THE POSITIVE DEFINITENESS OF A CLASS OF POLYNOMIALS FROM THE GLOBAL STABILITY ANALYSIS OF LOTKA-VOLterra SYSTEMS

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Abstract. In this paper, a class of Lotka-Volterra discrete diffusion systems is considered. A mechanical procedure for checking the positive definiteness of polynomials from the stability analysis of these diffusion systems is described. Some known results of lower dimensional systems are checked and the Hofbauer-So-Takeuchi conjecture is proved in the case of \( n = 4 \) based on the proposed procedure and the computer algebraic system MATHEMATICA.

Keywords – Lotka-Volterra systems, Global stability, Positive Definite, Polynomials

1. INTRODUCTION

Recently, the theory of monotone flow of solutions of systems of differential equations is well developed [?, ?, ?, ?] and has been applied to some concrete systems, e.g., [?], [?], [?], [?], [?], [?]. For Lotka-Volterra system with or without diffusion, [?], [?], [?], this theory plays a very important role in the analysis of permanence (which means that there is a compact region \( K \) in the interior of \( \mathbb{R}^+ \) such that all the solutions of the considered system with positive initial conditions ultimately enter \( K \)) and global stability of the system. In [?], it is shown, among other things, that two Lotka-Volterra competition diffusion systems which were suggested by computer simulation to have a globally stable equilibrium point [?] are really globally stable.

In [?], the global stability of the systems is proved by checking directly the uniqueness of a positive equilibrium point based on the recently developed theory of numerically determining solutions of systems of polynomial equations [?]. On the other hand, by using computers, it is showed in [?] that each product of \((-1)^i\) and the \(i\)-th leading principal minor of the Jacobian matrix of the considered system is positive definite. This is equivalent to the stability of the Jacobian matrix and implies, according to the monotonicity of the system, the uniqueness of a positive equilibrium point.

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In this paper, we modify the technique in [?] and [?] and propose a mechanical method of manipulating polynomials. One method is applied to check the known results of [?] mechanically and to show that the conjecture proposed in [?] is true in the case of \( n = 4 \), by showing that a class of polynomials (with the largest one having 40549 terms) are positive definite. In this paper, all the polynomials are considered in the interior of \( \mathbb{R}^+ \).

Section 2 contains some background concepts and fundamental results. In Section 3, we describe the mechanical manipulating procedure, modified from the method of [?] and [?], for checking the positive definiteness of polynomials. Some known results are illustrated by this procedure in Section 4. The Hofbauer-So-Takeuchi’s conjecture for \( n = 4 \) is proved in Section 5.

2. BACKGROUND CONCEPTS AND RESULTS

In this paper, we consider the global stability of the two-species Lotka-Volterra competitive discrete diffusion system

\[
\begin{align*}
\dot{u}_i &= u_i(r_i - au_i - bv_i) + \sum_{j=1}^{n} d_{ij}u_j, \\
\dot{v}_i &= v_i(s_i - cu_i - dv_i) + \sum_{j=1}^{n} e_{ij}v_j, \quad i = 1, \ldots, n.
\end{align*}
\]  

(1)

Here \( a, b, c \) and \( d \) are positive constants with \( ad - bc > 0 \), and \( d_{ij}, e_{ij} \) (\( i \neq j \)) are nonnegative constants with \( d_{ii}, e_{ii} < 0 \) and

\[
\begin{align*}
d_{ii} + \sum_{j \neq i} d_{ij} &\leq 0, \\
e_{ii} + \sum_{j \neq i} e_{ij} &\leq 0.
\end{align*}
\]  

(2)

The matrices \( D = (d_{ij})_{n \times n} \) and \( E = (e_{ij})_{n \times n} \) are supposed to be irreducible. For the biological meaning of this system and the parameters, see [?, ?].

