Singularities and Strata
of Parametrized Curves

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Part I: Singularities of Parametrized Plane Curves

A. Moving Lines and $\mu$-Bases

B. Singularities
- Multiplicity of a Singular Point
- Infinitely Near Singular Points
- The Genus Formula
- Blowing Up

C. Singularities and the $\mu$-Type
- Axial Moving Lines
- A Theorem
- Infinitely Near Triple Points for $n = 6$

D. Points of Multiplicity $c$ when $\mu = c$ and $n = 2c$
- The Setup
- The Stratification
- Hint of the Proof
A. Two Moving Lines

Imagine two moving lines in $\mathbb{P}^2$:

\[
p = Q_1(s, t)x + Q_2(s, t)y + Q_3(s, t)z = 0
\]
\[
q = Q_4(s, t)x + Q_5(s, t)y + Q_6(s, t)z = 0
\]

where $Q_i \in \mathbb{C}[s, t]$ are homogeneous polynomials such that $\text{deg}(Q_1) = \text{deg}(Q_2) = \text{deg}(Q_3)$ and $\text{deg}(Q_4) = \text{deg}(Q_5) = \text{deg}(Q_6)$.

We assume the lines are always distinct, i.e., the Hilbert-Burch matrix

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A = \begin{pmatrix}
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has rank 2 for all $(s, t) \in \mathbb{P}^1$. 

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The point of intersection of the two moving lines traces out a rational curve $C \subseteq \mathbb{P}^2$.

We assume the parametrization $\mathbb{P}^1 \to C$ is birational.
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A. Properties of Moving Lines

- \text{deg}(C) = \text{deg}(p) + \text{deg}(q) = n.
- The map \( \mathbb{P}^1 \to \mathbb{P}^2 \) is given by
  \[
  B = (a, b, c) = 2 \times 2 \text{ minors of } A = \begin{pmatrix} Q_1 & Q_4 \\ Q_2 & Q_5 \\ Q_3 & Q_6 \end{pmatrix}.
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- \( \gcd(a, b, c) = 1 \) by rank assumption on \( A \).
- \( I = \langle a, b, c \rangle \subseteq R = \mathbb{C}[s, t] \) has free resolution
  \[
  0 \to R(-n - \text{deg}(p)) \oplus R(-n - \text{deg}(q)) \xrightarrow{A} R(-n)^3 \xrightarrow{B} I \to 0.
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- \( p, q \) generate \( \text{Syz}(a, b, c) \cong R(-n - \text{deg}(p)) \oplus R(-n - \text{deg}(q)) \).
- All rational plane curves arise this way.

\textbf{Definition}

If \( \text{deg}(p) = \mu \leq n - \mu = \text{deg}(q) \), then \( p, q \) is a \( \mu \)-basis of the parametrization, and \( \mu \) is the \( \mu \)-type of the parametrization.
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B. Singularities

Here are two parametrized curves of degree $n = 4$:

![Curves](image)

**Definition**

Assume $P = (0, 0) \in \mathbb{C}^2$ lies on a curve $C = V(f)$. Write

$$f = f_k + f_{k+1} + \cdots,$$

where $f_i$ is homogeneous of degree $i$ and $f_k \neq 0$. The multiplicity of $P$ is $m_P = k$. 
B. Infinitely Near Singularities

Besides the “visible” singularities such as those on the previous slide, here are also infinitely near singularities that become visible only after a blowup.

Here is an Example.

Equation:
\[(x^2 + y^2 - 3x)^2 = 4x^2(2 - x)\]

Visible Singular points:
- **Node** at \((1, 0)\), multiplicity 2
- **Tacnode** at \((0, 0)\), multiplicity 2

When we blow up the origin, we still have a singularity, again of multiplicity 2. Hence there are three singular points, two visible and one infinitely near, all of multiplicity 2.
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B. The Genus Formula

Infinitely nearly points play an important role in the following theorem.

**Theorem**

Let \( C \subseteq \mathbb{P}^2 \) be irreducible curve of degree \( n \) and genus \( g \). Then

\[
g = \frac{(n - 1)(n - 2)}{2} - \sum_{P} m_P(m_P - 1)/2.
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The sum is over all visible and infinitely near singular points of \( C \).

**Corollary**

An irreducible curve is rational \( \iff (n - 1)(n - 2) = \sum_{P} m_P(m_P - 1) \).

**Example**

Rational, degree 4, multiplicities = 2: three singularities.
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B. Blowing Up

The blowup of $\mathbb{C}^2$ at $(0,0)$ separates tangent directions through $(0,0)$, as shown on the right.

Let $x, y$ be coordinates on $\mathbb{C}^2$. The blowup has two coordinate patches:

- $(\mathbb{C}^2)_1 \rightarrow \mathbb{C}^2$, $(x, t) \mapsto (x, tx)$.
- $(\mathbb{C}^2)_2 \rightarrow \mathbb{C}^2$, $(u, y) \mapsto (uy, y)$.

Consider the line $y = mx$ through $(0,0)$. In $(\mathbb{C}^2)_1$, keep $x$ and replace $y$ with $tx$, giving $tx = mx$. Cancel $x$ to get the horizontal line $t = m$. 

Image from What’s Happening in the Mathematical Sciences, Vol. 7
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C. Axial Moving Lines

**Theorem**

\[ \mu = 1 \iff \text{the curve has a singular point of multiplicity } n - 1. \]

**Proof.**

Assume there is a singular point of multiplicity \( n - 1 \) at \((0, 0, 1)\) \( \in \mathbb{P}^2 \). For \((s, t) \in \mathbb{P}^1\), the line \( p = sx + ty \) goes through \((0, 0, 1)\). By Bezout, it meets the curve in exactly one additional point. Since \( p = sx + ty \) follows this parametrization and has degree 1 is \( s, t \), we get \( \mu = 1 \).

