Towards Guaranteed Accuracy Computations in Control

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Asian Symposium on Computer Mathematics 2012

Organized Session: On The Latest Progress In Verified Computation

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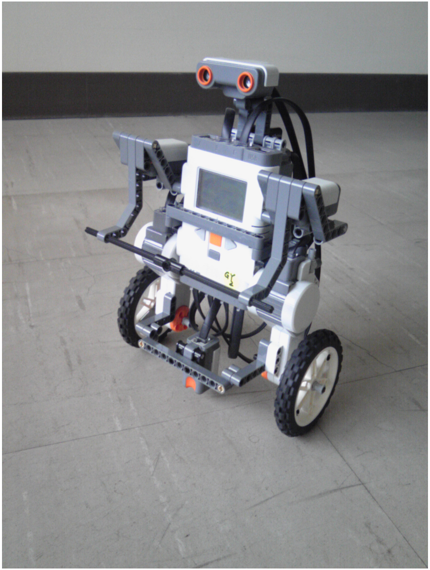
Outline

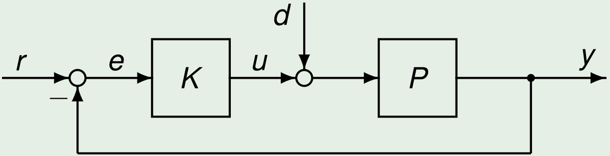
1. Control Theory
2. Design Approach
3. Guaranteed Accuracy Polynomial Spectral Factorization
4. Concluding Remarks

Outline

Feedback Control Systems



Feedback Loop



- P : *Plant* — System to be controlled dynamical system
- K : *Controller* — Control strategy

Aims

- Stabilization
- Disturbance attenuation
- Robustness — strong against uncertainty

(Post-)Modern Control Theory

Classical Control

- Qualitative
- Graphical approach
- (Simple) Algebraic computation

(Post-)Modern Control

For superior design....

- Mathematically oriented approach
- Extensive computation — *get along with computers*

- Mathematical modelling
- Mathematical formulation
- Quantitatively
- Optimization problems

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Sketch (1)

Dynamical system description

High-order (linear) differential equation

$$\text{E.g.,}\quad \ddot{y}(t) + 3\dot{y}(t) + 5y(t) = \dot{u}(t) + 2u(t)$$

Laplace Transform

- Transfer function

$$Y(s) = \mathcal{L}[y(t)]$$
$$U(s) = \mathcal{L}[u(t)]$$
$$P(s) = (\text{rational function in } s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+3s+5}$$

Set of 1st order differential eqns

- State-space representation
  - introduction of state $x(t)$

$$P: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

$$\begin{cases} \dot{x}(t) = \underbrace{\begin{bmatrix} -3 & -5 \\ 1 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t) \\ y(t) = \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{C} x(t) \end{cases}$$

Sketch (2)

Laplace Transformation

1. ***Polynomial Spectral Factorization***

$$f(s) \triangleq Y(s)Y(-s) + U(s)U(-s) = g(s)g(-s)$$

2. *Calculating the Controller*
   Set of linear equations

! In either approach, Step 1 is the harder.

Set of 1st order differential eqns

1. ***Algebraic Riccati equation***

$$XA + A^{\mathrm{T}}X - XBB^{\mathrm{T}}X + C^{\mathrm{T}}C = 0$$

$$YA^{\mathrm{T}} + AY - YC^{\mathrm{T}}CY + BB^{\mathrm{T}} = 0$$

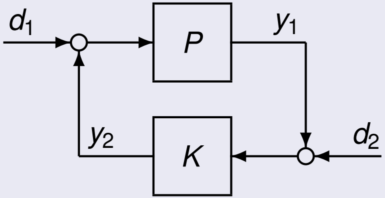
2. *Calculating the Controller*
   Straightforward matrix computation

$$K : \begin{cases} \dot{x}_K(t) = (A - YC^{\mathrm{T}}C - BB^{\mathrm{T}}X)x_K(t) + YC^{\mathrm{T}}y(t) \\ u(t) = -B^{\mathrm{T}}Xx_K(t) \end{cases}$$

Normalized LQG Control Problem

Problem Formulation

LQG = Linear Quadratic Gaussian



$$\min_{K \text{ stabilizing}} \left\| T_{zw}(P, K) \right\|_2$$

Given P, find a controller K
that minimizes the $\mathcal{H}_2$-norm of the transfer function matrix T_{zw}
from $w = (d_1 \; d_2)^T$ to $z = (y_1 \; y_2)^T$.

$\mathcal{H}_2$-norm

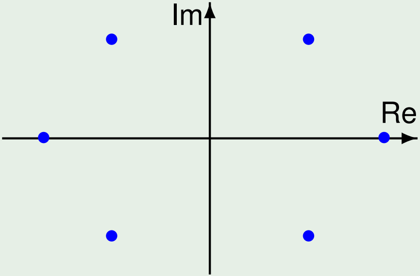
$$\left\| G(s) \right\|_2 \triangleq \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr} \left\{ G^*(i\omega) G(i\omega) \right\} d\omega \right)^{\frac{1}{2}}$$

$\left\| G(s) \right\|_2^2$: Energy of the system output to an impulse input signal

Outline

What is Polynomial Spectral Factorization?