In the sense of Smith [?], system (1) is a \( 2n \)-dimensional \( K \)-monotone system with respect to the following ordering on \( \mathbb{R}_+^n \times \mathbb{R}_+^n \):

\[
(u, v) \leq (\bar{u}, \bar{v}) \quad \text{iff} \quad u_i \leq \bar{u}_i, \quad v_i \geq \bar{v}_i.
\]

It is easy to check that solutions starting in \( \mathbb{R}_+^n \times \mathbb{R}_+^n \) remain there, that is, the state space \( \mathbb{R}_+^n \times \mathbb{R}_+^n \) is positively invariant. Clearly, the origin \( E_0 = (0, \ldots, 0; 0, \ldots, 0) \) is an equilibrium point. For any \( d_{ij} \), there exists a boundary equilibrium point in the positive \( u \) subspace with the form \( E_u^n = (\bar{u}_1, \ldots, \bar{u}_n; 0, \ldots, 0) \), where \( (\bar{u}_1, \ldots, \bar{u}_n) \) is the globally stable positive equilibrium point of the system [?]

\[
\dot{u}_i = u_i(r_i - au_i) + \sum_{j=1}^{n} d_{ij}u_j, \quad i = 1, \ldots, n.
\]
Similarly, for any $e_{ij}$, there exists another boundary equilibrium point $E^n_{u} = (0, ..., 0; \bar{v}_1, ..., \bar{v}_n)$, where $(\bar{v}_1, ..., \bar{v}_n)$ is the globally stable positive equilibrium point of

$$\dot{v}_i = v_i(s_i - dv_i) + \sum_{j=1}^{n} e_{ij} v_j, \quad i = 1, ..., n.$$  

The Jacobian matrix of system (1) is given by

$$J = \begin{pmatrix} d_{ij} + \delta_{ij}(f - au_i) & -\delta_{ij}bu_i \\ -\delta_{ij}cv_i & e_{ij} + \delta_{ij}(g - dv_i) \end{pmatrix},$$

where $f = r_i - au_i - bv_i$, $g = s_i - cu_i - dv_i$ and $\delta_{ij}$ is the Kronecker delta.

The spectrum of matrix $J$, written $\sigma(J)$, is the set of eigenvalues of $J$. Define the stability modulus of $J$, $s(J)$, as

$$s(J) = \max\{Re\lambda : \lambda \in \sigma(J)\}.$$  

For system (1), the following result is well-known [?, ?, ?, ?].

**LEMMA 1.** 1) If there is no positive equilibrium point, then one of $E^n_{u}$ and $E^n_{v}$ is globally stable;

2) If both $E^n_{u}$ and $E^n_{v}$ are unstable (i.e., $s(J(E^n_{u})) > 0$ and $s(J(E^n_{v})) > 0$), then system (1) is permanent and there exist positive equilibria $E^*$ and $E^*$ of system (1) with $E^n_{u} > E^* \geq E^*$ such that all the solutions of (1) in $intR^n_{+} \times R^n_{+}$ enter the ordered interval $[E^*, E^*]$. Furthermore, if the positive equilibrium point is unique, then it is globally stable.

If the $K$-monotone system (1) has two or more positive equilibrium points, then there must be a stable one and an unstable one [?, ?]. Therefore, if we can show that each equilibrium is locally asymptotically stable, then the uniqueness of the positive equilibrium point is guaranteed.

setting the right side hand of (1) to zero, we obtain the following *equilibrium equations* which are satisfied by all the positive equilibrium points of (1):

$$u_i(r_i - au_i - bv_i) + \sum_{j=1}^{n} d_{ij}u_j = 0,$$

$$v_i(s_i - cu_i - dv_i) + \sum_{j=1}^{n} e_{ij} v_j = 0, \quad i = 1, ..., n. \quad (3)$$

From [?, ?, ?, ?] and Lemma 1, we have

**LEMMA 2.** The following statements are equivalent:

1) System (1) is globally stable (which means that system (1) has a globally stable nonnegative equilibrium point);

2) Each positive equilibrium point is locally asymptotically stable;

3) At each positive equilibrium point $E$, $s(J(E)) < 0$;

4) At each positive equilibrium point $E$, the Jacobian matrix $J(E)$ is stable, i.e., the leading principal minors of $J(E)$ alternate in sign (starting from negative).
From Lemma 2, it is clear that in order to prove global stability of system (1), it suffices to show that under the equilibrium equations (3), the leading principal minors of $J(E)$ alternate in sign.

