Factorization of Differential Operators with Ordinary Differential Polynomial Coefficients

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Joint work with Yong Luo
Outline

- Problem
- Motivation
- Algorithm
- Conclusion
Problem

- $\mathcal{K}$: a differential field.
- $\mathcal{K}\{y\}$: the ring of univariate differential polynomials over $\mathcal{K}$.
- $\mathcal{K}\{y\}[\sigma]$: the ring of differential operators over $\mathcal{K}\{y\}$.

Problem: how to factor $L = \sigma^n + c^n - 1 \sigma^{n-1} + \cdots + c_1 \sigma + c_0$ over $\mathcal{K}\{y\}$?
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Specialization

- \( L \in \mathcal{K}\{y\}[\sigma] \), \( \gamma \) an element in some extension field of \( \mathcal{K} \).

- \( L(\gamma) \triangleq L|_{y=\gamma} \) is called the **specialization** of \( L \) with respect to \( \gamma \).

**Lemma:** \( L = L_1 \cdot L_2 \) over \( \mathcal{K}\{y\} \) \( \implies \) \( L(\gamma) = L_1(\gamma) \cdot L_2(\gamma) \).

**Example:**

\[
L = \sigma^2 + (y_1 + y_2)\sigma + y_3 + y_1y_2 = (\sigma + y_1) \cdot (\sigma + y_2)
\]

induces a factorization

\[
L\left(\frac{c}{t^2}\right) = \sigma^2 + c \cdot \frac{-2t + 6}{t^4}\sigma + \frac{24c \cdot t^2 - 12c^2}{t^7}
\]

\[
= (\sigma + y_1)|_{y=\frac{c}{t^2}} \cdot (\sigma + y_2)|_{y=\frac{c}{t^2}}
\]

\[
= (\sigma - \frac{2c}{t^3}) \cdot (\sigma + \frac{6c}{t^4})
\]

over \( \mathbb{Q}(t) \).
Decomposition of Differential Polynomials

\[ f \in \mathcal{K}\{y\}, \quad o_f \text{ denotes the order of } f. \]

\[ \mathcal{L} : \quad \mathcal{K}\{y\} \longrightarrow \mathcal{K}\{y\}[\sigma] \]

\[ f \longrightarrow \sum_{i=0}^{o_f} \frac{\partial f}{\partial y_i} \sigma^i \]

\[ \mathcal{L} \text{ is called the linearization operator over } \mathcal{K}\{y\}. \]

**Theorem:** \( f, g, h \in \mathcal{K}\{y\}. \) If \( f = g \circ h \), then \( \mathcal{L}(f) = (\mathcal{L}(g) \circ h) \cdot \mathcal{L}(h). \)

**Corollary:** If \( h \) is a right decomposition factor of \( f \), then \( \mathcal{L}(h) \) must be a right-side factor of \( \mathcal{L}(f) \).
Decomposition of Differential Polynomials

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**Corollary:** If \( h \) is a right decomposition factor of \( f \), then \( \mathcal{L}(h) \) must be a right-side factor of \( \mathcal{L}(f) \).

**Decomposition:** \( f \longrightarrow \mathcal{L}(f) \longrightarrow \mathcal{L}(h) \longrightarrow h \longrightarrow g \).
**Theorem** (Lüroth’s): \( k \subset K \subset k(x) \). Then \( K = k(g) \) for some \( g \in k(x) \).

**Theorem** (E. Noether): \( k \subset K \subset k(x) \). If there exists a polynomial in \( K \), then \( K = k(g) \) and \( g \) can be chosen to be a polynomial.

**Theorem** (J. F. Ritt): \( k \subset K \subset k\langle y \rangle \). Then \( K = k\langle g \rangle \) for some \( g \in k\langle y \rangle \).

**Problem**: \( k \subset K \subset k\langle y \rangle \). If there exists a differential polynomial \( f \) in \( K \), can the generator be chosen to be a differential polynomial?

It’s equivalent to consider the rational differential decomposition of \( f \).

**A necessary condition**: If \( \left( \sum_{0 \leq i \leq n} \frac{a_i}{p^{\alpha_i} \sigma^i} \right) \cdot \left( \sum_{0 \leq j \leq n} \frac{b_j}{p^{\beta_j} \sigma^j} \right) \in K\{y\}[\sigma] \) and \( a_i, b_i, p \in K\{y\}, \alpha_i > 0, \beta_i \geq 0 \), then \( p^{\beta_i} | b_i (0 \leq i \leq n) \) for all \( i \).
Algorithm

The total degree of $a_i \sigma^j$ is defined to be that of $a_i$.

Assume that $L \in \mathcal{K}\{y\}[\sigma]$ has a factorization $L = A \cdot B$. We write $L, A, B$ as the sum of their homogeneous parts respectively:

$$L = L_t + L_{t-1} + \cdots + L_1 + L_0$$
$$A = A_{t_1} + A_{t_1-1} + \cdots + A_1 + A_0$$
$$B = B_{t_2} + B_{t_2-1} + \cdots + B_1 + B_0$$
We compute $A_i, B_i (i = 0, 1, \cdots)$ step by step. When $\mathcal{K}$ is a constant field:

1. The computation of $A_0, B_0$ is equivalent to the factorization of a univariate algebraic polynomial.

2. $A_k, B_k$ are determined uniquely by $A_0, B_0, L_k$ (a nontrivial theorem).
Factorization Algorithm

Input: a differential operator \( L = \sigma^n + c_{n-1}\sigma^{n-1} + \cdots + c_1\sigma + c_0 \) over \( \mathbb{C}\{y\} \).