The moving line used in this proof is very special

**Definition**

A moving line \( p = Q_1(s, t)x + Q_2(s, t)y + Q_3(s, t)z \) in \( \mathbb{P}^2 \) is axial if the \( Q_i \) are relatively prime and there is a point (the axis) \((\alpha, \beta, \gamma) \in \mathbb{P}^2\) that lies on \( p = 0 \) for all \((s, t) \in \mathbb{P}^1\).
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C. Singularities and the $\mu$-Type

The case $\mu = 1$ from the previous slide generalizes nicely.

**Theorem (Chen/Goldman/Song, Chen/Liu/Wang)**

Assume $C \subseteq \mathbb{P}^2$ is a rational plane curve of degree $n$ and type $\mu$. Then the singular points of $C$ have:

- $(\mu = n - \mu)$ Multiplicity $\leq \mu$.
- $(\mu < n - \mu)$ Either multiplicity $\leq \mu$ or multiplicity $= n - \mu$, and there is at most one singular point of multiplicity $n - \mu$.

**Theorem (Chen/Goldman/Song)**

Assume $C \subseteq \mathbb{P}^2$ is a parametrized curve of degree $n$ and type $\mu$. If $P \in C$ has multiplicity $k$, then there is an axial moving line with axis $P$ and degree $n - k$ that follows the parametrization.
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C. The Case of Degree $n = 4$

Here are two examples with $n = 4$:

When $n = 4$, we always have $\mu = 1$ or $2$. Furthermore:

- $\mu = 2 \iff$ all singularities have multiplicity 2.
- $\mu = 1 \iff$ there is a point of multiplicity 3.
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\[
\begin{array}{ccc}
0.5 & 2 & 2 \\
1 & 0 & 1 \\
1.5 & -1 & -2 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\
-0.5 & -1 & -1.5 \\
\end{array}
\]

When \( n = 4 \), we always have \( \mu = 1 \) or 2. Furthermore:

- \( \mu = 2 \iff \) all singularities have multiplicity 2.
- \( \mu = 1 \iff \) there is a point of multiplicity 3.
Let $C \subseteq \mathbb{P}^2$ be a parametrized curve of degree $n = 6$ and type $\mu = 3$, and assume that $C$ has a triple point at $P = (0, 0, 1)$. By a previous theorem, we get a Hilbert-Burch matrix of the form

$$A = \begin{pmatrix} Q_1 & Q_4 \\ Q_2 & Q_5 \\ 0 & Q_6 \end{pmatrix}$$

with $\deg(Q_i) = 3$ and $\gcd(Q_1, Q_2) = 1$. Also set $Q = Q_1 Q_5 - Q_2 Q_4$. The parametrization is given by the $2 \times 2$ minors of $A$ with suitable signs, so

$$B = (a, b, c) = (Q_2 Q_6, -Q_1 Q_6, Q).$$

**Theorem**

The curve $C$ has an infinitely near triple point at $P$ $\iff$ $Q_6$ is a linear combination of $Q_1$ and $Q_2$. 
Let $C \subseteq \mathbb{P}^2$ be a parametrized curve of degree $n = 6$ and type $\mu = 3$, and assume that $C$ has a triple point at $P = (0, 0, 1)$. By a previous theorem, we get a Hilbert-Burch matrix of the form

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with $\deg(Q_i) = 3$ and $\gcd(Q_1, Q_2) = 1$. Also set $Q = Q_1 Q_5 - Q_2 Q_4$. The parametrization is given by the $2 \times 2$ minors of $A$ with suitable signs, so

$$B = (a, b, c) = (Q_2 Q_6, -Q_1 Q_6, Q).$$

**Theorem**

The curve $C$ has an infinitely near triple point at $P \iff Q_6$ is a linear combination of $Q_1$ and $Q_2$. 
C. Infinitely Near Triple Points for $n = 6$

Let $C \subseteq \mathbb{P}^2$ be a parametrized curve of degree $n = 6$ and type $\mu = 3$, and assume that $C$ has a triple point at $P = (0, 0, 1)$. By a previous theorem, we get a Hilbert-Burch matrix of the form

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with $\text{deg}(Q_i) = 3$ and $\gcd(Q_1, Q_2) = 1$. Also set $Q = Q_1 Q_5 - Q_2 Q_4$. The parametrization is given by the $2 \times 2$ minors of $A$ with suitable signs, so

$$B = (a, b, c) = (Q_2 Q_6, -Q_1 Q_6, Q).$$

**Theorem**

*The curve $C$ has an infinitely near triple point at $P$ if and only if $Q_6$ is a linear combination of $Q_1$ and $Q_2$.***
C. Begin the Proof of \( \Rightarrow \)

Affinely, the parametrization is 
\[
x = \frac{a}{c} = \frac{Q_2 Q_6}{Q}, \quad y = \frac{b}{c} = \frac{-Q_1 Q_6}{Q}.
\]

Blowing up \( \mathbb{C}^2 \) at \((0,0)\) gives two affine charts, one of which contains a triple point. Assume that the triple point lies in the chart \((\mathbb{C}^2)_1\) with variables \(x, t\) such that \(y = tx\). This gives the parametrization
\[
x = \frac{Q_2 Q_6}{Q}, \quad t = \frac{tx}{x} = \frac{y}{x} = \frac{-Q_1 Q_6}{Q_2 Q_6} = \frac{-Q_1}{Q_2}
\]
in \((\mathbb{C}^2)_1\). Using a common denominator gives
\[
x = \frac{Q_2^2 Q_6}{Q_2 Q}, \quad t = \frac{-Q_1 Q}{Q_2 Q}.
\]

Once that check that \(\gcd(Q_2^2 Q_6, -Q_1 Q, Q_2 Q) = 1\), so that we have a parametrization of degree 9.
C. Begin the Proof of \(\Rightarrow\)

Affinely, the parametrization is \(x = \frac{a}{c} = \frac{Q_2 Q_6}{Q}, \quad y = \frac{b}{c} = \frac{-Q_1 Q_6}{Q}\).

