A Simple Quantifier-free Formula of Positive Semidefinite Cyclic Ternary Quartic Forms

Jingjun Han [1]

School of Mathematical Sciences, Peking University, Beijing, 100871, China

October 27, 2012

[1] hanjingjunfdfz@gmail.com

1. Shing-Tung Yau High School Mathematics Award
2. *Criterions on Equality of Symmetric Inequalities method*
3. A quantifier-free formula of cyclic ternary quartic forms

1. Research Background

2. Research Result
   - Main Result
   - Preliminaries
   - Lemma 6
   - Lemma 8
   - Theorem 9
   - Lemma 10

3. Examples

4. Acknowledgements

Research Background

1. A. Tarski, who gave a first quantifier elimination method for real closed fields in the 1930s

Research Background

1. A. Tarski, who gave a first quantifier elimination method for real closed fields in the 1930s
2. G. E. Collins, who introduced a so-called cylindrical algebraic decomposition (CAD) algorithm for QE problem in the 1970s

Research Background

1. A. Tarski, who gave a first quantifier elimination method for real closed fields in the 1930s
2. G. E. Collins, who introduced a so-called cylindrical algebraic decomposition (CAD) algorithm for QE problem in the 1970s
3. Many researchers have studied a special quantifier elimination problem

$$(\forall x \in \mathbb{R})(x^4 + px^2 + qx + r \geq (>)0)$$

Research Background

1. A. Tarski, who gave a first quantifier elimination method for real closed fields in the 1930s
2. G. E. Collins, who introduced a so-called cylindrical algebraic decomposition (CAD) algorithm for QE problem in the 1970s
3. Many researchers have studied a special quantifier elimination problem

$$(\forall x \in \mathbb{R})(x^4 + px^2 + qx + r \geq (>)0)$$

4. González-Vega etc. and Yang etc. developed theory on root classification of polynomials independently, Yang named the theory *complete discrimination systems*.

Research Background

Choi etc, Harris, Timofte, Yao etc. have studied quantifier-free formula of symmetric forms.

Choi etc.,1987

$x_i \in R_+, i = 1, 2, \ldots, n$, the degree of form $T(x_1, x_2, \ldots, x_n)$ equals three, then $T \geq 0 \iff T(1, 0, 0, \ldots, 0) \geq 0$ $T(1, 1, 0, 0, \ldots, 0) \geq 0$, . . . , $T(1, 1, 1, \ldots, 1, 0) \geq 0$; $T(1, 1, 1, \ldots, 1) \geq 0$.

Harris, 1999

$x, y, z \in R_+$, the degree of form $P(x, y, z)$ equals 4 or 5, then $P(x, y, z) \geq 0 \iff P(x, 1, 1) \geq 0, P(x, 1, 0) \geq 0$.

Yao etc., 2008

Give an algorithm to compute a quantifier free formula of the positive semidefinite symmetric forms of degree d with n variables in $\mathbb{R}^n$($d \leq 5$, n is given.)

The author developed the so-called *Criterions on Equality of Symmetric Inequalities method*, which can prove these results.

Question

How about cyclic forms?

Han, 2011

For cyclic form P with degree 3 of 3 variables, namely

$$P(a,b,c)=m(a^3+b^3+c^3)+n(a^2b+b^2c+c^2a)+s(ab^2+bc^2+ca^2)+3tabc\geq 0$$

$, \forall a,b,c\in R_+, P\geq 0 \Longleftrightarrow P(1,1,1)\geq 0\ ; [\forall a\geq 0, P(a,1,0)\geq 0].$

The author once studied cyclic ternary quartic forms, namely

$$(\forall x, y, z \in \mathbb{R})[F(x, y, z) = \sum_{cyc} x^4 + k \sum_{cyc} x^2y^2 + l \sum_{cyc} x^2yz + m \sum_{cyc} x^3y + n \sum_{cyc} xy^3 \geq 0],$$

but not obtained a quantifier-free formula.

Difficulty

Two quantifiers (Due to homogenous).
Difficult to obtain automatically by previous methods or quantifier elimination tools.

