Graphs, Posets, Sudoku, and Gröbner Bases

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Overview

This course will cover four topics:

1. Graphs and Binomial Edge Ideals
2. Graphs and Coloring Ideals
3. Lattices and Join-Meet Ideals
4. Sudoku Puzzles

These topics are linked by the theory of Gröbner bases:

- Topics 1 and 2 use Gröbner bases in the statements of theorems.
- Topic 3 uses Gröbner bases in the proof of a theorem.
- Topic 4 is a fun application of Gröbner bases.

We begin with a quick review of Gröbner bases.
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Let \( R = k[x_1, \ldots, x_n] \) be a polynomial ring over a field \( k \).

**Definition**

A monomial order on \( R \) is a total order \( > \) on the monomials \( x^\alpha = x_1^{a_1} \cdots x_n^{a_n} \) such that

- \( x^\alpha > x^\beta \Rightarrow x^\alpha x^\gamma > x^\beta x^\gamma \) for all \( x^\gamma \).
- \( x^\alpha > 1 \) for all \( x^\alpha \neq 1 \).

A monomial order allows to define the leading term \( \text{LT}(f) = cx^\alpha \) of a nonzero polynomial \( f \in R \).

**Definition**

\( \{g_1, \ldots, g_t\} \) is a Gröbner basis of a nonzero ideal \( I \subseteq R \) if \( g_1, \ldots, g_t \in I \) and for every \( f \neq 0 \) in \( I \), \( \text{LT}(f) \) is divisible by some \( \text{LT}(g_i) \).

There are powerful algorithms for computing Gröbner bases.
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A Gröbner basis $G = \{g_1, \ldots, g_t\}$ of an ideal $I$ is a basis of $I$, i.e.,

$$I = \langle g_1, \ldots, g_t \rangle.$$ 

$G$ is reduced if every $g_i$ has the property that for all $j \neq i$, $\text{LT}(g_j)$ divides no term of $g_i$.

Once we fix a monomial order on $R$, every nonzero ideal $I \subseteq R$ has a unique reduced Gröbner basis.

(The Consistency Theorem) Over $\mathbb{C}$, a system of polynomial equations

$$f_1 = \cdots = f_s = 0$$

has no solutions if and only if $\{1\}$ is the reduced Gröbner basis of the ideal $I = \langle f_1, \ldots, f_s \rangle$. 
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A labeling of $G$ is a bijection $V(G) \simeq [n] = \{1, \ldots, n\}$. Given a labeling, we typically assume $V(G) = [n]$.

**Definition**

The binomial edge ideal of a labeled graph $G$ is the ideal $J_G$ in the polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ generated by the binomials

$$f_{ij} = x_i y_j - x_j y_i$$

for all $i, j$ such that $ij \in E(G)$ and $i < j$.

**Question**

When do the binomials $f_{ij}$ form a Gröbner basis of $J_G$?
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Closed Graphs

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A labeling of $G$ is **closed** if for all distinct edges $ji, ik \in E(G)$ with either $j > i < k$ or $j < i > k$, then $jk \in E(G)$.

A graph is **closed** if it has a closed labeling.

A labeling of $G$ gives a direction to each edge $ij \in E(G)$ where the arrow points from $i$ to $j$ when $i < j$. Then closed means the following:

Whenever the arrows point away from $i$ (on the left) or towards $i$ (on the right), closed means that $j$ and $k$ are connected by an edge.
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Theorem (Herzog, Hibi, Hreinsdóttir, Kahle, Rauh; Ohtani)

A labeling of $G$ is closed $\iff$ the $f_{ij}$ form a Gröbner basis of $J_G$ for the lex order satisfying

$$x_1 > \cdots > x_n > y_1 > \cdots > y_n.$$ 

Proof.