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t &= \frac{tx}{x} = \frac{y}{x} = \frac{-Q_1 Q_6}{Q_2 Q_6} = \frac{-Q_1}{Q_2}
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Blowing up $\mathbb{C}^2$ at $(0, 0)$ gives two affine charts, one of which contains a triple point. Assume that the triple point lies in the chart $(\mathbb{C}^2)_1$ with variables $x, t$ such that $y = tx$. This gives the parametrization

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Once that check that $\gcd(Q_2^2 Q_6, -Q_1 Q, Q_2 Q) = 1$, so that we have a parametrization of degree 9.
C. Continue the Proof of $\Rightarrow$

From $(Q_2^2 Q_6, -Q_1 Q, Q_2 Q)$, we get the Hilbert-Burch matrix

$$
\begin{pmatrix}
0 & -Q \\
Q_2 & 0 \\
Q_1 & Q_2 Q_6
\end{pmatrix}
$$

as follows:

- $t = -Q_1/Q_2$ gives the moving line $0 \cdot x + Q_2 t + Q_1 = 0$. We use this as the first column of Hilbert-Burch matrix.
- The parametrization equals the $2 \times 2$ minors of Hilbert-Burch matrix. This makes the second column easy to find.

Let the triple point be $(\alpha, \beta)$. Since it maps to $(0, 0)$ via $(x, t) \mapsto (x, tx)$, we have $\alpha = 0$, so that the triple point is $(0, \beta)$. 
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C. Finish the Proof of \( \Rightarrow \)

By the theorem, a triple point at \((0, \beta)\) implies that there is moving line of degree \(9 - 3 = 6\) with axis \((0, \beta)\) that follows the parametrization. From the Hilbert-Burch matrix, this gives the moving line

\[
A \begin{pmatrix} 0 \\ Q_2 \\ Q_1 \end{pmatrix} + \lambda \begin{pmatrix} -Q \\ 0 \\ Q_2 Q_6 \end{pmatrix} = \begin{pmatrix} -\lambda Q \\ AQ_2 \\ AQ_1 + \lambda Q_2 Q_6 \end{pmatrix}
\]

with \(\deg(A) = 3\) and \(\lambda \in \mathbb{C}\), and having \((0, \beta)\) as axis implies

\[
(-\lambda Q) \cdot 0 + (AQ_2) \beta + (AQ_1 + \lambda Q_2 Q_6) \equiv 0.
\]

This implies that \(Q_2\) divides \(AQ_1\), and since \(\gcd(Q_1, Q_2) = 1\), we get \(Q_2 = \gamma A, \gamma \in \mathbb{C}\), since \(\deg(A) = \deg(Q_2) = 3\). Divide by \(A\) to get

\[
\beta Q_2 + Q_1 + \lambda \gamma Q_6 = 0.
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Since \(\gcd(Q_1, Q_2) = 1\) implies that \(\lambda \gamma \neq 0\), we conclude that \(Q_6\) is a linear combination of \(Q_1\) and \(Q_2\). \(\square\)
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D. Points of Multiplicity $c$ when $n = 2c$ and $\mu = c$

The analysis of $\infty$-near triple points for $n = 6$ and $\mu = 3$ was inspired by a conversation with Ron Goldman. This led to joint work with Andy Kustin, Claudia Polini and Bernd Ulrich that is the final topic of Part I.

Let $C \subseteq \mathbb{P}^2$ be irreducible and rational of degree $n = 2c$ and $\mu = c$. By earlier results, all singular points have multiplicity $\leq c$.

**Proposition**

In this situation, there are at most three singular points multiplicity $c$, visible or infinitely near.

**Proof.**

Let $s = \#$ singular points of multiplicity $c$ (including $\infty$-near). Then

$$(2c - 1)(2c - 2) = (n - 1)(n - 2) = \sum_P m_P(m_P - 1) \geq s \cdot c(c - 1),$$

which easily implies $s \leq 3$. 

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Let $R = \mathbb{C}[s, t]$ and $R_{2c}$ be the vector space of homogeneous polynomials of degree $2c$. Note that $\dim R_{2c} = 2c + 1$.

Define $\mathcal{U} \subseteq R_{2c}^3$ to be

$$\mathcal{U} = \{(a, b, c) \in R_{2c}^3 \mid (a, b, c) \text{ gives a parametrization with } \mu = c\}$$

The set $\mathcal{U}$ is open and dense in $R_{2c}^3$. Thus $\dim \mathcal{U} = 2c + 1$.

Given $(a, b, c) \in \mathcal{U}$, the proposition implies that the corresponding rational curve has at most three points of multiplicity $c$.