Research Method

1. Apply *Criterions on Equality of Symmetric Inequalities method* to reduce number of quantifiers to one
2. Apply the theory of *complete discrimination systems* and function `RealTriangularize` in Maple15 (or other tools) for solving the reduced case (QEPCAD, Discover).

Research Result

$$F(x,y,z)=\sum_{cyc}x^4+k\sum_{cyc}x^2y^2+l\sum_{cyc}x^2yz+m\sum_{cyc}x^3y+n\sum_{cyc}xy^3,$$

then

$$(\forall x,y,z\in\mathbb{R})\qquad [F(x,y,z)\geq 0]$$

is equivalent to

$$\begin{aligned}&\vee(g_4=0\wedge f_2=0\wedge((g_1=0\wedge m\geq 1\wedge m\leq 4)\vee(g_1>0\wedge g_2\geq 0)\vee\\ &\quad(g_1>0\wedge g_3\geq 0)))\\ &\vee(g_4^2+f_2^2>0\wedge f_1>0\wedge f_3=0\wedge f_4\geq 0)\\ &\vee(g_4^2+f_2^2>0\wedge f_1>0\wedge f_3>0\wedge\\ &\quad((f_5>0\wedge(f_6<0\vee f_7<0))\vee(f_5=0\wedge f_7<0)))\end{aligned}$$

where

$$
\begin{aligned}
f_1 :=& 2 + k - m - n, \\
f_2 :=& 4k + m + n - 8 - 2l, \\
f_3 :=& 1 + k + m + n + l, \\
f_4 :=& 3(1 + k) - m^2 - n^2 - mn, \\
f_5 :=& -4k^3m^2 - 4k^3n^2 - 4k^2lm^2 + 4k^2lmn - 4k^2ln^2 \\
& - kl^2m^2 + 4kl^2mn - kl^2n^2 + 8klm^3 + 6klm^2n + 6klmn^2 \\
& + 8kln^3 - 2km^4 + 10km^3n - 3km^2n^2 + 10kmn^3 - 2kn^4 \\
& + l^3mn - 9l^2m^2n - 9l^2mn^2 + lm^4 + 13lm^3n - 3lm^2n^2 \\
& + 13lmn^3 + ln^4 - 7m^5 - 8m^4n - 16m^3n^2 - 16m^2n^3 - 8mn^4
\end{aligned}
$$

$$
\begin{aligned}
&-7n^5+16k^4+16k^3l-32k^2lm-32k^2ln+12k^2m^2\\
&-48k^2mn+12k^2n^2-4kl^3+4kl^2m+4kl^2n-12klm^2\\
&-60klmn-12kln^2+40km^3+48km^2n+48kmn^2+40kn^3\\
&-l^4+10l^3m+10l^3n-21l^2m^2+12l^2mn-21l^2n^2\\
&+10lm^3+48lm^2n+48lmn^2+10ln^3-17m^4-14m^3n\\
&-21m^2n^2-14mn^3-17n^4-16k^3+32k^2l-48k^2m\\
&-48k^2n+80kl^2-48klm-48kln+96km^2+48kmn+96kn^2\\
&-24l^3-24l^2m-24l^2n+24lm^2-24lmn+24ln^2-16m^3\\
&-48m^2n-48mn^2-16n^3-96k^2-64kl+64km+64kn+96l^2\\
&-32lm-32ln-16m^2-32mn-16n^2+64k-128l+64m+64n+128,\\
f_6 :=&4k^2+2kl-4km-4kn+l^2-7lm-7ln+13m^2-mn+13n^2\\
&-40k+20l+8m+8n-32,
\end{aligned}
$$