In [7], Hofbauer et al. posed the following conjecture.

**Hofbauer-So-Takeuchi Conjecture**: Under the assumption (2) with $ad - bc > 0$, the “n-patch” system (1) is globally stable, provided that the dispersal matrices $D = (d_{ij})_{n \times n}$ and $E = (e_{ij})_{n \times n}$ are symmetric.

In [7], the conjecture is shown to be true in the cases when $n = 2$ and $3$. The main result of the present paper is to show that the conjecture holds true in the case of $n = 4$.

3. THE PD-ALGORITHM

To illustrate our method, consider the following $4 \times 4$ matrix $m_{44}$ which is just the Jacobian of system (1) for $n = 2$.

$$m_{44} = \begin{pmatrix}
-d_{12} \frac{u_2}{u_1} - au_1 & d_{12} & -bu_1 & 0 \\
-d_{21} \frac{u_1}{u_2} - au_2 & 0 & -bu_2 & e_{12} \\
-cv_1 & 0 & -dv_1 & e_{21} \\
0 & -cv_2 & e_{12} & -e_{21} \frac{u_2}{u_1} - dv_2
\end{pmatrix}. \tag{4}
$$

We assume equal dispersal rate case, i.e., $d_{ij} = d_{ji}$ and $e_{ij} = e_{ji}$.

To prove the global stability of this system, by Lemma 2, we only need to show that each determinant of the $i$-th order principal submatrix of $m_{44}$ has sign $(-1)^i$ ($i = 1, 2, 3, 4$). It is easy to check that each determinant of the $i$-th order principal submatrix of $m_{44}$ has sign $(-1)^i$ for $i = 1, 2, 3, 4$.

Since $ad - bc > 0$, we can replace $d$ by $\frac{bc}{a} + r$ with a certain positive $r$ and expand $\text{Det}[m_{44}]$ with the positive denominator removed to obtain a polynomial denoted by $DM4(u_i, v_i; r, a, b, c, d_{ij}, e_{ij})$ of the variables $r$, $a, b, c, d_{ij}$ and $e_{ij}$ ($i < j; i, j = 1, 2, 3, 4$) and parameters $u_i$ and $v_i$ ($i, j = 1, 2, 3, 4$). $DM4$ has 15 terms. Next we check whether $DM4$ has a negative monomial or not. If the answer were negative, then the positive definiteness of $DM4$ would be proved. But, in this case, the answer is positive.

The PD-Algorithm for checking the positive definitness of $DM4$ can be described as follows.

Step 1. Write $DM4$ as a polynomial of the variable $r$ with maximal degree 2 as follows:

$$DM4(u_i, v_i; r, a, b, c, d_{ij}, e_{ij}) = DM40(u_i, v_i; b, c, d_{ij}, e_{ij}) + DM41(u_i, v_i; a, b, c, d_{ij}, e_{ij})r + DM42(u_i, v_i; a, b, c, d_{ij}, e_{ij})r^2,$$

where $DM40$, $DM41$ and $DM42$ have 4, 8 and 3 terms, respectively. Since all the terms in $DM41$ and $DM42$ are positive, we only need to consider $DM40$. Note that the variables $a$ and $r$ are absent from $DM40$. 