**Problem**

Decompose $\mathcal{U}$ into pieces according to the number of points of multiplicity $c$. What can we say about the pieces?
Problem to Study

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Problem

Decompose $\mathcal{U}$ into pieces according to the number of points of multiplicity $c$. What can we say about the pieces?
The Partition For \( n = 2c, \mu = c \)

Partition

\[ U = \{(a, b, c) \in \mathbb{R}_{2c}^3 \mid (a, b, c) \text{ gives a parametrization with } \mu = c\} \]

into disjoint subsets as follows:

- \( S_{\emptyset} = \{(a, b, c) \in U \mid \text{no points of mult } c\} \)
- \( S_c = \{(a, b, c) \in U \mid \text{one point of mult } c\} \)
- \( S_{c,c} = \{(a, b, c) \in U \mid \text{two distinct points of mult } c\} \)
- \( S_{c,c,c} = \{(a, b, c) \in U \mid \text{three distinct points of mult } c\} \)
- \( S_{c:c} = \{(a, b, c) \in U \mid \text{one point & one } \infty \text{-near point of mult } c\} \)
- \( S_{c:c,c} = \{(a, b, c) \in U \mid \text{one point & one } \infty \text{-near point of mult } c, \text{ and one additional point of mult } c\} \)
- \( S_{c:c:c} = \{(a, b, c) \in U \mid \text{one point & two } \infty \text{-near points of mult } c\} \)

We call the \( S_i \) the \textbf{strata} of \( U \).
The Main Result

**Theorem (Cox, Kustin, Polini, Ulrich)**

The strata $S_i$ are open in their closures, irreducible, and fit into the diagram:

$$
\begin{align*}
S_{c:c:c}^{3c+7} & \rightarrow S_{c:c,c}^{3c+8} & S_{c:c}^{4c+6} & \rightarrow S_{c,c}^{4c+7} & \rightarrow S_{c}^{5c+5} & \rightarrow S_{\emptyset}^{6c+3} \\
S_{c,c,c}^{3c+9} & \rightarrow S_{c:c,c}^{3c+8} & S_{c:c}^{4c+6} & \rightarrow S_{c,c}^{4c+7} & \rightarrow S_{c}^{5c+5} & \rightarrow S_{\emptyset}^{6c+3}
\end{align*}
$$

Furthermore:

- Arrows mean "is contained in the closure of".
- Superscripts indicate the dimension of each strata.
- Recall $\dim \mathcal{U} = \dim R_{2c}^3 = 3(2c + 1) = 6c + 3$. 

A Hint of the Proof

To compute the dimension of $S_{c:c,c}$, the first step is to show that for suitable coordinates in $\mathbb{P}^2$ and a suitable basis of the moving lines, we have the normal form

$$A = \begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \\ 0 & Q_2 \end{pmatrix}$$

where the point of multiplicity $c$: $c$ is $(0, 0, 1)$ and the other point of multiplicity $c$ is $(1, 0, 0)$. Note:

- $\deg(Q_i) = c$ since $\mu = c$.
- The entry in the $Q_6$ position is $Q_2 = 0 \cdot Q_1 + 1 \cdot Q_2$.

Let $N_{c:c,c} = \left\{ \begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \\ 0 & Q_2 \end{pmatrix} \right\}$ be the set of normal forms.
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Let $N_{c:c,c} = \left\{ \begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \\ 0 & Q_2 \end{pmatrix} \right\}$ be the set of normal forms.
A Hint of the Proof, Continued

We have the normal form
\[
\begin{pmatrix}
Q_1 & 0 \\
Q_2 & Q_3 \\
0 & Q_2
\end{pmatrix}
\]
and the maps

\[
GL(3) \times N_{c:c,c} \times GL(2) \xrightarrow{\Phi} \{\text{H-B matrices for } c:c,c\} \xrightarrow{\Psi} S_{c:c,c},
\]

where \(\Phi\) is matrix multiplication and \(\Psi(A) = 2 \times 2\) minors of \(A\).

Furthermore:
- \(\Phi\) and \(\Psi\) are surjective.
- The generic fiber of \(\Phi\) has dimension 5 (takes proof).
- The generic fiber of \(\Psi\) has dimension 3 (easy to see).

Hence \(S_{c:c,c}\) is irreducible of dimension

\[
\dim S_{c:c,c} = 9 + 3(c + 1) + 4 - 5 - 3 = 3c + 8.
\]
References


We will now take a 10 minute break before resuming the course.
References


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Part II: Strata of Parametrized Curves in $\mathbb{P}^d$

A. $\mu$-Strata of Parametrized Curves in $\mathbb{P}^2$
   - Definition and Dimension of the $\mu$-Strata
   - Zariski Closure of the $\mu$-Strata

B. $\mu$-Types of Parametrized Curves in $\mathbb{P}^d$

C. $\mu$-Strata of Parametrized Curves in $\mathbb{P}^d$
   - Definition and Dimension of the $\mu$-Strata
   - Zariski Closure of the $\mu$-Strata

D. Proper Parametrizations

E. The Smallest $\mu$-Stratum
   - Smooth and Singular in $\mathbb{P}^3$
   - Ancestor Ideals and Rational Normal Scrolls

F. Comments on the Proofs
A. Strata of Parametrized Plane Curves

Let $R = \mathbb{C}[s, t]$ and $R_n = \{ F \in R \mid F \text{ is homogeneous, } \deg(F) = n \}$. To study parametrizations in $\mathbb{P}^2$ with the same $\mu$-type, let

$$\mathcal{P}_n \subseteq R_n^3$$

consist of relatively prime, linearly independent $(a, b, c)$ for which the parametrization is generically one-to-one. Note $\mathcal{P}_n$ is open in $R_n^3$.