- **Given** : $f(s) = f(-s) = -s^6 + 9s^4 - 4s^2 + 36$;
  Self-reciprocal polynomial
- **Task** : Decompose $f(s)$ as a product of a *stable* polynomial and an *anti-stable* polynomial ('*mirror image*')
  $$f(s) = \underbrace{\left(s^3 + 5s^2 + 8s + 6\right)}_{g(s)} \underbrace{\left(-s^3 + 5s^2 - 8s + 6\right)}_{g(-s)}$$
- *Self-reciprocal* — *stable* and *unstable* roots symmetrically



Stable: all the roots in the left half plane

Self-reciprocal : $f(s)$
Stable : $g(s)$
— *spectral factor*
Mirror image : $g(-s)$

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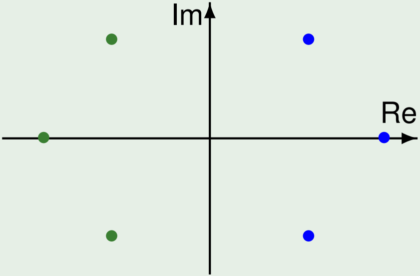

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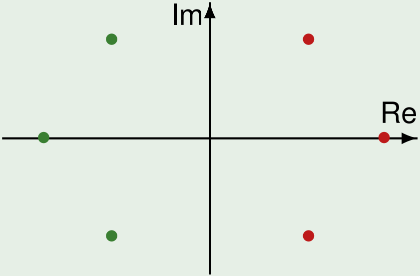

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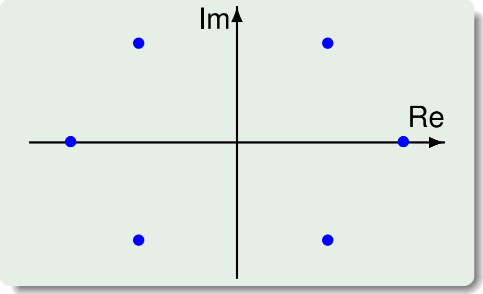
'Full' Guaranteed Approach

By Means of Verified Polynomial Root Computation

- Each Root in the left half plane is found as an interval on the real axis / a box in the complex plane.
- Express the spectral factor as a product of linear factors and expand it to get bounds for coefficients

$$\begin{aligned} g(s) &= (s - p_1)(s - p_2)(s - p_3) \\ &= s^3 - (p_1 + p_2 + p_3)s^2 + (p_1 p_2 + p_2 p_3 + p_3 p_1)s - p_1 p_2 p_3 \end{aligned}$$



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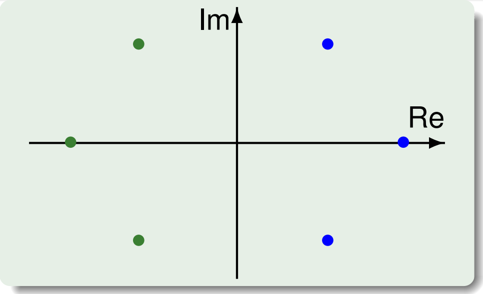

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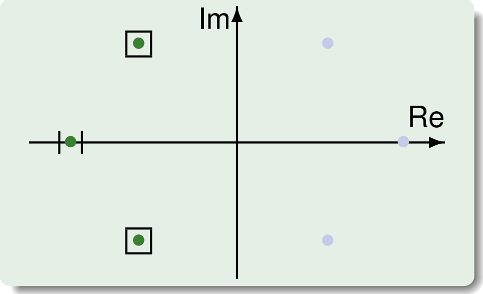


Hybrid Approach

- 'Full' Guaranteed Approach — *always works*
- Eventually want to get bounds for coefficients of the spectral factor
- To get tighter bounds for coefficients, the Krawczyk method can be employed.

Suggested Approach

1. Compute coefficients of the spectral factor using an ordinary (*unverified*) numerical method
2. Give (*heuristically*) bounds for coefficients, and *check whether a solution is included in the bounds*

- *Use 'Full' Guaranteed Approach as a backup*

Polynomial Spectral Factorization

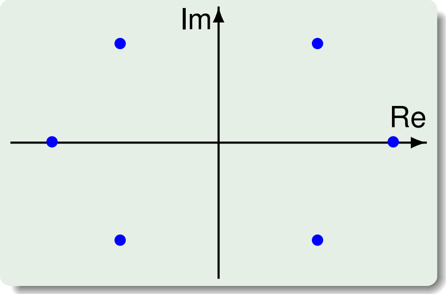
Problem Formulation

- **Given** : an even polynomial in s (polynomial in s^2)
$$f(s) = (-1)^n s^{2n} + a_{2n-2}s^{2n-2} + a_{2n-4}s^{2n-4} + \cdots + a_0$$
- **Task** : Find a polynomial
$$g(s) = s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0$$
such that $\boxed{f(s) = (-1)^n g(s)g(-s)}$
and $g(s)$ has roots in the open *left half plane* only.

