

Numerical Reparametrization of Rational Parametric Plane Curves

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Problem

Proper reparametrization is a basic simplifying process for rational parameterized curves. There are complete results proposed for the curves with exact coefficients but few papers discuss the situations with numerical coefficients.

We focus on the numerical problem since it has practical background.



Let \mathbb{C} the field of the complex numbers, and \mathcal{C} a rational plane algebraic curve over \mathbb{C} . A parametrization \mathcal{P} of \mathcal{C} is proper if and only if the map

$$\mathcal{P} : \mathbb{C} \longrightarrow \mathcal{C} \subset \mathbb{C}^2; t \longmapsto \mathcal{P}(t).$$

is birational.

If all but finitely many points on \mathcal{C} are generated by k parameter values, then $\text{index}(\mathcal{P}(t)) = k$ is the improper index of \mathcal{C} .



Example 1

$x = \frac{2t}{t^2+1}$, $y = \frac{t^2-1}{t^2+1}$ is a *proper* parametrization of the unit circle $x^2 + y^2 = 1$, while $x = \frac{2t^2}{t^4+1}$, $y = \frac{t^4-1}{t^4+1}$ is an *improper* parametrization of the same circle, since any point (x, y) of the circle has two corresponding parameters $t = \pm\sqrt{\frac{x}{1-y}}$.

For simplification, it is important to check the properness of a parametrization and find the proper reparametrization if it is improper.



Symbolic proper reparametrization algorithm

Algorithm 1 (Exists for Symbolic cases)

Given a rational affine parametrization

$\mathcal{P}(t) = (p_{1,1}(t)/p_{1,2}(t), p_{2,1}(t)/p_{2,2}(t))$, in reduced form, of a plane algebraic curve \mathcal{C} , the algorithm computes a rational proper parametrization $\mathcal{Q}(s)$ of \mathcal{C} , and a rational function $R(t)$ such that $\mathcal{P}(t) = \mathcal{Q}(R(t))$.



Approximate proper re-parametrization

In design of engineering and computer aided design, people often obtain rational parametrizations with float coefficients with errors. A perturbed improper unit circle is

$$\begin{aligned} x &= \frac{1.999t^2 + 3.999t + 2.005 - 0.003t^4 + 0.001t^3}{2.005 + 0.998t^4 + 4.002t^3 + 6.004t^2 + 3.997t}, \\ y &= \frac{0.001 - 0.998t^4 - 4.003t^3 - 5.996t^2 - 4.005t}{2.005 + 0.998t^4 + 4.002t^3 + 6.004t^2 + 3.997t}. \end{aligned} \quad (1)$$



It is a curve with degree four in precise consideration. However, in the neighbor region of a generic point, there is an another part of the curve passing through. In other words, the curve is approximate diplex(two to one), or, called *approximate improper*.

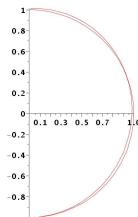


Figure 1 : A numerical curve

It is naturally to find a single curve(approximate proper) to replace the origin curve.



In 1986, T.W. Sederberg gave a heuristic algorithm to find a reparametrization of numerical improper curves. No more detailed discussions were proposed. And there are few papers discussed this problem.

Hence, we try to

- Define the approximate improper index,
- Compute the approximate improper index,
- Compute the approximate proper reparametrization,
- Estimate the error of the origin curve and the reparameterized one



Some notions

$\mathbb{F} = \overline{\mathbb{C}(s)}$ the algebraic closure of $\mathbb{C}(s)$.

For a given tolerance $\epsilon > 0$, and polynomials $A, B \in \mathbb{C}[t, s] \setminus \mathbb{C}$, we say that $A \approx_{\epsilon} B$, if $A(t, s) = B(t, s) + U(t, s)$, $U \in \mathbb{C}[t, s]$, where $\|U\| \leq \epsilon \|A\|$, and $\|\cdot\|$ denotes the infinity norm.

$$\mathcal{P}(t) = \left(\frac{p_{1,1}(t)}{p_{1,2}(t)}, \frac{p_{2,1}(t)}{p_{2,2}(t)} \right) \in \mathbb{C}(t)^2, \quad \epsilon \gcd(p_{j,1}, p_{j,2}) = 1, \quad j = 1, 2$$

be a rational parametrization of a given plane algebraic curve \mathcal{C} .

$$\mathcal{Q}(s) = \left(\frac{q_{1,1}(s)}{q_{1,2}(s)}, \frac{q_{2,1}(s)}{q_{2,2}(s)} \right) \in \mathbb{C}(s)^2, \quad \epsilon \gcd(q_{j,1}, q_{j,2}) = 1, \quad j = 1, 2$$

also a rational parametrization of a plane curve.



