Resultant-free computation of indefinite hyperexponential integrals

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October 26, 2012

Problem

For a hyperexponential function $h(x)$, find $\int h\,dx$, if h is hyperexponential integrable.

1. h is said to be hyperexponential over $k(x)$ if h'/h is in $k(x)$.

   Analytically, a hyperexp. function h looks like

   $$\exp\left(\int f\,dx\right).$$

2. h is hyperexponential integrable if there exists hyperexp. g s.t. $h = g'$.

   Like $\exp(x^2)$ is not, but $x\exp(x^2)$ is.

The Almkvist–Zeilberger algorithm 1

- Given $f \in k(x)$, if

  $$f = \frac{p'}{p} + \frac{q}{r}, \quad \text{where } \gcd(r, q - ir') = 1 \text{ for all integer } i \geq 0,$$

  then $(p, q, r) \in k[x]^3$ is a differential Gosper form.

$f = u/v \in k(x)$ such that $u, v \in k[x]$ and $\gcd(u, v) = 1$.

1. Start $p := 1, q := u$ and $r := v$;
2. Find integer root of $R := \text{resultant}_x(v, u - zv') \in k[z]$;
3. $w := \gcd(v, u - jv')$;
4. $p \leftarrow pw^j, \quad r \leftarrow r/w, \quad q \leftarrow \frac{q - jw'(r/w)}{w}$.

Remark

The algorithm before for computing differential Gosper form contains

- Resultant computation:

$$R := \mathrm{resultant}_x(v, u - zv').$$

- Find integer roots of R.

The Almkvist–Zeilberger algorithm 2

For a hyperexponential h, we want to decide whether $h = g'$ for some hyperexp. g.

- Find differential Gosper form of h

$$\frac{h'}{h} = \frac{p'}{p} + \frac{q}{r};$$

- $g = \frac{r}{p} y h$, where $y \in k[x]$;
- Estimate its degree, then use undetermined coefficients method compute a polynomial solution of

$$y'r + y(q + r') = p.$$

Ingredients in our algorithm

Given

$$h = exp\left(\int \frac{u}{v}dx\right)$$

compute

$$\frac{u}{v} = \frac{p'}{p} + \frac{q}{r}.$$

Lemma. The factors of p are the multiplicity-one factors of v.
Proof. Use an elementary divisibility argument.

Revised algorithm to compute differential Gosper forms

1. Compute the product v_1 of multiplicity-one factors of v;
2. Factor $v_1 = p_1 \cdots p_n$ into irreducible factors.
3. Set $p := 1$. For i from 1 to n, test whether there exists $\lambda_i \in \mathbb{N}$ such that

$$p_i \mid (u - \lambda_i p_i'(v/p_i)).$$

   If such a nonnegative integer λ_i exists, then update $p := p_i^{\lambda_i} p$.
4. Set $q/r := f - p'/p$, and return (p, q, r).

Remark

- Irreducible factorization is not that terrible in practice;
- We avoid the resultant computation and integer-root finding.

Universal denominators

For a linear differential equation with rational-function coefficients in x, a polynomial $U \in k[x]$ is a universal denominator for rational solutions of this equations if the denominator of any rational solution divides U.

$$\text{rational solutions} \quad \rightsquigarrow \quad \text{polynomial solutions}$$

Given h. Then $f = \frac{h'}{h}$. Find g s.t. $g' = h$ is equivalent to find the rational solutions of

$$y' + fy = 1.$$

If we have the universal denominator, then it suffices to compute a polynomial solution of the above equation.

A key observation

Consider the differential equation

$$y' + fy = 1, \quad \text{where } f = \frac{u}{v} \in k(x).$$

Lemma. If $a/b \in k(x)$ is a rational solution, then all factors of b are multiplicity-one factors of v.

Proof.

- One way is to use divisibility;
- The other is order estimation, based on the fact

$$\text{Let } p \in k[x] \text{ irreducible, } p^m \mid b \quad \Rightarrow \quad p^{m+1} \mid \operatorname{den}((a/b)').$$

Algorithm for universal denominators

1. Compute the product v_1 of multiplicity-one factors of v;
2. Factor $v_1 = p_1 \cdots p_n$ into irreducible factors.
3. Set $b := 1$. For i from 1 to n, test whether there exists $\lambda_i \in \mathbb{N}$ such that

$$p_i \mid (u - \lambda_i p_i'(v/p_i)).$$

   If such a nonnegative integer λ_i exists, then update $b := p_i^{\lambda_i} b$.
4. Return b.

An example

Given $h = (1+2x)\exp(x)/(2\sqrt{x})$.

$$\frac{h'}{h} = \frac{4x^2+4x-1}{2x(2x+1)},$$

so $u = 4x^2+4x-1$ and $v = 2x(2x+1)$.
The irreducible and multiplicity-one factors of v are x and $2x+1$.
Then

$$\mathrm{rem}(4x^2+4x-1-4\lambda x, 2x+1) = 0$$

yields $\lambda = 1$ while

$$\mathrm{rem}(4x^2+4x-1-2\lambda(2x+1), x) = 0$$

yields $\lambda = 1/2$. So the universal denominator b is $2x+1$.

An example (continue)

It suffices to calclulate the polynomial solution of

$$(4x^2 - 1)a + 2(2x^2 + x)a' = 2x(4x^2 + 4x + 1).$$

The upper bound is $\deg(v) + \deg(b) - \deg(u) = 1$, because $\deg(u) = 2 > \deg(v) - 1 = 1$. Using the method of undetermined coefficients we get $a = 2x$. Hence h is hyperexponential integrable and

$$g = \sqrt{x}\exp(x).$$

Summary

- Compute differential Gosper form without resultant computation and integer-root finding.
- A new way to find universal denominator.

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Thank you!