Output: a set \( S \) which consists of all nonequivalent factorizations \((A, B)\) of \( L \).

S1: \( S := \emptyset \). Write \( L \) as the sum of the homogeneous parts:
\[
L = L_t + L_{t-1} + \cdots + L_1 + L_0,
\]
where \( t \) is the total degree of \( L \). Let
\[
T = \{(A_0, B_0) : A_0 \cdot B_0 = L_0, A_0 \text{ and } B_0 \text{ are monic}\}.
\]

S2: If \( T \neq \emptyset \), choose a \((V_1, V_2) \in T\), \( T := T - \{(V_1, V_2)\} \), go to next step. Otherwise, output \( S \) and terminate the algorithm.

S3: Solve \((A_i, B_i)\) with inputs \( A_0 = V_1, B_0 = V_2 \) and \( L_i \). If all of the solutions of \( A_i, B_j \) exists, \( A := \sum_{0 \leq i \leq t} A_i, B := \sum_{0 \leq j \leq t} B_j, \)
\[
S := S \cup \{(A, B)\}. \]
Go to S2.
Factorization Algorithm

**Input:** a differential operator \( L = \sigma^n + c_{n-1}\sigma^{n-1} + \cdots + c_1\sigma + c_0 \) over \( \mathbb{C}\{y\} \).

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S := S \cup \{(A, B)\}. \text{ Go to S2.}
\]

- In the worst case, the number of choices of \((A_0, B_0)\) are exponential.
- When \( A_0, B_0 \) are chosen, \( A, B \) can be obtained with linear complexity (w.r.t the number of terms of \( L \)).
Example

\[ L = \sigma^4 + (yy_2 + y_1^2)\sigma^3 + (yy_1^2y_2 + 2yy_3 + 2y_1y_2 + y_2 + y_1 + 3)\sigma^2 + (yy_1^2y_3 + y_1^3y_2 + yy_2^2 + y_1^3 + yy_4 + 2y_1y_3 + y_2^2 + 2y_2)\sigma + (y_1^2y_2 + y_1y_2 + y_3 + y_2 + 2y_1 + 2) \in \mathbb{Q}\{y\}[\sigma]. \]

Firstly we get the homogeneous parts of \( L \) and equations of \( A_i, B_i \):

\[
\begin{align*}
L_0 &= \sigma^4 + 3\sigma^2 + 2 \\
    &= A_0 \cdot B_0 \\
L_1 &= (y_2 + y_1)\sigma^2 + 2y_2\sigma + (y_3 + y_2 + 2y_1) \\
    &= A_0 \cdot B_1 + A_1 \cdot B_0 \\
L_2 &= (yy_2 + y_1^2)\sigma^3 + (2yy_3 + 2y_1y_2)\sigma^2 + (yy_4 + 2y_1y_3 + 2yy_2 + y_2^2 + y_1^2)\sigma \\
    &= A_0 \cdot B_2 + A_2 \cdot B_0 + A_1 \cdot B_1 \\
L_3 &= (yy_2^2 + y_1^3)\sigma + y_1^2y_2 \\
    &= A_0 \cdot B_3 + A_3 \cdot B_0 + A_1 \cdot B_2 + A_2 \cdot B_1 \\
L_4 &= yy_1^2y_2\sigma^2 + (yy_1^2y_3 + y_1^3y_2)\sigma \\
    &= A_0 \cdot B_4 + A_4 \cdot B_0 + A_1 \cdot B_3 + A_3 \cdot B_1 + A_2 \cdot B_2
\end{align*}
\]
By \( L_0 = \sigma^4 + 3\sigma^2 + 2 = (\sigma^2 + 1) \cdot (\sigma^2 + 2) \), we have

\[
T = \{ (\sigma^2 + 1, \sigma^2 + 2), (\sigma^2 + 2, \sigma^2 + 1) \}.
\]

Firstly, we choose \((A_0, B_0) = (\sigma^2 + 2, \sigma^2 + 1) \in T\), then we have

\[
(\sigma^2 + 2) \cdot B_1 + A_1 \cdot (\sigma^2 + 1) = L_1 \implies A_1 = y_2, B_1 = y_1.
\]

Substituting \(A_0, A_1, B_0, B_1\) to the third equation, we have

\[
A_0 \cdot B_2 + A_2 \cdot B_0 = L_2 - y_1 y_2 \implies A_2 = y_1^2 \sigma, B_2 = y y_2 \sigma.
\]

Similarly, we get

\[
A_3 = B_3 = 0, \quad A_4 = B_4 = 0
\]

in the fourth and fifth equations respectively. Thus

\[
A = \sigma^2 + y_1^2 \sigma + y_2 + 2, \quad B = \sigma^2 + yy_2 \sigma + y_1 + 1
\]

and \(L = A \cdot B\) is a factorization of \(L\).

If we choose \((\sigma^2 + 2, \sigma^2 + 1)\) to begin the computation, it will return no factorization.
Conclusion

1. Propose some problems related to the factorization of differential operators over $\mathcal{K}\{y\}$.

2. Give a factorization algorithm for monic differential operators over $\mathcal{C}\{y\}$. 
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Thank you!