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Let

$$S_{\epsilon}^{\mathcal{P}\mathcal{Q}}(t, s) = \epsilon \gcd(H_1^{\mathcal{P}\mathcal{Q}}, H_2^{\mathcal{P}\mathcal{Q}})$$

where

$$H_j^{\mathcal{P}\mathcal{Q}}(t, s) = p_{j,1}(t)q_{j,2}(s) - q_{j,1}(s)p_{j,2}(t), j = 1, 2$$

In these conditions, we say that $\mathcal{P}(t) \sim_{\epsilon} \mathcal{Q}(s)$ if $S_{\epsilon}^{\mathcal{P}\mathcal{Q}}(t, s) \approx_{\epsilon} 0$,
for $t \in \mathbb{F}$.



The *approximate improper index* is generalized from the concept of symbolic improper index. In geometric view, it is the number of times of \mathcal{P} passing by a neighborhood of a generic point at the given plane curve. We denote it by $\epsilon\text{index}(\mathcal{P})$.

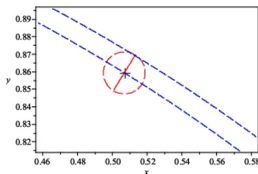


Figure 2 : Neighborhood to a generic point



Taking into account the intuitive idea, we observe that one may compute the approximate index by finding the approximate common solutions, $t \in \mathbb{F}$, of $H_1^{\mathcal{PP}}(t, s) = 0$ and $H_2^{\mathcal{PP}}(t, s) = 0$.

To simplify the computation, we can fix $s = s_0 \in \mathbb{C}$ as a specialization and find the approximate common solutions for two univariate polynomials $H_1^{\mathcal{PP}}(t, s_0) = 0$ and $H_2^{\mathcal{PP}}(t, s_0) = 0$.

However, it is possible that the number of approximate common solutions may be greater than the approximate index for some s_0 . The situation could happen at the singular points.



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Approximate improper index

Definition 1

We define the approximate improper index of \mathcal{P} , $\epsilon\text{index}(\mathcal{P})$, as

$$\epsilon\text{index}(\mathcal{P}) = \min_{\substack{s_0 \in \mathbb{C} \\ p_{j,2}(s_0) \neq 0, j=1,2}} \#\{t \in \mathbb{C} \mid H_1^{\mathcal{P}\mathcal{P}}(t, s_0) \approx_{\epsilon} 0, H_2^{\mathcal{P}\mathcal{P}}(t, s_0) \approx_{\epsilon} 0\}.$$

\mathcal{P} is said to be approximate improper or ϵ -improper if $\epsilon\text{index}(\mathcal{P}) > 1$. Otherwise, it is said to be approximate proper or ϵ -proper.



Taking into account that the behavior of the fibre change at the singular points, the number of s_0 is at least the number of singularities plus one. From the genus formula, one gets that a bound for the number of singularities is given by $(n-1)(n-2)$, where n is the degree of the input curve.

In practical computation, we can select some different s_0 randomly and count the number of approximate common solutions of $H_1^{\mathcal{PP}}(t, s_0) = 0$ and $H_2^{\mathcal{PP}}(t, s_0) = 0$. The minimal one is the approximate index.



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Proposition 1

It holds that

$$\epsilon\text{index}(\mathcal{P}) = \deg_t(S_\epsilon^{\mathcal{P}\mathcal{P}}).$$

In addition, $\epsilon\text{index}(\mathcal{P}) = 1$ if and only if $S_\epsilon^{\mathcal{P}\mathcal{P}}(t, s) \approx_\epsilon (t - s)$.

