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Factoring Finite-rank Linear Functional Systems

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1 Problem and goal

Finite-rank (linear functional) system S:

- a set of linear homogeneous differential and difference equations
- $\mathrm{sol}(S)$ is finite-dim over some suitable constant field

Goal: an algorithm for factoring a finite-rank system S, i.e., finding "subsystems" L s.t

$$\mathrm{sol}(L) \subsetneq \mathrm{sol}(S).$$

Result of this poster: a factorization algorithm applicable for

- Systems of linear differential equations
- Reflexive systems of linear difference equations
- Reflexive systems of linear differential-difference equations

2 Example

$$L = \left\{\partial_x^3 + \frac{x^3-3x}{x^2-1}\partial_x^2,\ \partial_y\partial_x + \frac{3yx^2}{x^2-1}\partial_x^2 + 3yx\partial_x + x\partial_y,\right.$$

$$\left.\partial_y^2 + \frac{9y^2x+3x+y}{x^2-1}\partial_x^2 + 3y\partial_y + (3+9y^2)\partial_x\right\}.$$

We find

$$\mathrm{sol}(L) = \mathrm{sol}(\{\partial_x + x,\ \partial_y^2 + 3y\partial_y + y\}) \oplus \mathrm{sol}(\{\partial_x,\ \partial_y\})$$

$$\oplus\, \mathrm{sol}(\{\partial_x - 2/(2x-3y^2),\ \partial_y + 6y/(2x-3y^2)\})\,.$$

3 Previous work

Systems	Factoring
LODE's	Beke (1894), Schwarz (89) Bronstein (94), van Hoeij (97) Giesbrecht & Zhang (03), Cluzeau (03)
LOΔE's	Bronstein & Petkovšek (96) van Hoeij (99) (first order factors) Abramov & Paule & Petkovšek (98)
LPDE's	Li & Schwarz & Tsarev (02, 03)
LP$\{D+\Delta\}$E's	Labahn & Li (04) (first order factors)

4 Basic definitions

$\mathcal{F} = \mathcal{C}(x_1, \ldots, x_n)$; for $i = 1, \ldots, n$, let σ_i be an automorphism of $\mathcal{F}$ and let δ_i be an additive mapping of $\mathcal{F}$ with the property:

$$\delta_i(ab) = \sigma_i(a)\delta_i(b) + \delta_i(a)b \quad \forall\, a, b \in \mathcal{F}.$$

Assume $\quad \sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i,\ \sigma_i \circ \delta_j = \delta_j \circ \sigma_i,\ \delta_i \circ \delta_j = \delta_j \circ \delta_i,\ \forall i,\ j.$

An *Ore polynomial ring* [Chyzak & Salvy, 98] is defined to be $\mathcal{A} = \mathcal{F}[\partial_1, \ldots, \partial_n]$ with multiplicative rules:

$$\partial_i\partial_j = \partial_j\partial_i, \quad \partial_i a = \sigma_i(a)\partial_i + \delta_i(a), \quad \forall\, a \in \mathcal{F},\ \forall\, i,\ j.$$

$\mathcal{A} = \mathcal{F}[\partial_1, \ldots, \partial_n]$ is *orthogonal* [Labahn & Li, 04] if,

$$\sigma_i = 0, \qquad \delta_i = \frac{\partial}{\partial x_i}, \qquad \text{for } i = 1, \ldots, p,$$
$$\sigma_j : x_j \mapsto x_j + 1, \qquad \delta_j = 0, \qquad \text{for } j = p+1, \ldots, n.$$

For a (left) ideal I of $\mathcal{A}$, $\mathrm{rank}(I) := \dim_{\mathcal{F}}(\mathcal{A}/I)$.

- A system $S \subset \mathcal{A}$ is *of finite rank* if $\mathrm{rank}((S)) < \infty$.
- S is *reflexive* if

$$\partial_j f \in (S) \ \text{ with } f \in \mathcal{A} \text{ and } j \in \{p+1, \ldots, n\} \implies f \in (S).$$

- S is reflexive $\iff \mathrm{rank}((S)) = \dim_{\mathcal{C}} \mathrm{sol}(S)$.