For $1 \leq \mu \leq \lfloor n/2 \rfloor$, we have the $\mu$-stratum

$$\mathcal{P}_n^{\mu} = \{(a, b, c) \in \mathcal{P}_n \mid (a, b, c) \text{ has type } \mu \}.$$

**Theorem (Chen, Cox, Sederberg, 1998)**

$\mathcal{P}_n^{\mu}$ is open in its Zariski closure, irreducible, and has dimension

$$\dim(\mathcal{P}_n^{\mu}) = \begin{cases} 3(n + 1) & \text{if } \mu = \lfloor n/2 \rfloor \text{ (the generic case)} \\ 2n + 2\mu + 4 & \text{if } \mu < \lfloor n/2 \rfloor. \end{cases}$$
A. The Zariski Closure of a $\mu$-Stratum

The $\mu$-stratum $P^\mu_n$ is not closed. Let $\overline{P}^\mu_n$ be its Zariski closure in $P_n$.

Conjecture (Chen, Cox, Sederberg, 1998)

For every $1 \leq \mu \leq \lfloor n/2 \rfloor$, we have

$$\overline{P}^\mu_n = P_1^n \cup \cdots \cup P^\mu_n.$$  

Two Proofs

- In 2004, Carlos D’Andrea published a proof of (1).
- In 2004, Anthony Iarrobino published *Ancestor ideals of a vector space of forms*. The results of his paper imply (1) and a lot more.

However, it was not until 2012 that people realized the connection between (1) and Iarrobino’s paper. Around that time, he and I began a joint project to apply his results to parametrized curves in projective space of any dimension.
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A. The Zariski Closure of a $\mu$-Stratum

The $\mu$-stratum $\mathcal{P}_n^\mu$ is not closed. Let $\overline{\mathcal{P}}_n^\mu$ be its Zariski closure in $\mathcal{P}_n$.

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However, it was not until 2012 that people realized the connection between (1) and Iarrobino’s paper. Around that time, he and I began a joint project to apply his results to parametrized curves in projective space of any dimension.
Let $R = \mathbb{C}[s, t]$ and $R_n$ be as before. We will consider parametrizations

$$\varphi = (a_0, \ldots, a_d) : \mathbb{P}^1 \longrightarrow \mathbb{P}^d$$

where $a_0, \ldots, a_d \in R_n$ satisfy:

- $\gcd(a_0, \ldots, a_d) = 1$.
- $a_0, \ldots, a_d$ are linearly independent.

The first condition guarantees that $\varphi$ is defined on all of $\mathbb{P}^1$. The second condition implies that the image curve $C = \varphi(\mathbb{P}^1) \subseteq \mathbb{P}^d$ does not lie in a hyperplane.

**Definition**

$\mathcal{P}_{n,d} \subseteq R_n^{d+1}$ is the set of all $(a_0, \ldots, a_d) \in R_n^{d+1}$ such that $a_0, \ldots, a_d$ are relatively prime and linearly independent.
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$\mathcal{P}_{n,d} \subseteq R^{d+1}_n$ is the set of all $(a_0, \ldots, a_d) \in R^{d+1}_n$ such that $a_0, \ldots, a_d$ are relatively prime and linearly independent.
B. The $\mu$-Type of a Parametrization

A parametrization $(a_0, \ldots, a_d) \in \mathcal{P}_{n,d}$ gives the homogeneous ideal $I = \langle a_0, \ldots, a_d \rangle \subseteq R$. By the Hilbert Syzygy Theorem, there are integers $\mu_1, \ldots, \mu_d$ and an exact sequence

$$0 \rightarrow \bigoplus_{i=1}^{d} R(-n - \mu_i) \rightarrow R(-n)^{d+1} \rightarrow I \rightarrow 0.$$ 

- $\mu_i \geq 1$ for all $i$ by linear independence of the $a_i$.
- $\mu_1 + \cdots + \mu_d = n$.

**Definition**

If $\mu_1 \leq \cdots \leq \mu_d$, then the $\mu$-type of $(a_0, \ldots, a_d)$ is the $d$-tuple $\mu = (\mu_1, \ldots, \mu_d)$.

**Example**

If $(a, b, c) \in \mathcal{P}_{n,2}$ has type $\mu$ in the sense of Part I, then $\mu \leq n - \mu$, and the $\mu$-type of $(a, b, c)$ is $\mu = (\mu, n - \mu)$. 
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C. The $\mu$-Strata

**Definition**

A *$d$-part partition* of $n$ consists of integers $1 \leq \mu_1 \leq \cdots \leq \mu_d$ such that $n = \mu_1 + \cdots + \mu_d$.

**Definition**

If $\mu = (\mu_1, \ldots, \mu_d)$ is a $d$-part partition of $n$, then

$$\mathcal{P}_{n,d}^\mu = \{ (a_0, \ldots, a_d) \in \mathcal{P}_{n,d} \mid (a_0, \ldots, a_d) \text{ has type } \mu \}.$$ 

We call $\mathcal{P}_{n,d}^\mu$ a $\mu$-stratum.

**Comment on English Usage**

In English, a plural is often created by adding “s” to the singular, such as “course” and “courses”. But words with Latin origins often use “um” for singular and “a” for plural, such as “local maximum” and “local maxima”. The same is true for “stratum” and “strata”.