Remark 1

We observe that $S_\epsilon^{\mathcal{P}\mathcal{P}}$ is not unique but all of them have the same degree ($\epsilon\text{index}(\mathcal{P}) = \deg_t(S_\epsilon^{\mathcal{P}\mathcal{P}})$).

The approximate greatest common divisor, $S_\epsilon^{\mathcal{P}\mathcal{P}}(t, s_0)$ can be computed during the computation of $\epsilon\text{index}(\mathcal{P})$.



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Numerical and ϵ -proper reparametrization

Definition 2

Let $\mathcal{P}(t) = (p_1(t), p_2(t)) \in \mathbb{C}(t)^2$ be a rational parametrization of a curve \mathcal{C} . We say that another parametrization $\mathcal{Q}(s) = (q_1(s), q_2(s)) \in \mathbb{C}(s)^2$ is an ϵ -numerical reparametrization of $\mathcal{P}(t)$ if there exists $R(t) = M(t)/N(t) \in \mathbb{C}(t) \setminus \mathbb{C}$, with $\text{egcd}(M, N) = 1$, such that $\mathcal{P} \sim_{\epsilon} \mathcal{Q}(R)$. In addition, if $\epsilon\text{index}(\mathcal{Q}) = 1$, then we say that \mathcal{Q} is an ϵ -proper reparametrization of \mathcal{P} .



In these conditions of Definition 2, we have the following theorem.

Theorem 1

Let $\mathcal{Q}(s) \in \mathbb{C}(s)^2$ be ϵ -proper, and $R(t) = M(t)/N(t) \in \mathbb{C}(t) \setminus \mathbb{C}$, with $\epsilon \gcd(M, N) = 1$. Then, it holds that

$$S_{\epsilon}^{\mathcal{P}\mathcal{P}}(t, s) \approx_{\epsilon} M(t)N(s) - M(s)N(t).$$

In addition, $\epsilon \text{index}(\mathcal{P}) = \deg_t(S_{\epsilon}^{\mathcal{P}\mathcal{P}}) = \deg(R)$.



The following theorem shows an important relation between $\epsilon\text{index}(\mathcal{P})$ and $\epsilon\text{index}(\mathcal{Q})$.

Theorem 2

It holds that $\epsilon\text{index}(\mathcal{P}) = \epsilon\text{index}(\mathcal{Q}) \deg(R)$.

Corollary 1

The parametrization \mathcal{Q} is ϵ -proper if and only if $\deg_t(S_\epsilon^{\mathcal{P}\mathcal{P}}) = \deg(R)$.



One can get another corollary from Theorem 2

Corollary 2

Let $S_{\epsilon}^{\mathcal{P}\mathcal{P}}(t, s) \approx_{\epsilon} M(t)N(s) - M(s)N(t)$, $M(t), N(t) \in \mathbb{C}[t]$.
 Then \mathcal{Q} is ϵ -proper and $\mathcal{P} \sim_{\epsilon} \mathcal{Q}(R)$, where
 $R(t) = M(t)/N(t) \in \mathbb{C}(t) \setminus \mathbb{C}$.

Similar to the symbolic cases, we write

$M(t)N(s) - M(s)N(t) = C_m(t)s^m + C_{m-1}(t)s^{m-1} + \cdots + C_0(t)$
 and set $R(t) = \frac{C_{i_0}(t)}{C_{j_0}(t)}$, where C_{i_0}, C_{j_0} are not both constant and
 $\epsilon\gcd(C_{i_0}, C_{j_0}) = 1$.



The following theorem constructs the reparametrization $\mathcal{Q}(s)$.

Theorem 3

For $k = 1, 2$, let $L_k(s, x_k) = \text{Res}_t(G_k(t, x_k), sC_{j_0}(t) - C_{i_0}(t))$, where $G_k(t, x_k) = x_k p_{k,2}(t) - p_{k,1}(t)$. If for $k = 1, 2$,

$$L_k(s, x_k) = (x_k q_{k,2}(s) - q_{k,1}(s))^\ell + \epsilon^\ell W_k(s, x_k),$$

$W_k \in \mathbb{C}[s, x_k]$, $\|W_k\| \leq \|L_k\|$, where $\text{egcd}(q_{k,1}, q_{k,2}) = 1$, then $\mathcal{Q}(s) = \left(\frac{q_{1,1}(s)}{q_{1,2}(s)}, \frac{q_{2,1}(s)}{q_{2,2}(s)} \right)$ is an ϵ -numerical reparametrization of \mathcal{P} .