$$
\begin{aligned}
f_7 :=& -768 + 352k^2 - 332l^2 + 180n^2 + 180m^2 + 56k^3 - 8k^4 \\
& + 14l^3 + 132n^3 + 132m^3 + 42n^4 + 42m^4 - 480k - 60lmn - 192n \\
& + 32klmn - 192m + 912l + l^4 - 354kmn + 158kln + 158klm + 26k^2mn \\
& - 11kln^2 + 22k^2lm + 22k^2ln - 45kmn^2 - 90lm^2n - 45km^2n \\
& - 11klm^2 + 23l^2mn - 90lmn^2 + kl^2m + kl^2n + 36mn - 480km + 592kl \\
& - 480kn - 60lm - 60ln + 8k^3m + 8k^3n - 20k^2l + 32k^2n + 32k^2m \\
& - 12k^3l + 234mn^2 + 234m^2n - 192ln^2 - 258kn^2 - 192lm^2 - 258km^2 \\
& + 116l^2m + 116l^2n + 87m^3n + 87mn^3 - 15kn^3 + 90m^2n^2 - 30ln^3 \\
& - 15km^3 - 30lm^3 + 25l^2m^2 + 25l^2n^2 - 14k^2m^2 - 14k^2n^2 \\
& - 146kl^2 - 10l^3m - 10l^3n - 2k^2l^2 + 3kl^3, \\
g_1 :=& k - 2m + 2, g_2 := 4k - m^2 - 8, g_3 := 8 + m - 2k, g_4 = m - n.
\end{aligned}
$$

By Theorem 9, it suffices to find a quantifier-free formula of

$$(\forall t \in \mathbb{R})[g(t) := 3(2+k-m-n)t^4 + 3(4+m+n-l)t^2 + k + 1 + m + n + l - \sqrt{27(m-n)^2 + (4k+m+n-8-2l)^2}\,t^3 \geq 0].$$

Case 1 $\sqrt{27(m-n)^2 + (4k+m+n-8-2l)^2} = 0$, that is $m = n$ and $4k + m + n - 8 - 2l = 0$. Hence

$$\begin{aligned} g(t) =& 3(2+k-2m)t^4 + 3(4+2m-l)t^2 + k + 1 + 2m + l \\ =& 3(2+k-2m)t^4 + 3(8+m-2k)t^2 + 3(k+m-1). \end{aligned}$$

If $2 + k - 2m = 0$, then

$$\forall t \in \mathbb{R} \quad g(t) \geq 0 \Longleftrightarrow 1 \leq m \leq 4.$$

If $2 + k - 2m > 0$, then

$$\forall t \in \mathbb{R} \quad g(t) \geq 0 \Longleftrightarrow (g_1 > 0 \wedge g_2 \geq 0) \vee (g_1 > 0 \wedge g_3 \geq 0).$$

Case 2 $\sqrt{27(m-n)^2+(4k+m+n-8-2l)^2} \neq 0$ and $1+k+m+n+l=0$. In this case, it is easy to show that $2+k-m-n>0$. Thus,

$$
\begin{aligned}
&\forall t \in \mathbb{R}, \quad g(t) \geq 0 \\
\Longleftrightarrow &\forall t \in \mathbb{R}, \quad 3(2+k-m-n)t^2+3(4+m+n-l) \\
&\quad -\sqrt{27(m-n)^2+(4k+m+n-8-2l)^2}\,t \geq 0 \\
\Longleftrightarrow &27(m-n)^2+(4k+m+n-8-2l)^2 \\
&\leq 36(2+k-m-n)(4+m+n-l) \\
\Longleftrightarrow &3(1+k) \geq m^2+n^2+mn.
\end{aligned}
$$

Case 3 $\sqrt{27(m-n)^2+(4k+m+n-8-2l)^2} \neq 0$ and $1+k+m+n+l \neq 0$. In this case, by Lemma 10, we know that for all $x \in \mathbb{R}$, $g \geq 0$ holds if and only if

$$f_1 > 0 \wedge f_3 > 0 \wedge ((f_5 > 0 \wedge (f_6 \leq 0 \vee f_7 \leq 0)) \vee (f_5 = 0 \wedge f_7 < 0)).$$

The theorem is proved. ☺

Given a polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

we write the derivative of $f(x)$ as

$$f'(x) = 0 \cdot x^n + na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1}.$$