M is a *∂-finite module* if M is an $\mathcal{A}$-module which is also a finite-dim $\mathcal{F}$-vector space (a generalization of differential modules [van der Put & Singer, 03]).

A ∂-finite module M is *reflexive* if

$$\partial_j m = 0 \text{ with } m \in M \text{ and } j \in \{p+1, \ldots, n\} \implies m = 0.$$

5 Translation into module language

$$\text{System } S \implies \text{ideal } (S) \iff \mathcal{A}\text{-module } \mathcal{A}/(S)$$
$$S \text{ reflexive} \iff \mathcal{A}/(S) \text{ reflexive}$$

Advantages of module language:

- more concise: bases of vector spaces instead of bases of ideals
- more general: applicable to systems in several unknowns
- more powerful: convenient for using multi-linear algebra

Let $e_1, \ldots, e_s$ be a basis of M over $\mathcal{F}$ and

$$\partial_i(e_1, \ldots, e_s) = (e_1, \ldots, e_s)A_i, \quad i = 1, \ldots, n.$$

Fact: M reflexive $\iff A_{p+1}, \ldots, A_n$ invertible.

6 Idea: use exterior power

- M is reflexive:

M reflexive, $\partial_i(e_1, \ldots, e_s) = (e_1, \ldots, e_s)A_i$ $\overset{\wedge}{\Longrightarrow}$ $\wedge^d M$ reflexive, $\partial_i(f_1, \ldots, f_t) = (f_1, \ldots, f_t)B_i$

d-dim $N \subsetneq M$ $\overset{\wedge}{\Longrightarrow}$ 1-dim $\wedge^d N \subsetneq \wedge^d M$

$\mathcal{F}w_1 \oplus \cdots \oplus \mathcal{F}w_d$ $\Longleftarrow$ [Compoint & Weil, 04] $\Longrightarrow$ $\mathcal{F}w$ with $w = w_1 \wedge \cdots \wedge w_d$

$\Updownarrow$

hyperexponential-scaled sols of *integrable system* of $\wedge^d M$: $\{\partial_i Z = -B_i Z,\ i = 1, \ldots, p,\ \partial_j Z = B_j^{-1} Z,\ j = p+1, \ldots, n\}$

def $\downarrow$

$H = hY$ with $Y \in \mathcal{F}^t$ and h *hyperexponential over $\mathcal{F}$*, i.e., $\frac{\partial_i(h)}{h} \in \mathcal{F}$, for $i = 1, \ldots, n$.

Hyperexp sols of

$\Downarrow$ [Labahn & Li, 04]

$$H = e^{\int r_1}\left(\prod r_2\right)w, \quad \text{with } r_1,\ r_2 \in \mathcal{F},\ w \in \wedge^d M.$$

Consider a map $\varphi_w : M \longrightarrow \wedge^{d+1} M, \quad v \mapsto v \wedge w.$

Theorem: $w = w_1 \wedge \cdots \wedge w_d \iff \dim_{\mathcal{F}}(\ker(\varphi_w)) = d.$

- M is not reflexive: computing 1-dim submodules directly.

7 Example

$\mathcal{F} = \mathcal{C}(x, k)$, $\mathcal{A} = \mathcal{F}[\partial_x, E_k]$. A rank-four ideal I generated by

$$\partial_x^2 - \frac{2((k-x)^2+k)}{x(k-x)}\partial_x + \frac{((k-x)^3 - 3xk + 3k^2 + 2k)}{x^2(k-x)}, \quad E_k^2 - \frac{2x(x-k-1)}{x-k-2}E_k + \frac{x^2(x-k)}{x-k-2}.$$

All first order factors of I:

$$\left(\partial_x - \frac{\partial_x(h)}{h}, \quad E_k - \frac{E_k(h)}{h}\right)$$

where $h = \frac{c_1 + c_2 k + c_3 x}{x-k}\exp(-x)x^{k+1}$.

8 Future Study

LPDE's	* improve efficiency
LP$\{D+\Delta\}$E's	* find any order factors, i.e., find any dim submodules of non-reflexive modules * improve efficiency