($\Leftarrow$) Suppose the $f_{ij}$ form a Gröbner basis and $i < j < l$ with $ij, il \in E(G)$. Then $J_G$ contains

$$f = y_l f_{ij} - y_j f_{il} = y_l (x_i y_j - x_j y_i) - y_j (x_i y_l - x_l y_i) = -x_j y_i y_l + x_l y_i y_j$$

and $\text{LT}(f) = -x_j y_i y_l$. This is divisible by $\text{LT}(f_{rs}) = x_r y_s$, $r < s$, $rs \in E(G)$. The only possibility is $\text{LT}(f_{jl}) = x_j y_l$ since $i < j < l$. Hence $jl \in E(G)$.

($\Rightarrow$) Use the Buchberger Criterion. Exercise!
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($\Rightarrow$) Use the Buchberger Criterion. **Exercise!**
When Does a Graph Have a Closed Labeling?

Not Closed

Closed!
Three Properties

A graph $G$ is

- **Chordal** if every cycle has a chord.

- **Claw-free** if

  ![Graph Diagram]

  is not an induced subgraph of $G$.

- **Narrow** if every shortest path $P$ of maximal length has the property that every vertex of $G$ either lies on $P$ or is adjacent to $P$.

**Proposition (Herzog, Hibi, Hreinsdóttir, Kahle, Rauh)**

A closed graph is chordal and claw-free.

**Exercise:** Prove this.
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A connected graph is closed ⇐⇒ it is chordal, claw-free and narrow.

Remark

The conditions of the theorem are independent of each other:

- A claw is narrow and chordal but not claw-free.
- A 4-cycle is narrow and claw-free but not chordal.
- The graph
  ![Triforce](image)
  is chordal and claw-free but not narrow (Legend of Zelda triforce).
Characterize Closed Graphs

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Definition

A *k*-coloring of a graph $G$ is a function from $V(G)$ to a set of $k$ colors such that adjacent vertices have distinct colors.

Example

This graph has a 3-coloring.
Graph Ideal

**Definition**

The \textit{k-coloring ideal} of \(G\) is the ideal \(I_{G,k} \subseteq \mathbb{C}[x_i \mid i \in V]\) generated by:

- \(x_i^k - 1\) for all \(i \in V(G)\)
- \(x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_ix_j^{k-2} + x_j^{k-1}\) for all \(ij \in E(G)\).

**Lemma**

\(V(I_{G,k}) \subseteq \mathbb{C}^n\) consists of all \(k\)-colorings of \(G\) for the set of colors consisting of the \(k^{th}\) roots of unity

\[\mu_n = \{1, \zeta_k, \zeta_k^2, \ldots, \zeta_k^{k-1}\}, \quad \zeta_k = e^{2\pi i / k}.\]

Proof. \[
\frac{(x_i^k - 1) - (x_j^k - 1)}{x_i - x_j} = x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_j^{k-1}.
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Two Observations

- $G$ has a $k$-coloring $\iff V(I_{G,k}) \neq \emptyset$.
- By the Consistency Theorem, there is a Gröbner basis criterion for the existence of a $k$-coloring.

3-Colorings

For 3-colorings, the ideal $I_{G,3}$ is generated by

- for all $i \in V(G)$: $x_i^3 - 1$
- for all $ij \in E(G)$: $x_i^2 + x_ix_j + x_j^2$.

These equations can be hard to solve!

Theorem

3-colorability is NP-complete.
The Existence of Colorings

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This example of a graph with a 3-coloring is due to Chao and Chen (1993).

Hillar and Windfeldt (2008) compute the reduced Gröbner basis of the graph ideal $I_{G,3}$ for lex with $x_1 > \cdots > x_{12}$.

The reduced Gröbner basis is:

\[
\begin{align*}
\{ & x_3^2 - 1, \ x_7 - x_{12}, \ x_4 - x_{12}, \ x_3 - x_{12}, \\
& x_1^2 + x_{11}x_{12} + x_{12}^2, \ x_9 - x_{11}, \ x_6 - x_{11}, \ x_2 - x_{11}, \\
& x_{10} + x_{11} + x_{12}, \ x_8 + x_{11} + x_{12}, \ x_5 + x_{11} + x_{12}, \\
& x_1 + x_{11} + x_{12} \}\.
\end{align*}
\]

Note $x_8 - x_{10}, \ x_5 - x_{10}, \ x_1 - x_{10} \in I_{G,3}$. 
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$$x_1 + x_{11} + x_{12} \}.$$

Note $x_8 - x_{10}, \ x_5 - x_{10}, \ x_1 - x_{10} \in I_{G,3}$. 
Uniquely $k$-Colorable Graphs

The Chao/Chen graph has essentially only one 3-coloring.