Repeating the above procedure, we can eliminate all the variables \( b, c, d_{12} \) and \( e_{12} \) in \( DM4 \) to obtain a set \( RS \) of polynomials which contains only the parameters \( u_i \) and \( v_i \). In the above case, \( RS = \{ f \} \), where

\[
f = u_2^3v_1^3 - u_1u_2^2v_1^2v_2 - u_1^2u_2v_1v_2^2 + u_1^3v_2^3.
\]

Clearly, \( f = (u_2v_1 - u_1v_2)^2(u_2v_1 + u_1v_2) \geq 0 \). Since all the terms in \( DM41 \) and \( DM42 \) are positive, \( DM4 \) is positive definite.

In the general case, we are dealing with a more complicated polynomial. The set \( RS \) so obtained consists of finitely many polynomials of \( u_i \) and \( v_i \), i.e., \( RS = \{ f_1, ..., f_n \} \). Clearly, the positive definiteness of \( RS = \{ f_1, ..., f_n \} \) (i.e., all \( f_i \) are positive semidefinite and at least one is positive definite) implies that of the original polynomial. After obtaining the set \( RS \), we need to simplify it.

**Step 2.** Finding the quotation space \( RS1 \) of \( RS \).

**Definition.** Two polynomials \( f_1(x_1, ..., x_n) \) and \( f_2(y_1, ..., y_m) \) are equivalent if \( n = m \) and there is a replacement \( y_i \rightarrow x_j \) \((i, j = 1, ..., n)\) such that \( f_1(x_1, ..., x_n) = f_2(y_1, ..., y_n) \).

According to the above definition, we can get the quotation space \( RS1 \) of \( RS \) with respect to the equivalent relation in the definition. Obviously, the positive definiteness of polynomials in \( RS1 \) is equivalent to that of \( RS \).

**Step 3.** Removing the known polynomials which appear in the \( RS \)'s of the preceding minors.

In a concrete manipulation, we will consider all the leading principal minors of a matrix, and some polynomials may appear in many \( RS \)'s of these minors. Clearly, we do not need to consider those positive semidefinite polynomials already appeared in smaller minors, when we deal with a larger one. After this step, we obtain \( RS2 \) from \( RS1 \). Clearly, the positive definiteness of \( RS2 \) implies that of \( RS1 \).

**Step 4.** Removing the positive factors.

After getting \( RS2 \), we factorize each \( f_i \) in \( RS2 \) and removing its positive factors. This gives us a set \( RS3 \) of polynomial whose positive definiteness is equivalent to that of \( RS2 \).

**Step 5.** Eliminating unrelated positive terms.

For each polynomial \( f_i \) in \( RS3 \), we can eliminate those monomials in which all the variables are independent of the variables in the negative monomials to obtain a simplified \( RS4 \). Clearly, the positive definiteness of \( RS4 \) implies that of \( RS3 \).

After the above five steps, we obtain a set (denoted by \( PS \)) of polynomials whose positive definiteness will be checked. 1) If \( PS = \{ \} \) (empty), then the considered \( i \)-th principal minor has sign \((-1)^i\). 2) If \( PS = \{-1\} \), then this procedure cannot determine the sign of the minor. 3) In general, \( PS = \{ f_1, ..., f_n \} \). That each \( f_i \) being nonnegative and at least one being positive ensures the positive definiteness of the minor.

It should be noticed that only in case 1), the answer is positive. In cases 2) and 3) with some indefinite polynomials, we have no conclusion.

**REMARK.** In the above procedure, we can choose any order of independent variables \( r, a, b, c, d_{ij} \) and \( e_{ij} \) (or \( k_{ij} \) in the following examples).
4. EXAMPLES

Our aim in this section is to illustrate how the method developed in Section 3 can be applied to the stability analysis in certain known $2 \times 3$-dimensional systems \[ ? \].

**Definition.** A matrix is said to have Property (P), if its leading principal minors alternate in sign (starting from negative).

We consider the general form of matrix $m_{6i}$ which is the Jacobian matrix (2) of system (1) in the case of $n = 3$. Its principal submatrices are denoted by $m_{6i} (i = 1, 2, 3, 4, 5)$. It is easy to know that for $i = 1, 2, 3$, the determinants of $m_{6i}$ have sign $(-1)^i$. By using the PD-Algorithm, we can check if the Jacobian possess the Property (P) in the following cases easily.