Discriminant Matrix

The Sylvester matrix of $f(x)$ and $f'(x)$

$$\begin{vmatrix} a_0 & a_1 & a_2 & \ldots & a_n & & & \\ 0 & na_0 & (n-1)a_1 & \ldots & a_{n-1} & & & \\ & a_0 & a_1 & \ldots & a_{n-1} & a_n & & \\ & 0 & na_0 & \ldots & 2a_{n-1} & a_n & & \\ & \vdots & \vdots & \ddots & \vdots & \vdots & & \\ & & & & a_0 & a_1 & \ldots & a_n \\ & & & & 0 & na_0 & \ldots & a_{n-1} \end{vmatrix}$$

is called the *discrimination matrix* of $f(x)$, and denoted by $Discr(f)$.

Discriminant Sequence

Denoted by D_k the determinant of the submatrix of $Discr(f)$ formed by the first $2k$ rows and the first $2k$ columns. For $k = 1, \dots, n$, we call the n-tuple

$$\{D_1(f), D_2(f), \dots, D_n(f)\}$$

the *discriminant sequence* of polynomial $f(x)$.

Sign list

We call list

$$[\mathrm{sign}(D_1(f)), \mathrm{sign}(D_2(f)), \dots, \mathrm{sign}(D_n(f))]$$

the *sign list* of the *discriminant sequence* $\{D_1(f), D_2(f), \dots, D_n(f)\}$

Revised sign list

Given a *sign list*

$$[s_1, s_2, \ldots, s_n],$$

we construct a new list

$$[\epsilon_1, \epsilon_2, \ldots, \epsilon_n]$$

as follows (which is called the *revised sign list*): if $[s_1, s_2, \ldots, s_n]$ is a section of the given list, where $s_i \neq 0$, $s_{i+1} = s_{i+2} = \ldots = s_{i+j-1} = 0$, $s_{i+j} \neq 0$, then we replace the subsection

$$[s_{i+1}, s_{i+2}, \ldots, s_{i+j-1}]$$

by

$$[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \ldots],$$

i.e., let

$$\epsilon_{i+r} = (-1)^{[\frac{r+1}{2}]} \cdot s_i$$

for $r = 1, 2, \ldots, j-1$. Otherwise, let $\epsilon_k = s_k$ i.e., no change for other terms.

Theorem 1 González-Vega etc. 1989, Yang etc. 1996

Given a polynomial with real coefficients, $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$. If the number of the sign changes of the *revised sign list* of

$$\{D_1(f), D_2(f), \ldots, D_n(f)\}$$

is v, then the number of the pairs of distinct conjugate imaginary root of $f(x)$ equals v. Furthermore, if the number of non-vanishing members of the *revised sign list* is l, then the number of the distinct real roots of $f(x)$ equals $l - 2v$.

Research Result Lemma 6

Lemma 6

Let $x, y, z \in \mathbb{C}$, $x + y + z = 1$ and $xy + yz + zx, xyz \in \mathbb{R}$. The necessary and sufficient condition of $x, y, z \in \mathbb{R}$ is $xyz \in [r_1, r_2]$, where

$$r_1 = \frac{1}{27}(1 - 3t^2 - 2t^3), r_2 = \frac{1}{27}(1 - 3t^2 + 2t^3)$$

and $t = \sqrt{1 - 3(xy + yz + zx)} \geq 0$.

Remark

This Lemma implies that if $x, y, z \in \mathbb{R}$ and $x + y + z = 1$, then $\sqrt{1 - 3(xy + yz + zx)} = t \geq 0$, and the range of xyz is $[r_1, r_2]$.

