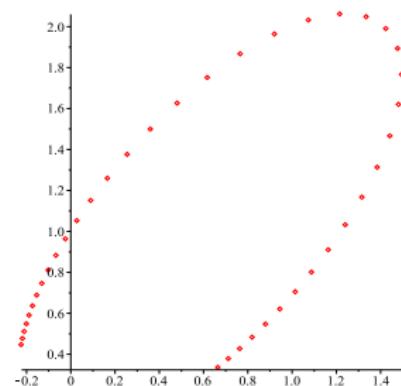
C^0 reparameterization p^*



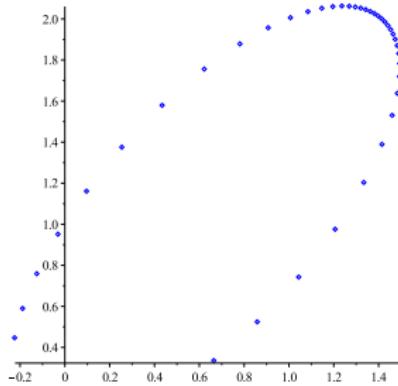
ω_p and ω_{p^*}

Plot with C^1 Piecewise Reparameterization

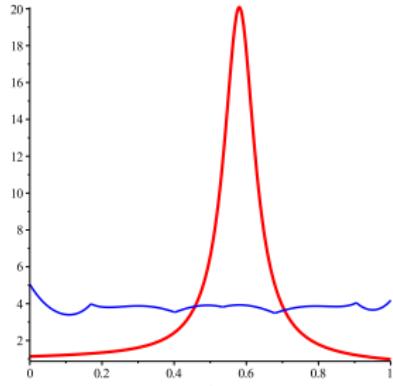
Let $p = \left(\frac{t^3 - 6t^2 + 9t - 2}{2t^4 - 16t^3 + 40t^2 - 32t + 9}, \frac{t^2 - 4t + 4}{2t^4 - 16t^3 + 40t^2 - 32t + 9} \right)$,
 $\bar{u} = 0.99$ and $N_{\text{pt}} = 40$.



Original p



C^1 reparameterization p^*



ω_p and ω_{p^*}

Experimental Results

Table: Experimental data with ReparameterizationN and ReparameterizationU.
 d = degree, u = uniformity, N = number of pieces, T = time (seconds).

Curve	d	Original u	ReparameterizationN($p, 1$)		ReparameterizationN($p, 3$)		ReparameterizationU($p, 0.9$)		
			u	T	u	T	u	N	T
C1	8	0.808	0.808	0.047	0.886	0.625	0.991	6	0.265
C2	12	0.960	0.960	0.016	1.000	0.203	0.960	1	0.016
C3	6	0.906	0.919	0.031	1.000	0.219	0.906	1	0.016
C4	6	0.796	0.796	0.031	0.911	0.437	0.996	8	0.250
C5	6	0.879	0.879	0.015	0.961	0.328	0.997	4	0.110
C6	50	0.647	0.706	1.422	0.973	8.203	0.971	5	10.578
C7	70	0.181	0.418	3.157	0.550	20.204	0.960	6	31.390
C8	100	0.184	0.184	6.188	0.989	119.141	0.970	2	95.828
C9	120	0.682	0.683	9.109	0.999	66.813	0.994	2	37.188
C10	150	0.253	0.479	29.765	-	> 3000	-	-	> 3000

Outline

1 Problem

2 Methods

3 Implementation

4 Examples and Experiments

5 Summary

Summary

- Introduce the definition of angular speed uniformity for parameterizations of plane curves;
- Prove the existence of arc-angle (maybe irrational) parameterizations;
- Propose two methods to improve the uniformity of angular speed;
- Present a software package for computing piecewise rational reparameterizations with improved uniformities.

Future Work

- Investigate efficient methods for computing a rational approximation of the arc-angle reparameterization for any given rational parameterization whose angular speed may become zero over $[0, 1]$;
- Generalize the existing methods to parametric space curves and parametric surfaces...

Thanks for your attention!