David A. Cox  (Amherst College)  
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C. The µ-Strata

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David A. Cox (Amherst College)
C. The Dimension of the $\mu$-Strata

**Theorem (Iarrobino)**

Let $\mu$ be a $d$-part partition of $n$. Then $\mathcal{P}_{n,d}^\mu$ is Zariski open in a subvariety of $R_{n}^{d+1} \simeq \mathbb{C}(d+1)(n+1)$, and $\mathcal{P}_{n,d}^\mu$ is irreducible with

$$\dim(\mathcal{P}_{n,d}^\mu) = (d + 1)(n + 1) - \sum_{i > j} \max(0, \mu_i - \mu_j - 1).$$

**Example**

When $d = 2$, we have $\mu = (\mu, n - \mu)$, so that

$$\dim(\mathcal{P}_{n,2}^\mu) = 3(n + 1) - \max(0, (n - \mu) - \mu - 1)$$

$$= \begin{cases} 
3(n + 1) & \text{if } n - \mu = \mu, \text{ i.e., } \mu = \lfloor n/2 \rfloor \\
2n + 2\mu + 4 & \text{if } n - \mu > \mu, \text{ i.e., } \mu < \lfloor n/2 \rfloor.
\end{cases}$$

This agrees with what we saw earlier.
C. The Dimension of the $\mu$-Strata

**Theorem (Iarrobino)**

Let $\mu$ be a $d$-part partition of $n$. Then $P^\mu_{n,d}$ is Zariski open in a subvariety of $R^{d+1}_n \cong \mathbb{C}^{(d+1)(n+1)}$, and $P^\mu_{n,d}$ is irreducible with

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This agrees with what we saw earlier.
We compare different $\mu$-types as follows.

**Definition**

Given $\mu$-types $\mu$ and $\mu'$, we define $\mu \leq \mu'$ provided

\[
\mu_1 \leq \mu'_1, \mu_1 + \mu_2 \leq \mu'_1 + \mu'_2, \ldots, \mu_1 + \cdots + \mu_d \leq \mu'_1 + \cdots + \mu'_d.
\]

**Remark**

We write $\mu$-types in ascending order. In representation theory and number theory, partitions of $n$ are usually written in descending order. The standard partial order is also defined differently but is essentially the reverse of the order defined here.

**Theorem (Iarrobino)**

The Zariski closure of $\mathcal{P}^\mu_{n,d}$ in $\mathcal{P}_{n,d}$ is

\[
\overline{\mathcal{P}^\mu_{n,d}} = \bigcup_{\mu' \leq \mu} \mathcal{P}^{\mu'}_{n,d}.
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C. Example: Degree $n$ in $\mathbb{P}^2$

Recall that when $d = 2$, we have $\mu = (\mu, n - \mu)$. At the beginning of Part II, we wrote $\mathcal{P}_{n,2}^\mu$ more simply as $\mathcal{P}_n^\mu$.

**Conjecture (Chen, Cox, Sederberg, 1998)**

For every $1 \leq \mu \leq \lfloor n/2 \rfloor$, we have

$$\overline{\mathcal{P}}_n^\mu = \mathcal{P}_n^1 \cup \cdots \cup \mathcal{P}_n^\mu.$$

**Proof.**

The key observation is that if $\mu = (\mu, n - \mu)$ and $\mu' = (\mu', n - \mu')$, then

$$\mu' \leq \mu \iff \mu' \leq \mu, \quad \mu' + (n - \mu') \leq \mu + (n - \mu) \iff \mu' \leq \mu.$$

The conjecture now follows immediately from Iarrobino’s theorem. \qed
Recall that when $d = 2$, we have $\mu = (\mu, n - \mu)$. At the beginning of Part II, we wrote $P^{\mu}_{n,2}$ more simply as $P^\mu_n$.

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**Proof.**

The key observation is that if $\mu = (\mu, n - \mu)$ and $\mu' = (\mu', n - \mu')$, then

$$\mu' \leq \mu \iff \mu' \leq \mu, \frac{\mu' + (n - \mu')}{n} \leq \frac{\mu + (n - \mu)}{n} \iff \mu' \leq \mu.$$

The conjecture now follows immediately from Iarrobino’s theorem. \qed
When $d = 3$, the smallest $n$ with incomparable $\mu$-types is $n = 9$:

$$(1, 4, 4) \text{ and } (2, 2, 5).$$

We write the stratum $\mathcal{P}_{9,3}^{(\mu_1, \mu_2, \mu_3)}$ as

$$(\mu_1, \mu_2, \mu_3)_{\text{dim}},$$

where “dim” is its dimension.

The closure of a stratum consists of the stratum and everything strictly below it in the diagram.
C. Example: Degree 9 in $\mathbb{P}^3$

When $d = 3$, the smallest $n$ with incomparable $\mu$-types is $n = 9$:

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D. Non-Proper Parametrizations

\( \mathbb{P}^1 \to \mathbb{P}^d \) is proper if it is generically one-to-one, i.e., its generic degree is 1. One expects that “most” parametrizations should be proper, i.e., being non-proper should be a rare phenomenon. Here is the result.

**Theorem (Cox, Iarrobino)**

- If \( \mu = (\mu_1, \ldots, \mu_d) \), then \( \mathcal{P}_{n,d}^{\mu} \) contains parametrizations of generic degree \( k > 1 \) if and only if \( k \mid \mu_i \) for all \( i \).

- Let \( k > 1 \) divide \( \mu \). Then the parametrizations in \( \mathcal{P}_{n,d}^{\mu} \) of generic degree \( k \) form a constructible subset of \( \mathcal{P}_{n,d}^{\mu} \) with irreducible Zariski closure of codimension

\[
(k - 1)(m(d + 1) - S - 2), \quad S = \sum_{i > j} \max(0, \tilde{\mu}_i - \tilde{\mu}_j),
\]

where \( m = n/k \) and \( \mu = k(\tilde{\mu}_1, \ldots, \tilde{\mu}_d) \). Furthermore:

- The codimension is \( \geq (k - 1)(d(d - 1) + 2m - 2) \geq 1 \).

- A generic parametrization in \( \mathcal{P}_{n,d}^{\mu} \) is proper.
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Theorem (Cox, Iarrobino)

- If $\mu = (\mu_1, \ldots, \mu_d)$, then $\mathcal{P}_{n,d}^\mu$ contains parametrizations of generic degree $k > 1$ if and only if $k \mid \mu_i$ for all $i$.