Definition

A graph $G$ is **uniquely $k$-colorable** if it has a unique $k$-coloring up to the permutation of the colors.

Hillar and Windfeldt show that unique $k$-colorability is easy to detect using Gröbner bases.

We start with a $k$-coloring of $G$ that uses all $k$ colors. Assume the $k$ colors occur among the last $k$ vertices. Then:

- Use variables $x_1, \ldots, x_{n-k}, y_1, \ldots, y_k$ with lex order

$$x_1 > \cdots > x_{n-k} > y_1 > \cdots > y_k.$$ 

- Use these variables to label the vertices of $G$. 

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- Use these variables to label the vertices of $G.$
Some Interesting Polynomials

Consider the following polynomials:

\[
y_k^k - 1
\]
\[
h_j(y_j, \ldots, y_k) = \sum_{\alpha_j+\ldots+\alpha_k=j} y_j^{\alpha_j} \cdots y_k^{\alpha_k}, \quad j = 1, \ldots, k - 1
\]
\[
x_i - y_j, \quad \mathrm{color}(x_i) = \mathrm{color}(y_j), \quad j \geq 2
\]
\[
x_i + y_2 + \cdots + y_k, \quad \mathrm{color}(x_i) = \mathrm{color}(y_1).
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In this notation, the Gröbner basis given earlier is:

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A Theorem

- $G$ has vertices $x_1, \ldots, x_{n-k}, y_1, \ldots, y_k$.
- $G$ has a $k$-coloring where $y_1, \ldots, y_k$ get all the colors.
- $\mathbb{C}[x_1, \ldots, x_{n-k}, y_1, \ldots, y_k]$ with lex $x_1 > \cdots > x_{n-k} > y_1 > \cdots > y_k$.

Using this data, we create:

- The coloring ideal $I_{G,k} \subseteq \mathbb{C}[x_1, \ldots, x_{n-k}, y_1, \ldots, y_k]$.
- The $n$ polynomials $g_1, \ldots, g_n$ given by
  
  $y_k^k - 1, \quad h_j(y_j, \ldots, y_k) \quad (j = 2, \ldots, k - 1), \quad y_1 + \cdots + y_k$
  $x_i - y_j \quad (x_i \text{ has color } y_j, \ j \geq 2), \quad x_i + y_2 + \cdots + y_k \quad (x_i \text{ has color } y_1)$.

Theorem (Hillar and Windfeldt)

The following are equivalent:

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Comments

- The theorem on the previous slide assumes that we know $k$ vertices that will carry distinct colors.
- Hillar and Windfeldt have a version of the theorem that does make this assumption.

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We will now take a 10 minute break before resuming the course.
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Lattices

Graphs are not the only combinatorial objects that give interesting ideals. Here we explore ideals associated to finite lattices.

A poset is a partially ordered set. All posets are assumed to be finite.

**Definition**

Let \((L, \geq)\) be a poset and fix \(a, b \in L\).

- \(a\) and \(b\) have a **join** if \(\{a, b\}\) has a least upper bound in \(L\) with respect to \(\geq\). If a join exists, it is unique and is denoted \(a \lor b\).

- \(a\) and \(b\) have a **meet** if \(\{a, b\}\) has a greatest lower bound in \(L\) with respect to \(\geq\). If a meet exists, it is unique and is denoted \(a \land b\).

**Definition**

A lattice is a poset \(L\) such that for every \(a, b \in L\), \(a \lor b\) and \(a \land b\) exist.
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**Definition**

A **lattice** is a poset $L$ such that for every $a, b \in L$, $a \lor b$ and $a \land b$ exist.
A lattice $L$ is:
- **distributive** if $a \land (b \lor c) = (a \land b) \lor (a \land c)$ for all $a, b, c \in L$.
- **modular** if $a \leq b$ implies $a \lor (c \land b) = (a \lor c) \land b$ for all $c \in L$.