Research Result Lemma 6

Consider the polynomial

$$f(X) = X^3 - (x+y+z)X^2 + (xy+yz+zx)X - xyz,$$

according to the theory of *complete discrimination systems*, the equation $f(X) = 0$ has three real roots if and only if

$$D_3(f) \geq 0 \wedge D_2(f) \geq 0,$$

where

$$D_2(f) = (x+y+z)^2 - 3(xy+yz+zx) = 1 - 3(xy+yz+zx) = t^2,$$

$$D_3(f) = (x-y)^2(y-z)^2(z-x)^2 = \frac{1}{27}(4D_2(f)^3 - (3D_2(f) - 1 + 27xyz)^2).$$

Using the substitution $t = \sqrt{D_2(f)}$ and $xyz = r$,

$$x, y, z \in \mathbb{R} \iff (x-y)^2(y-z)^2(z-x)^2 \geq 0 \wedge (x+y+z)^2 \geq 3xy+3yz+3zx,$$

$$\iff 4t^6 - (3t^2 - 1 + 27r)^2 \geq 0 \wedge t \geq 0$$

$$\iff \frac{1}{27}(1 - 3t^2 - 2t^3) \leq r \leq \frac{1}{27}(1 - 3t^2 + 2t^3) \wedge t \geq 0.$$

Research Result Lemma 8

Lemma 8

The inequality $F(x,y,z) \geq 0$ holds for any $x,y,z \in \mathbb{R}$ if and only if

$$2\sum_{cyc} x^4 + 2k\sum_{cyc} x^2y^2 + 2l\sum_{cyc} x^2yz + (n+m)\sum_{cyc} x^3y + (m+n)\sum_{cyc} xy^3$$
$$\geq |(m-n)(x+y+z)(x-y)(y-z)(z-x)|$$

holds for all $x,y,z \in \mathbb{R}$.

Research Result Lemma 8

$F(x,y,z) \geq 0$ is equivalent to: for all $x,y,z \in \mathbb{R}$,

$$2\sum_{cyc} x^4 + 2k\sum_{cyc} x^2y^2 + 2l\sum_{cyc} x^2yz + (n+m)\sum_{cyc} x^3y + (m+n)\sum_{cyc} xy^3$$
$$\geq (m-n)(x+y+z)(x-y)(y-z)(z-x).$$

On the other hand, $F(x,y,z) \geq 0 \Longrightarrow F(x,z,y) \geq 0$,

$$2\sum_{cyc} x^4 + 2k\sum_{cyc} x^2y^2 + 2l\sum_{cyc} x^2yz + (n+m)\sum_{cyc} x^3y + (m+n)\sum_{cyc} xy^3$$
$$\geq (n-m)(x+y+z)(x-y)(y-z)(z-x)$$

Research Result Theorem 9

Theorem 9

The positive semidefinite cyclic ternary quartic form

$$\forall x, y, z \in \mathbb{R} \qquad F(x, y, z) \geq 0$$

holds if and only if the following inequality holds.

$$(\forall t \in \mathbb{R})[g(t) := 3(2 + k - m - n)t^4 + 3(4 + m + n - l)t^2 + k + 1 + m + n + l - \sqrt{27(m-n)^2 + (4k + m + n - 8 - 2l)^2}\,t^3 \geq 0].$$

Research Result Theorem 9

- $(\forall t \geq 0)[g(t) \geq 0] \iff (\forall t \in \mathbb{R})[g(t) \geq 0]$
- $(\forall x, y, z \in \mathbb{R})[F(x, y, z) \geq 0] \iff (\forall t \geq 0)[g(t) \geq 0]$

According to Lemma 8, $F(x, y, z) \geq 0 \iff$

$$2\sum_{cyc} x^4 + 2k\sum_{cyc} x^2y^2 + 2l\sum_{cyc} x^2yz + (n+m)\sum_{cyc} x^3y + (m+n)\sum_{cyc} xy^3$$
$$-|(m-n)(x+y+z)(x-y)(y-z)(z-x)| \geq 0$$

for all $x, y, z \in \mathbb{R}$. Denote it as $G(x, y, z) \geq 0$.