- Let $k > 1$ divide $\mu$. Then the parametrizations in $\mathcal{P}_{n,d}^\mu$ of generic degree $k$ form a constructible subset of $\mathcal{P}_{n,d}^\mu$ with irreducible Zariski closure of codimension

$$ (k - 1)(m(d + 1) - S - 2), \quad S = \sum_{i > j} \max(0, \tilde{\mu}_i - \tilde{\mu}_j), $$

where $m = n/k$ and $\mu = k(\tilde{\mu}_1, \ldots, \tilde{\mu}_d)$. Furthermore:

- The codimension is $\geq (k - 1)(d(d - 1) + 2m - 2) \geq 1$.

- A generic parametrization in $\mathcal{P}_{n,d}^\mu$ is proper.
D. Non-Proper Parametrizations

\( \mathbb{P}^1 \to \mathbb{P}^d \) is proper if it is generically one-to-one, i.e., its generic degree is 1. One expects that “most” parametrizations should be proper, i.e., being non-proper should be a rare phenomenon. Here is the result.

**Theorem (Cox, Iarrobino)**

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D. Example: Degree 9 in $\mathbb{P}^3$

Looking at the $\mu$-strata for $n = 9$ and $d = 3$ shows that non-proper parametrizations occur only for $\mu = (3, 3, 3)$ and have generic degree $k = 3$.

Since $\tilde{\mu} = (1, 1, 1)$ and $m = n/k = 3$, the theorem implies that the generic degree 3 locus has codimension

$$(k - 1)(m(d + 1) - S - 2) = (3 - 1)(3 \cdot (3 + 1) - 0 - 2) = 2 \cdot 10 = 20$$

in $\mathcal{P}^{(3,3,3)}_{9,3}$.

Since $\dim \mathcal{P}^{(3,3,3)}_{9,3} = 40$, the non-proper parametrizations have dimension 20. Comparing this to the other $\mu$-strata, we see that non-proper parametrizations are really rare in this case!
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E. The Smallest $\mu$-Stratum

The smallest $\mu$-type $\mu_{\text{min}} = (1, \ldots, 1, n - d + 1)$ features in:

- Goldman, Jia, Wang, Axial moving planes and singularities of rational space curves, 2009. They study $\mu$-bases in $\mathbb{P}^3$ and define axial moving planes which spin about a line (the axis). A curve of type $(1, 1, n - 2)$ has axial moving planes $p, q$ of degree 1 in $s, t$. Furthermore:
  - The curve is smooth $\iff$ the axes of $p, q$ are disjoint.
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$$A_1 = \begin{pmatrix} s & 0 & r_0 \\ t & 0 & r_1 \\ 0 & s & r_2 \\ 0 & t & r_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} s & 0 & r_0 \\ t & s & r_1 \\ 0 & t & r_2 \\ 0 & 0 & r_3 \end{pmatrix}.$$  

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$$A_1 = \begin{pmatrix} s & 0 & r_0 \\ t & 0 & r_1 \\ 0 & s & r_2 \\ 0 & t & r_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} s & 0 & r_0 \\ t & s & r_1 \\ 0 & t & r_2 \\ 0 & 0 & r_3 \end{pmatrix}. $$

Some observations about the normal forms:
- The first two columns of $A_1, A_2$ give axial moving planes (the 0s).
- For $A_1$, the axes are disjoint, but not for $A_2$ (location of the 0s).
- The parametrization is given by the $3\times3$ minors of the Hilbert-Burch matrix. Thus, for suitable $h_1, h_2$, we have
  \[ (a_0, a_2, a_2, a_3) = \begin{cases} (th_1, -sh_1, -th_2, sh_2) & \text{for } A_1 \\ (t^2 h_1, -sth_1, s^2 h_1, h_2) & \text{for } A_2 \end{cases} \]
- For $A_1$, the curve lies on $xw = yz$; for $A_2$, it lies on $xz = y^2$. 
E. Conclusion for \((1, 1, n - 2)\)

When we combine the Goldman/Jia/Wang and Kustin/Polini/Ulrich papers, we get the following nice result.

**Theorem**

Curves of type \(\mu_{\text{min}} = (1, 1, n - 2)\) in \(\mathbb{P}^3\) come in two flavors:

- The curve is smooth ⇔ its axes are disjoint ⇔ it lies on \(xw = yz\).
- The curve is singular ⇔ its axes intersect ⇔ it lies on \(xz = y^2\).

There is a lot more to say! The full story of \(\mu_{\text{min}} = (1, \ldots, 1, n - d + 1)\) is quite wonderful and involves

- ancestor ideals, and
- rational normal scrolls.

To introduce these ideas, we explore the algebra and geometry of \((1, 1, n - 2)\) for the smooth case, with Hilbert-Burch matrix \(A_1\).
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David A. Cox (Amherst College)
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E. Algebra and Geometry of $A_1$

\[(a_0, a_1, a_2, a_3, a_4) = (th_1, -sh_1, -th_2, sh_2)\]

**Algebra**

\[\langle h_1, h_2 \rangle\] is the ancestor ideal of \(\langle a_0, a_1, a_2, a_3, a_4 \rangle\) since for \(m \geq n\),

\[\langle h_1, h_2 \rangle_m = \langle a_0, a_1, a_2, a_3, a_4 \rangle_m.\]

**Theorem (Iarrobino, 2004)**

For \(\mu = (\mu_1, \ldots, \mu_d)\), the number of minimal generators of the ancestor ideal is

\[\tau = d + 1 - \#\{i \mid \mu_i = 1\}.\]

For \((1, 1, n-2)\), \(\tau = 3 + 1 - 2 = 2.\)

**Geometry**

\((a_0, a_1, a_2, a_3, a_4)\) is the sum

\[h_1(t, -s, 0, 0) + h_2(0, 0, -t, s).\]

This curve lies on the rational normal scroll \(S_{1,1}\), which consists of all sums

\[u(t, -s, 0, 0) + v(0, 0, -t, s)\]

for \((s, t), (u, v) \in \mathbb{P}^1\). \(S_{1,1}\) is the surface \(xw = yz\) seen earlier.