**Example**

The power set $\mathcal{P}(A)$ of a finite set $A$ is partially ordered by inclusion. It is a lattice where join is $\cup$ and meet is $\cap$, and is distributive since

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

**Comments**
- Every distributive lattice is modular.
- The converse can fail: there exist modular lattices that are not distributive. We will soon see an example.
Distributive and Modular Lattices

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A lattice \( L \) is:

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Join-Meet Ideals

Definition

Let $L$ be a finite lattice and let $R$ be the polynomial ring whose variables are the elements of $L$. Then the join-meet ideal of $L$ is

$$I_L = \langle a b - (a \lor b)(a \land b) \mid a, b \in L \rangle \subseteq R.$$ 

A natural question concerns how properties of the lattice $L$ relate to properties of the ideal $I_L$. Here is a nice example.

Theorem (Hibi, 1987)

The join-meet ideal $I_L$ is prime if and only if the lattice $L$ is distributive.

We now discuss some of the interesting relations between Gröbner bases and join-meet ideals.
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We first give a Gröbner basis criterion for a lattice to be distributive.

**Theorem (Hibi, Qureshi)**

Let $L$ be a lattice. The following are equivalent:

1. $L$ is distributive.
2. $I_L$ is prime.
3. \[
\{ a \cdot b - (a \lor b)(a \land b) \mid a, b \in L \text{ incomparable} \}
\] is a Gröbner basis for $I_L$ for any monomial order satisfying $a \cdot b > (a \lor b)(a \land b)$ when $a, b$ are incomparable.

**Proof.**

(1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) was proved by Hibi in 1987.

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If $L$ is modular but not distributive, then $I_L$ is not prime. The next thing would be for $I_L$ to be radical. Recall:

- The radical of an ideal $I$ is $\sqrt{I} = \{ f \in R \mid f^m \in I \text{ for some } m \}$.
- $I$ is a radical ideal if $\sqrt{I} = I$.

There is a nice Gröbner basis criterion for an $I$ to be radical.

**Proposition**

Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis of $I$. If $\text{LT}(g_i)$ is square-free for every $i$, then $I$ is a radical ideal.

**Proof.**

If $\text{LT}(g_i)$ is square-free, then $\text{LT}(g_i) \mid \text{LT}(f^m) \Rightarrow \text{LT}(g_i) \mid \text{LT}(f)$.

**Exercise:** Complete the proof.
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**Proposition**

Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis of $I$. If $\text{LT}(g_i)$ is square-free for every $i$, then $I$ is a radical ideal.

**Proof.**

If $\text{LT}(g_i)$ is square-free, then $\text{LT}(g_i) \mid \text{LT}(f^m) \Rightarrow \text{LT}(g_i) \mid \text{LT}(f)$. 

**Exercise:** Complete the proof.
Modular Non-Distributive Lattices

If $L$ is modular but not distributive, then $I_L$ is not prime. The next thing would be for $I_L$ to be radical. Recall:

- The **radical** of an ideal $I$ is $\sqrt{I} = \{ f \in R \mid f^m \in I \text{ for some } m \}$.
- $I$ is a **radical ideal** if $\sqrt{I} = I$.

There is a nice Gröbner basis criterion for an $I$ to be radical.

**Proposition**

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**Exercise:** Complete the proof.
A Family of Examples

We next show that $I_L$ is radical for some modular non-distributive lattices $L$. Gröbner bases play a key role in the proof.

Here are two lattices. The one on the left is distributive; the one on the right is modular but not distributive. **Exercise: Prove this.**

The lattice on the right will be denoted $L_k$. 

![Diagram of two lattices]
$L_k$ is a Radical Modular Lattice

**Theorem (Ene and Hibi)**

The ideal $I_{L_k}$ is radical.