Research Result Theorem 9

Substituting $x+y+z, xy+yz+zx, xyz$ with p, q, r, we have

$$\sum_{cyc} x^4 = p^4 - 4p^2q + 2q^2 + 4pr \qquad \sum_{cyc} x^2y^2 = q^2 - 2pr$$

$$\sum_{cyc} x^2yz = pr \qquad \sum_{cyc} x^3y + xy^3 = q(p^2 - 2q) - pr$$

$$|(x-y)(y-z)(z-x)| = \sqrt{(x-y)^2(y-z)^2(z-x)^2}$$

$$= \sqrt{\frac{4(p^2-3q)^3 - (2p^3 - 9pq + 27r)^2}{27}}.$$

$$\begin{aligned} G(x,y,z) = &2p^4 + np^2q - 8p^2q + mp^2q + 2kq^2 - 2nq^2 - 2mq^2 \\ &+ 4q^2 + 2lpr + 8pr - npr - mpr - 4kpr \\ &- |m-n|p\sqrt{\frac{4(p^2-3q)^3 - (2p^3 - 9pq + 27r)^2}{27}} \geq 0 \end{aligned}$$

Research Result Theorem 9

Sufficiency.
If $p = 0$, then the inequality $G(x, y, z) \geq 0$ becomes

$$2(2 + k - m - n)q^2 \geq 0.$$

$(2 + k - m - n)$ is *leading coefficient* of $g(t) \Longrightarrow 2 + k - m - n \geq 0$
If $p \neq 0$, since the inequality is homogenous, assume $p = 1$.

$$(x + y + z)^2 \geq 3(xy + yz + zx),$$

thus we have $q \leq \frac{1}{3}$. Using the substitution $t = \sqrt{1 - 3q} \geq 0$, the inequality $G(x, y, z) \geq 0$ is equivalent to

$$\begin{aligned} &2(2 + k - m - n)t^4 + (16 - 4k + m + n)t^2 - 2 + 2k + m + n+ \\ &9(8 - 4k + 2l - m - n)r \geq \sqrt{3}|m - n|\sqrt{4t^6 - (3t^2 - 1 + 27r)^2}, \end{aligned} \tag{1}$$

where $t \geq 0$, $r \in [r_1, r_2]$.

Research Result Theorem 9

Since $\frac{2}{3}g(t) \geq 0$ is equivalent to

$$2(2+k-m-n)t^4 + (16-4k+m+n)t^2 - 2 + 2k + m + n$$
$$+ 9(8-4k+2l-m-n)r \geq \frac{2\sqrt{27(m-n)^2 + (8-4k+2l-m-n)^2}t^3}{3}$$
$$+ \frac{(8-4k+2l-m-n)(3t^2-1+27r)}{3},$$

thus, in order to prove $G(x,y,z) \geq 0$, it is sufficient to prove that

$$\begin{aligned} &\sqrt{3}|m-n|\sqrt{4t^6-(3t^2-1+27r)^2} \\ \leq &\frac{2\sqrt{27(m-n)^2+(8-4k+2l-m-n)^2}t^3}{3} \\ &+ \frac{(8-4k+2l-m-n)(3t^2-1+27r)}{3} \end{aligned} \tag{2}$$

Square both sides and collect terms, the sufficiency is proved.

$$H^2(r) = (\frac{2(8-4k+2l-m-n)}{3}t^3$$

Research Result Theorem 9

Necessity. $G(x,y,z) \geq 0 \Longrightarrow (\forall t \geq 0)[g(t) \geq 0]$. For any $t \geq 0$, if there exist $x, y, z \in \mathbb{R}$ such that $H(r) = 0$, $x+y+z = 1$ and $1-3(xy+yz+zx) = t^2$, then the equation of inequality (4) could be attained. Choosing such $x, y, z \in \mathbb{R}$, inequality (1) becomes

$$
\begin{aligned}
&2(2+k-m-n)t^4 + (16-4k+m+n)t^2 - 2 + 2k + m + n \\
&+ 9(8-4k+2l-m-n)r \geq \frac{2\sqrt{27(m-n)^2 + (8-4k+2l-m-n)^2}t^3}{3} \\
&+ \frac{(8-4k+2l-m-n)(3t^2-1+27r)}{3},
\end{aligned}
$$

$\Longleftrightarrow (\forall t \geq 0)[g(t) \geq 0]$. Suffices to show that there exist such $x, y, z \in \mathbb{R}$.