- By comparing the coefficients of the both sides of
$$\boxed{f(s) = (-1)^n g(s)g(-s)},$$
a set of algebraic equations in b_j is obtained.
- Krawczyk method easy to apply

Catch

- The set of algebraic equations to be solved has *multiple* solutions.
  $\Longrightarrow$ Need to make sure that we will get the right solution
- E.g., $-s^6 - 3s^4 + 2s^2 + 9$
  $$= \underbrace{(s^3 + s^2 + 2s + 3)}_{\text{Roots: } -1.2757,\ 0.13784 \pm 1.5273i}(-s^3 + s^2 - 2s + 3)$$
- What we have got :
  A set of polynomials whose coefficients are bounded by intervals



- An *infinite* number of polynomials
- How can we guarantee the stability of the enclosed solution?

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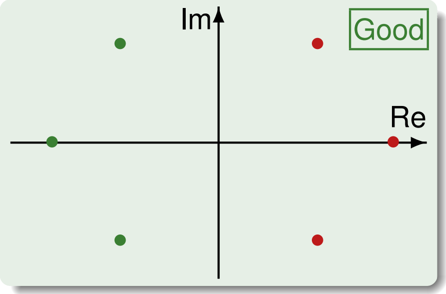

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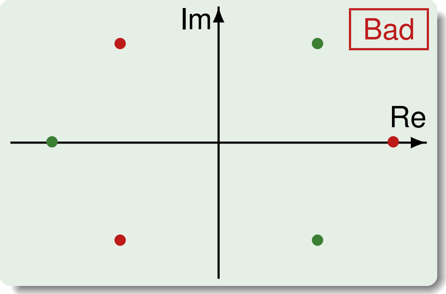

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Robust Stability Condition for 'Interval' Polynomials

Kharitonov's Theorem

All the polynomials in

$$\{p(s,\mathbf{a}) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0 \mid \mathbf{a} = (a_i),\ a_i \in [a_i^-, a_i^+]\},\quad a_n^- > 0$$

are stable iff the following four polynomials are stable:

$$\begin{aligned}
p^{++}(s) &= a_n^\bullet s^n + \cdots + a_4^+ s^4 + a_3^- s^3 + a_2^- s^2 + a_1^+ s + a_0^+ \\
p^{-+}(s) &= a_n^\bullet s^n + \cdots + a_4^+ s^4 + a_3^+ s^3 + a_2^- s^2 + a_1^- s + a_0^+ \\
p^{--}(s) &= a_n^\bullet s^n + \cdots + a_4^- s^4 + a_3^+ s^3 + a_2^+ s^2 + a_1^- s + a_0^- \\
p^{+-}(s) &= a_n^\bullet s^n + \cdots + a_4^- s^4 + a_3^- s^3 + a_2^+ s^2 + a_1^+ s + a_0^-
\end{aligned}$$

Stability Guarantee

Stability of each of the four polynomials can be examined by

- Algebraic method — Routh-Hurwitz test

  Check positivity of principal minors of the '*Hurwitz*' matrix
- Guaranteed accuracy root computation

All the polynomials in the obtained set are stable.

$\Longrightarrow$ The enclosed solution yields a stable polynomial.

Suggested Approach

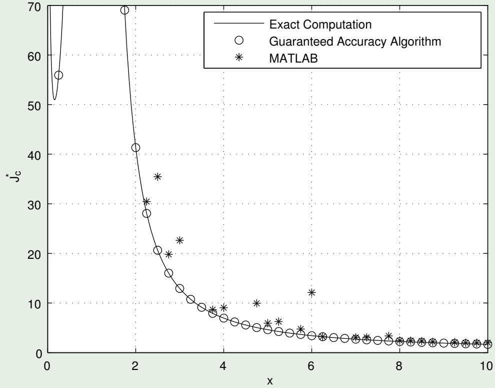
1. Compute coefficients of the spectral factor

   using an ordinary (*unverified*) numerical method
2. Give (*heuristically*) bounds for coefficients,

   and *check whether a solution is included in the bounds*
3. *Examine that the enclosed solution yields a stable spectral factor*

- *Use 'Full' Guaranteed Approach as a backup*

Example



$\mathcal{H}_2$ Tracking Problem:

Optimal Cost $J_{\mathrm{c}}^{\star}$

for Plant

$$P_x(s) = \frac{s - x}{s(s-1)}$$

$$(x > 0)$$

$$J_{\mathrm{c}}^{\star} = \frac{\sqrt{2(x+1)}(x^2 + 6x + 1) + 5x^2 + 10x + 1}{x(x-1)^2}$$

Guaranteed Accuracy Algorithm returns correct answers.

Outline

Concluding Remarks

What Has Been Achieved...

- Guaranteed accuracy polynomial spectral factorization
- Reliable control systems design

Future Work

- Effective implementation
- $\ldots$