Before beginning the proof, we note that $I_{L_k}$ contains

$$x_{k+1}z - x_ky_{k+1}, \; y_kz - x_ky_{k+1}, \; x_{k+1}y_k - x_ky_{k+1}.$$  

Hence $I_{L_k}$ also contains the polynomials

$$(x_{k+1}z - x_ky_{k+1}) - (y_kz - x_ky_{k+1}) = x_{k+1}z - y_kz = (x_{k+1} - y_k)z$$

and (guided by the Buchberger Criterion)

$$y_k(x_{k+1}z - x_ky_{k+1}) - z(x_{k+1}y_k - x_ky_{k+1}) + (z - y_k)(y_kz - x_ky_{k+1}),$$

which simplifies to $y_kz^2 - y_k^2z$. 
$L_k$ is a Radical Modular Lattice

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which simplifies to $y_k z^2 - y_k^2 z$. 

We now sketch the proof that $I_{L_k}$ is radical.

**Proof.**

- **Step 1:** Write down a Gröbner basis of $I = I_{L_k}$. The basis includes the polynomials $y_k^2 z - y_k z^2$ and $(x_{k+1} - y_k)z$ from the previous slide. Since $\text{LT}(y_k^2 z - y_k z^2)$ is not square-free, we cannot use the Proposition to conclude that $I$ is radical.

- **Step 2:** Prove that $I = \langle I, x_{k+1} - y_k \rangle \cap \langle I, z \rangle$ using $(x_{k+1} - y_k)z \in I$ and Gröbner bases.

- **Step 3:** Prove that $\langle I, x_{k+1} - y_k \rangle$ and $\langle I, z \rangle$ have Gröbner bases with square-free leading terms.

- **Step 4:** By the Proposition, $\langle I, x_{k+1} - y_k \rangle$ and $\langle I, z \rangle$ are radical.

- **Step 5:** Then we are done since the intersection of radical ideals is again a radical ideal!
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References

- T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, in *Commutative Algebra and Combinatorics (Kyoto, 1985)*, North-Holland, Amsterdam, 1987, 93–109,


Classic Sudoku

Sudoku instructions:

Fill out the grid so that every row, every column and every $3 \times 3$ box contains the digits 1, 2, 3, 4, 5, 6, 7, 8, 9.

This Sudoku took me about 20 minutes (I am an average solver).
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This Sudoku took me about 20 minutes (I am an average solver).
A Sudoku gives a graph with:

- vertices = 81 squares
- edges = links between
  - squares in same $3 \times 3$
  - squares in same row
  - squares in same column

Every vertex has degree

- $8$ (from same $3 \times 3$) +
- $6$ (from same row, not in $3 \times 3$) +
- $6$ (from same column, not in $3 \times 3$)

$= 20$

Since $\sum_{\text{vertices}} \text{degree of vertex} = 81 \cdot 20 = 2 \cdot \#\text{edges}$, we see that the Sudoku graph has $810$ edges!
The Sudoku Graph

A Sudoku gives a graph with:

- vertices = 81 squares
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  - squares in same $3 \times 3$
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Every vertex has degree
8 (from same $3 \times 3$) +
6 (from same row, not in $3 \times 3$) +
6 (from same column, not in $3 \times 3$)
= 20

Since $\sum_{\text{vertices}} \text{degree of vertex} = 81 \cdot 20 = 2 \cdot \# \text{edges}$, we see that the Sudoku graph has 810 edges!
We can cast a Sudoku puzzle as a graph coloring problem!

Colors: \{1, 2, \ldots, 9\}

Goal: Extend the partial coloring to a full coloring.

In a properly constructed Sudoku, the partial coloring has a unique extension to a complete coloring.
We can cast a Sudoku puzzle as a graph coloring problem!

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Colors: \{1, 2, \ldots, 9\}

Goal: Extend the partial coloring to a full coloring.