Research Result Theorem 9

Notice that

$$
\begin{aligned}
&H(r_1)H(r_2)\\
=&\left(\frac{2(8-4k+2l-m-n)}{3}t^3-2t^3\sqrt{3(m-n)^2+\frac{(8-4k+2l-m-n)^2}{9}}\right)\\
&\left(\frac{2(8-4k+2l-m-n)}{3}t^3+2t^3\sqrt{3(m-n)^2+\frac{(8-4k+2l-m-n)^2}{9}}\right)\\
=&-12t^6(m-n)^2\leq 0,
\end{aligned}
$$

where

$$r_1=\frac{1}{27}(1-3t^2-2t^3), r_2=\frac{1}{27}(1-3t^2+2t^3).$$

Therefore, for any given $t=\sqrt{1-3(xy+yz+zx)}\geq 0$, there exists $r_0\in[r_1,r_2]$, such that $H(r_0)=0$. By Lemma 6, such $x,y,z\in\mathbb{R}$ exist and we prove the necessity.

Theorem 9, Motivation

$$
\begin{aligned}
&F(x,y,z) \geq 0 \Longleftrightarrow G(x,y,z) \geq 0 \Longleftrightarrow \\
&2(2+k-m-n)t^4 + (16-4k+m+n)t^2 - 2 + 2k + m + n + \qquad (3)\\
&9(8-4k+2l-m-n)r \geq \sqrt{3}|m-n|\sqrt{4t^6-(3t^2-1+27r)^2},
\end{aligned}
$$

We want to eliminate r, notice that

$$A\sqrt{D^2-B^2} \leq \sqrt{A^2+C^2}D + CB$$

$$
\begin{aligned}
&\sqrt{3}|m-n|\sqrt{4t^6-(3t^2-1+27r)^2}\\
\leq &\frac{2\sqrt{27(m-n)^2+(8-4k+2l-m-n)^2}t^3}{3} \qquad (4)\\
&+\frac{(8-4k+2l-m-n)(3t^2-1+27r)}{3}
\end{aligned}
$$

there exists $t \geq 0$, $r \in [r_1, r_2]$, such that the equality of above inequality holds.

Thus,

$$g(t) \geq 0 \Longleftrightarrow G(x,y,z) \geq 0.$$

Research Result Lemma 10

Let $a_0 > 0$, $a_4 > 0$, $a_1 \neq 0$, $a_1, a_2 \in \mathbb{R}$, we consider the following polynomial

$$f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_4.$$

The *discriminant sequence* of $f(x)$ is

$$D_f = [D_1(f), D_2(f), D_3(f), D_4(f)],$$

where

$$\begin{aligned} D_1(f) =& {a_0}^2, \\ D_2(f) =& -8a_0^3a_2 + 3a_1^2a_0^2, \\ D_3(f) =& -4a_0^3a_2^3 + 16a_0^4a_2a_4 + a_0^2a_1^2a_2^2 - 6a_0^3a_1^2a_4, \\ D_4(f) =& -27a_0^2a_1^4a_4^2 + 16a_0^3a_2^4a_4 - 128a_0^4a_2^2a_4^2 - \\ & 4a_0^2a_1^2a_2^3a_4 + 144a_0^3a_2a_1^2a_4^2 + 256a_0^5a_4^3. \end{aligned}$$

For all $x \in \mathbb{R}$, $f(x) \geq 0$ holds if and only if one of the following cases holds,

$$(1) D_4(f) > 0 \wedge (D_2(f) < 0 \vee D_3(f) < 0),$$

$$(2) D_4(f) = 0, D_3(f) < 0.$$

Research Result Lemma 10

$\Longrightarrow$: If $f(x) \geq 0$ holds for all $x \in \mathbb{R}$, then the number of distinct real roots of $f(x)$ is less than 2. If it equals 2, then the roots of $f(x)$ are all real. If it equals 0, then $f(x)$ has no real root.