Remark 2

Note that, if in Theorem 3, one has that

$L_k(s, x_k) = (x_k q_{k,2}(s) - q_{k,1}(s))^\ell + \bar{\epsilon}^\ell W_k(s, x_k)$, then

$\mathcal{Q}(s) = \left(\frac{q_{1,1}(s)}{q_{1,2}(s)}, \frac{q_{2,1}(s)}{q_{2,2}(s)} \right)$ is an $\bar{\epsilon}$ -numerical reparametrization of \mathcal{P} .

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From Theorem 3, and using Corollary 2, we easily gets the following corollary.

Corollary 3

Let \mathcal{Q} be the ϵ -numerical reparametrization of \mathcal{P} computed in Theorem 3.

- ① It holds that \mathcal{Q} is ϵ -proper.
- ② It holds that $\deg(\mathcal{P}) = \deg(\mathcal{Q}) \deg(R)$.
- ③ $q_k(s) = q_{k,1}(s)/q_{k,2}(s)$ could be obtained by simplifying the rational function $\frac{\tilde{q}_{k,1}(s)}{\tilde{q}_{k,2}(s)} = \frac{-\text{coeff}(L_k, x_k, \ell-1)/\ell}{\text{coeff}(L_k, x_k, \ell)}$, $k = 1, 2$.



Consider the parametrization obtained in Corollary 3, precisely,
 $\tilde{\mathcal{Q}}(t) = (\tilde{q}_1, \tilde{q}_2) = \left(\frac{\tilde{q}_{1,1}}{\tilde{q}_{1,2}}, \frac{\tilde{q}_{2,1}}{\tilde{q}_{1,2}} \right)$. We observe that $\deg(\mathcal{P}) = \deg(\tilde{\mathcal{Q}})$
 and the simplification of $\tilde{\mathcal{Q}}$ provides the rational parametrization
 $\mathcal{Q}(s) = \left(\frac{q_{1,1}(s)}{q_{1,2}(s)}, \frac{q_{2,1}(s)}{q_{1,2}(s)} \right)$.

Theorem 4

Let \mathcal{C} be the curve parametrized by \mathcal{P} , and let $\tilde{\mathcal{D}}$ be the curve defined by $\tilde{\mathcal{Q}}$. It holds that the implicit equations defining the curves \mathcal{C} and $\tilde{\mathcal{D}}$ have the same homogeneous form of maximum degree, and hence both curves have the same points at infinity.



Finally, we consider the simplified \mathcal{Q} which is an ϵ reparametrization of \mathcal{P} .

Theorem 5

The following statements hold:

- ① *Let $I = (d_1, d_2) \subset \mathbb{R}$ and $d = \max\{|d_1|, |d_2|\}$. Let $M \in \mathbb{N}$ be such that for every $s_0 \in I$, it holds that $|q_{i,2}(R(s_0))| \geq M$, and $|p_{i,2}(s_0)| \geq M$ for $i = 1, 2$. Then, for every $s_0 \in I$,*

$$|p_i(s_0) - q_i(R(s_0))| \leq 1/M^2 \epsilon C, \quad i = 1, 2,$$

where $|\cdot|$ denotes the absolute value, and $C = \frac{d^{\deg(\mathcal{P})+1}}{(d-1)^{1/\ell}}, (d > 1); \frac{1}{(1-d)^{1/\ell}}, (d < 1); \ell^{1/\ell} \deg(\mathcal{P})^{1/\ell}, (d = 1)$.



Theorem 6 (5.continued)

② \mathcal{C} is contained in the offset region of \mathcal{D} at distance $2M^2\epsilon^{1/\ell}$. \mathcal{C} .

If Theorem 3 holds and then, \mathcal{Q} is an ϵ -proper reparametrization of \mathcal{P} (see Corollary 3).

If Remark 2 of Theorem 3 holds, and then \mathcal{Q} an $\bar{\epsilon}$ -proper reparametrization of \mathcal{P} . In this case, the formula obtained to the error bound in Theorem 5 is the same but it involves $\bar{\epsilon}$ instead of ϵ .