In a properly constructed Sudoku, the partial coloring has a unique extension to a complete coloring.
To solve this Sudoku, use:

- 81 variables $x_{ij}$, $1 \leq i, j \leq 9$.
- Relabel the 9 variables for red squares as $y_1, \ldots, y_9$.
- The 9-coloring equations.
- Eight further equations
  \[ h_8(y_8, y_9) = h_7(y_7, y_8, y_9) = \cdots = h_1(y_1, \ldots, y_9) = 0 \]
  that make $y_1, \ldots, y_9$ distinct.
- The 16 equations $x_{31} = y_7$, $x_{33} = y_6$, $x_{37} = y_2$, \ldots

Assuming a unique solution, a Gröbner basis will consist of polynomials of the form $x_{i1} - y_i$, etc. that tell us how to fill in the blank squares!
**Problem:** The Gröbner basis method for solving a $9 \times 9$ Sudoku is extremely slow. *Singular* takes 20 minutes to solve a Sudoku.

**Alternative:** $4 \times 4$ Sudoku puzzles are much easier. We will consider:

The first has a **unique solution** while the second has **two solutions**.
The 4 × 4 Case

The Graph

The 4 × 4 Sudoku graph has 16 vertices, each of which has degree

\[3 (2 \times 2) + 2 \text{ (row)} + 2 \text{ (column)} = 7\]

Since \(16 \cdot 7 = 2 \cdot 56\), the 4 × 4 Sudoku graph has 56 edges.

The Equations

We will use 16 variables \(x_{11}, \ldots, x_{44}\), and the 4-coloring equations are

for the vertex \(ij\): \(x_{ij}^4 - 1 = 0\)

if an edge connects vertices \(ij, kl\): \(x_{ij}^3 + x_{ij}^2 x_{kl} + x_{ij} x_{kl}^2 + x_{kl}^3 = 0\)

There are \(16 + 56 = 72\) equations. The “colors” are 1, \(-1\), \(i\), \(-i\), where \(i = \sqrt{-1}\).
The $4 \times 4$ Case

The Graph

The $4 \times 4$ Sudoku graph has 16 vertices, each of which has degree

$$3 \times (2 \times 2) + 2 \times \text{(row)} + 2 \times \text{(column)} = 7$$

Since $16 \times 7 = 2 \times 56$, the $4 \times 4$ Sudoku graph has 56 edges.

The Equations

We will use 16 variables $x_{11}, \ldots, x_{44}$, and the 4-coloring equations are

for the vertex $ij$:

$$x_{ij}^4 - 1 = 0$$

if an edge connects vertices $ij, kl$:

$$x_{ij}^3 + x_{ij}^2 x_{kl} + x_{ij} x_{kl}^2 + x_{kl}^3 = 0$$

There are $16 + 56 = 72$ equations. The “colors” are $1, -1, i, -i$, where $i = \sqrt{-1}$. 
In Mathematica

\[ \text{In[1]} := S := \{-1 + x11^4, x11^3 + x11^2 x12 + x11 x12^2 + x12^3, -1 + x12^4, x11^3 + x11^2 x13 + x11 x13^2 + x13^3, x12^3 + x12^2 x13 + x12 x13^2 + x13^3, -1 + x13^4, x11^3 + x11^2 x14 + x11 x14^2 + x14^3, x12^3 + x12^2 x14 + x12 x14^2 + x14^3, x13^3 + x13^2 x14 + x13 x14^2 + x14^3, -1 + x14^4, x11^3 + x11^2 x21 + x11 x21^2 + x21^3, x12^3 + x12^2 x21 + x12 x21^2 + x21^3, -1 + x21^4, x11^3 + x11^2 x22 + x11 x22^2 + x22^3, x12^3 + x12^2 x22 + x12 x22^2 + x22^3, x21^3 + x21^2 x22 + x21 x22^2 + x22^3, -1 + x22^4, x13^3 + x13^2 x23 + x13 x23^2 + x23^3, x14^3 + x14^2 x23 + x14 x23^2 + x23^3, x21^3 + x21^2 x23 + x21 x23^2 + x23^3, x22^3 + x22^2 x23 + x22 x23^2 + x23^3, -1 + x23^4, x13^3 + x13^2 x24 + x13 x24^2 + x24^3, x14^3 + x14^2 x24 + x14 x24^2 + x24^3, x21^3 + x21^2 x24 + x21 x24^2 + x24^3, x22^3 + x22^2 x24 + x22 x24^2 + x24^3, x23^3 + x23^2 x24 + x23 x24^2 + x24^3, -1 + x24^4, \ldots, -1 + x44^4 \} \]

\[ \text{In[2]} := \text{Length}[S] \]

\[ \text{Out[2]} = 72 \]

Thanks to Trevor Hyde '12
The First Example

We will use 1 $\leftrightarrow$ 1, 2 $\leftrightarrow$ $i$, 3 $\leftrightarrow$ $-1$, 4 $\leftrightarrow$ $-i$.