- $D_4(f) < 0$ (no solution)
- $D_4(f) > 0 \Longrightarrow D_2(f) < 0$ or $D_3(f) < 0$
- $D_4(f) > 0$ and $D_2(f) = 0$
- $D_4(f) = 0$ and $D_3(f) \geq 0$ (no solution)

Research Result Lemma 10

If $D_4(f) < 0$ and $D_2(f) > 0$, then the number of non-vanishing members of *revised sign list*, l, equals 4. Since $D_4(f)D_2(f) < 0$, then the number of the sign changes of *revised sign list*, v, equals 1, thus $l - 2v = 2$. The number of distinct real roots of $f(x)$ equals two and the number of the pairs of distinct conjugate imaginary root of $f(x)$ $v = 1$, which is impossible. Using function `RealTriangularize`, we can prove that the semi-algebraic system $a_4 > 0, D_4(f) < 0, D_2(f) \leq 0$ has no real solution. Therefore, $D_4(f) \geq 0$. Since $D_1(f) \geq 0$, the number of the sign changes of *revised sign list* $v \leq 2$.

Research Result Lemma 10

If $D_4(f) > 0$, thus $l = 4$. Notice that the number of real roots of $f(x)$, namely $l - 2v \leq 2$, so $v \geq 1$, from which, we get

$$D_2(f) \leq 0 \vee D_3(f) \leq 0.$$

Using function `RealTriangularize`, we can prove that both the semi-algebraic system $a_4 > 0, a_0 > 0, D_4(f) > 0, D_2(f) \geq 0, D_3(f) = 0, a_1 \neq 0$ and the semi-algebraic system $a_4 > 0, a_0 > 0, D_4(f) > 0, D_3(f) \geq 0, D_2(f) = 0, a_1 \neq 0$ have no real solution. Hence, if $D_4(f) > 0$ and $D_2(f) = 0$, then $D_3(f) < 0$; if $D_4(f) > 0$ and $D_3(f) = 0$, then $D_2(f) < 0$. Thus, when $D_4(f) > 0$, either $D_2(f) < 0$ or $D_3(f) < 0$ holds.

Research Result Lemma 10

If $D_4(f) = 0$ and $D_3(f) > 0$, then $l = 3$. The number of sign changes of *revised sign list* v equals either 2 or 0. From $0 \leq l - 2v \leq 2$, we have $v = 1$, which leads to contradiction. That implies if $D_4(f) = 0$, then $D_3(f) \leq 0$. Using function `RealTriangularize`, we can prove that the semi-algebraic system $a_4 > 0, a_0 > 0, D_4(f) = 0, D_3(f) = 0, a_1 \neq 0$ has no real solution. Hence, when $D_4(f) = 0$, we have $D_3(f) < 0$.

Research Result Lemma 10

$\Longleftarrow$: If $D_4(f) > 0 \wedge (D_2(f) < 0 \vee D_3(f) < 0)$, then the number of sign changes of *revised sign list* $v = 2$, so the number of distinct real roots of $f(x)$, $l - 2v$, equals 0, which means for any $x \in \mathbb{R}$, $f(x) > 0$.
If $D_4(f) = 0$ and $D_3(f) < 0$, then $l = 3$, the number of the sign changes of *revised sign list* $v = 2$. Thus, the number of distinct real roots of $f(x)$, $l - 2v$, equals 1, and the number of the pairs of distinct conjugate imaginary root of $f(x)$, v, equals 1, so f has a real root with multiplicity two, which means for any $x \in \mathbb{R}$, $f(x) \geq 0$.

Some Examples

Acknowledgements

This research was partly supported by the Headmaster's Scholarship for Undergraduate Students of Peking University, NSFC-11271034 and the project SYSKF1207 from ISCAS. The author would like to thank the anonymous referees for their valuable comments on a previous version of this paper.

Thank you !

For any given $t \geq 0$ and $r \in [r_1, r_2]$ there exist $x, y, z \in \mathbb{R}$ such that $x + y + z = 1$, $xy + yz + zx = \frac{1-t^2}{3}$ and $xyz = r$.