Algorithm 2

GIVEN a tolerance $\epsilon > 0$, and a rational parametrization $\mathcal{P}(t) = \left(\frac{p_{1,1}(t)}{p_{1,2}(t)}, \frac{p_{2,1}(t)}{p_{2,2}(t)} \right)$, $\epsilon \gcd(p_{i,1}, p_{i,2}) = 1, i = 1, 2$, of a plane algebraic curve \mathcal{C} , the algorithm

OUTPUTS a rational parametrization

$\mathcal{Q}(s) = \left(\frac{q_{1,1}(s)}{q_{1,2}(s)}, \frac{q_{2,1}(s)}{q_{2,2}(s)} \right)$, $\epsilon \gcd(q_{i,1}, q_{i,2}) = 1, i = 1, 2$ with

$\epsilon \text{index}(\mathcal{Q}) = 1$, and such that $\mathcal{P} \sim_{\epsilon} \mathcal{Q}(R)$, where $R(t) = \frac{M(t)}{N(t)}$, $\epsilon \gcd(M, N) = 1$.

\mathcal{Q} is an ϵ -proper reparametrization of \mathcal{P} (or \mathcal{Q} is an $\bar{\epsilon}$ -proper reparametrization of \mathcal{P})



Example 2

Let the tolerance $\epsilon = 0.2$, and the curve \mathcal{C} is the perturbed circle (see Figure 1). We compute the numerical reparametrization $\tilde{\mathcal{Q}}$ and the simplified one \mathcal{Q} . [Check the equality in Theorem 3](#), \mathcal{Q} is an ϵ -proper reparametrization of \mathcal{P} . $\mathcal{Q}(s) = \left(\frac{q_{1,1}(s)}{q_{1,2}(s)}, \frac{q_{2,1}(s)}{q_{2,2}(s)} \right) =$

$$\left(\frac{-0.00141685446739421312s^2 - 0.456393086560853146s + 0.2329464493}{0.469669800583654318s^2 - 0.47610245769809506s + 0.2350816567}, \frac{-1.87867573893595918s^2 + 1.89405343499205368s - 0.01488394423}{0.469669800583654318s^2 - 0.47610245769809506s + 0.2350816567} \right).$$



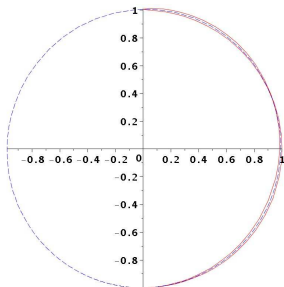


Figure 3 : Parametrization \mathcal{P} v.s. Parametrization $\tilde{\mathcal{Q}}$

They have the same homogeneous form of maximum degree of their implicit equations.



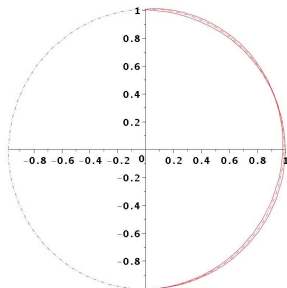


Figure 4 : Parametrization \mathcal{P} v.s. Parametrization \mathcal{Q}

The curve \mathcal{P} is contained in the offset region of \mathcal{Q} at distance 0.01313222334.



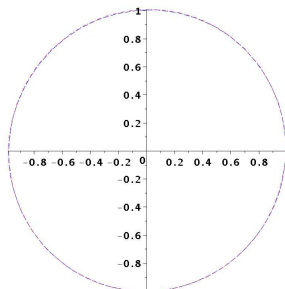


Figure 5 : Parametrization $\tilde{\mathcal{Q}}$ v.s. Parametrization \mathcal{Q}

\mathcal{Q} is simplified from $\tilde{\mathcal{Q}}$ by removing the approximate gcds from the numerators and denominators of $\tilde{\mathcal{Q}}$.



Conclusion

For a given numerical curve, we can determine whether it is approximate improper with respect to a given precision and, in the affirmative case, an ϵ -proper reparametrization can be found. More important, the input curve lies on the certain offset region of the reparameterized one.



Thanks !