Then we compute in Mathematica:

\[
\begin{align*}
\text{In[3]} & := x_{11} := 1; \ x_{21} := -i; \\
& \quad x_{41} := -1; \ x_{14} := -1; \ , \ x_{44} := 1
\end{align*}
\]

\[
\begin{align*}
\text{In[4]} & := \text{GroebnerBasis}[S, \{x_{12}, x_{13}, \\
& \quad x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33} \\
& \quad x_{34}, x_{42}, x_{43}\}] \\
\end{align*}
\]

\[
\begin{align*}
\text{Out[4]} & = \{-i + x_{43}, i + x_{42}, i + x_{34}, \\
& \quad 1 + x_{33}, -1 + x_{32}, -i + x_{31}, \\
& \quad -i + x_{24}, -1 + x_{23}, 1 + x_{22}, i + x_{13}, -i + x_{12}\}
\end{align*}
\]

Thus $x_{43} = i \leftrightarrow 2$, which we fill in. Then we fill in the rest. Solved!
The First Example

We will use $1 \leftrightarrow 1$, $2 \leftrightarrow i$, $3 \leftrightarrow -1$, $4 \leftrightarrow -i$.

Then we compute in Mathematica:

In[3] := $x_{11} := 1$; $x_{21} := -i$;
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Out[4] = {$-i + x_{43}$, $i + x_{42}$, $i + x_{34}$, $1 + x_{33}$, $-1 + x_{32}$, $-i + x_{31}$, $-i + x_{24}$, $-1 + x_{23}$, $1 + x_{22}$, $i + x_{13}$, $-i + x_{12}$}

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\( x41 := -1; \ x14 := -1; \ x44 := 1 \)

In[4] := GroebnerBasis[S, \{x12, x13, x22, x23, x24, x31, x32, x33, x34, x42, x43\}]

Out[4] = \{\(-i + x43, \ i + x42, \ i + x34, \ 1 + x33, \ -1 + x32, \ -i + x31, \ -i + x24, \ -1 + x23, \ 1 + x22, \ i + x13, \ -i + x12\}\}

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The Second Example

We first clear $x21$ and then proceed as before:

\[
\begin{align*}
\text{In}[5] & := \text{Clear}[x21] \\
\text{In}[6] & := \text{GroebnerBasis}[S, \{x12, x13, x21, x22, x23, x24, x31, x32, x33, x34, x42, x43\}] \\
\text{Out}[6] & = \{1 + x43^2, x42 + x43, x34 + x43, x31 - x43, x24 - x43, x21 + x43, x13 + x43, x12 - x43, 1 + x33, -1 + x32, -1 + x23, 1 + x22\}
\end{align*}
\]

\[1 + x43^2 = 0\]
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```
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```

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```

```
Out[6] = \{1 + x_{43}^2, x_{42} + x_{43}, x_{34} + x_{43}, x_{31} - x_{43},
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Out[6] = \{1 + x_{43}^2, x_{42} + x_{43}, x_{34} + x_{43}, x_{31} - x_{43}, 
\frac{x_{24} - x_{43}}{x_{21} + x_{43}}, x_{13} + x_{43}, x_{12} - x_{43}, 
1 + x_{33}, -1 + x_{32}, -1 + x_{23}, 1 + x_{22}\}

\[1 + x_{43}^2 = 0\]
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Conclusion

We have seen some interesting combinatorial objects:

- Closed Graphs.
- Graph Colorings and Sudoku.
- Distributive and Modular Lattices.

Each of these led to an ideal in a polynomial ring, and to understand the ideals, the key player was the theory of Gröbner bases.

In this course, I have presented a small sample of the amazing things that you can do with Gröbner bases.

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