This all generalizes!
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This all generalizes!
The rational normal scroll $S_{a,b} \subseteq \mathbb{P}^{a+b+1}$ consists of all sums

$$u(s^a, s^{a-1}t, \ldots, t^a, 0, \ldots, 0) + v(0, \ldots, 0, s^b, s^{b-1}t, \ldots, t^b).$$

The “edges” of $S_{a,b}$ are rational normal curves of degrees $a$ and $b$ parametrized by $(s, t)$, and its “lines” parametrized by $(u, v)$.

Given a curve of type $\mu_{\min} = (1, \ldots, 1, n - d + 1)$, its ancestor ideal has

$$\tau = d + 1 - \#\{i \mid \mu_i = 1\} = d + 1 - (d - 1) = 2$$

minimal generators $h_1, h_2$. Set $a = n - \deg(h_1)$, $b = n - \deg(h_2)$, so

$$d + 1 = \dim \text{Span}(a_0, \ldots, a_d) = \dim R_a h_1 \oplus R_b h_2 = (a + 1) + (b + 1).$$

Hence $a + b = d - 1$. 
E. Stratify the Minimal $\mu$-Stratum

**Theorem (Cox, Iarrobino)**

Let $(a_0, \ldots, a_d)$ have type $(1, \ldots, 1, n - d + 1)$. If $h_1, h_2$ are minimal generators of the ancestor ideal and $a = n - \deg(h_1)$, $b = n - \deg(h_2)$, then $a + b = d - 1$ and the curve lies on the rational normal scroll $S_{a,b}$.

**Theorem (Iarrobino, 2004)**

The set of curves of type $(1, \ldots, 1, n - d + 1)$ has dimension given by $d^2 + d + 2n$, and the subset that lie on $S_{a,b}$, $a \leq b$, has dimension $d^2 + d + 2n - \max(0, b - a - 1)$.

When $d = 3$

$(a, b) = (1, 1)$ gives $xw = yz$, and $(a, b) = (0, 2)$ gives $xz = y^2$.

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- Instead of \((a_0, \ldots, a_d) \in R^{d+1}_n\) linearly independent, he uses the vector space \(V = \text{Span}(a_0, \ldots, a_d) \subseteq R_n\) of dimension \(d + 1\). Thus the Grassmannian \(\text{Grass}(d + 1, R_n)\) plays a central role.

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- Given a Hilbert function \(HF\), Iarrobino considers the “stratum”

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- Iarrobino computes \(\dim GA_{HF}(d + 1, n)\) and proves that

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- Instead of \((a_0, \ldots, a_d) \in R_n^{d+1}\) linearly independent, he uses the vector space \(V = \text{Span}(a_0, \ldots, a_d) \subseteq R_n\) of dimension \(d + 1\). Thus the Grassmannian \(\text{Grass}(d + 1, R_n)\) plays a central role.

- \(V \in \text{Grass}(d + 1, R_n)\) gives the ideal \(\langle V \rangle \subseteq R\) and Hibert function \(HF_V(m) = \dim(R/\langle V \rangle)_m\).

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Given $V = \text{Span}(a_0, \ldots, a_d)$, then the ideal $I = \langle V \rangle = \langle a_0, \ldots, a_d \rangle$ has free resolution

$$0 \to \bigoplus_{i=1}^{d} R(-n - \mu_i) \to R(-n)^{d+1} \to R \to R/I \to 0.$$ 

Since $HF_V(m) = \dim(R/\langle V \rangle)_m = \dim(R/I)_m$, it follows that

$$HF_V(m) = \dim R_m - (d + 1) \dim R_{m-n} + \sum_{i=1}^{d} \dim R_{m-n-\mu_i}.$$ 

Since $\dim R_\ell = \max(0, \ell + 1)$, it follows that

Knowing the Hilbert function $HF$

is equivalent to

Knowing the $\mu$-type $\mu = (\mu_1, \ldots, \mu_d)$. 

HF_V \leq HF_{V'} \iff \mu \geq \mu' \iff G_V \leq G_{V'}

Set \( G_V(m) = \sum_{i=1}^{d} \dim R_{m-n-\mu_i} \).

Then

\[
HF_V(m) = \dim R_m - (d + 1) \dim R_{m-n} + G_V(m).
\]

Thus

\[
HF_V \leq HF_{V'} \iff G_V \leq G_{V'}.
\]

This holds \( \iff \) the graph of \( G_V \) (see right) is below the graph of \( G_{V'} \).

It is a fun exercise to show that this is equivalent to \( \mu \geq \mu' \).
F. $HF_V \leq HF_{V'} \iff \mu \geq \mu'$

Set $G_V(m) = \sum_{i=1}^{d} \dim R_{m-n-\mu_i}$.

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$$HF_V \leq HF_{V'} \iff G_V \leq G_{V'}.$$ 

This holds $\iff$ the graph of $G_V$ (see right) is below the graph of $G_{V'}$.

It is a fun exercise to show that this is equivalent to $\mu \geq \mu'$. 

The Graph of $G_V$
References