Case 1. Coordinated migration, i.e.,

$$d_{ji}/d_{ij} = e_{ji}/e_{ij} = k_{ij} \neq 0,$$

for $1 \leq i < j \leq 3$ with $k_{12}k_{23} = k_{13}$. (5)

Substituting $d_{ji} = d_{ij}k_{ij}$ and $e_{ji} = e_{ij}k_{ij}$ with (5) into $m_{6i}$ and manipulating $m_{6i} (i = 4, 5, 6)$ lead to

$$PD[m_{64}] = \{ \}, \quad PD[(-1)^5m_{65}] = \{ \}, \quad PD[m_{66}] = \{ f_{11} \}.$$

Since $f_{11} = 2u_2^2v_1^2 - 3u_1v_2v_1^2 + 2u_1^2v_2^2 > 0$, the Jacobian has Property (P) in this case.

Case 2. Linearly linked with coordinate dispersion, i.e.,

$$d_{21}/d_{12} = e_{21}/e_{12} = k_{12}, \quad d_{32}/d_{23} = e_{32}/e_{23} = k_{23},$$

and

$$d_{13} = d_{31} = e_{13} = e_{31} = 0.$$ (7)

By using the conditions (6) and (7), we can change $m_{6i}$ to functions involving the independent variables $k_{12}$ and $k_{23}$ instead of $d_{21}, d_{32}, e_{21}$ and $e_{32}$. The final RS's of $m_{6i}$ $i = 4, 5, 6$ take the form

$$PD[m_{64}] = \{ \}, \quad PD[(-1)^5m_{65}] = \{ -1 \}, \quad PD[M_{66}] = \{ f_{12} \},$$

where $f_{12} = f_{11}$. Clearly, the Jacobian has Property (P) in this case.

Case 3. Equal dispersal rate, i.e., $d_{ij} = e_{ij}$ for $i \neq j; i, j = 1, 2, 3$. The $PD[m_{6i}] i = 4, 5, 6$ take the form

$$PD[m_{64}] = \{ \}, \quad PD[(-1)^5m_{65}] = \{-1\},$$

$$PD[m_{66}] = \{ 18 \text{ polynomials} \}.$$

Since $\{-1\}$ appears in $PD[(-1)^5m_{65}]$, we do not know if the Jacobian has Property (P) or not by the PD-algorithm.
Case 4. Cyclically Linked, i.e., $d_{13} = d_{32} = d_{21} = 0 = e_{13} = e_{32} = e_{21}$. The result sets of $PD[m_{6i}]$ $i = 4, 5, 6$ take the form

$$PD[m_{64}] = \{\}, \quad PD[(-1)^5m_{65}] = \{-1\}, \quad PD[m_{66}] = \{-1\}.$$ 

Since, $\{-1\}$ appears in both $PD[(-1)^5m_{65}]$ and $PD[m_{66}]$, we do not know whether the Jacobian has Property ($P$). In fact, Hofbauer et al. [7] have given an example to show that the Property ($P$) does not hold in this case in general. In their example, the parameters are taken as, for the dispersion: $d_{11} = e_{11} = -200, d_{22} = d_{33} = e_{22} = e_{33} = -100, d_{12} = d_{23} = d_{31} = e_{12} = e_{23} = e_{31} = 0, d_{21} = d_{32} = e_{21} = e_{32} = 1$ and $d_{13} = e_{13} = 50$; for the interaction coefficients: $a = 1.001$, and $b = c = d = 1$; for growth rates: $r_1 = 52.001, r_2 = 102.502, r_3 = 310009/3000, s_1 = 152, s_2 = 102$ and $s_3 = 103$. It is checked that [7] $(u^*; v^*) = (1, 2, 3, 1, 1, 1)$ is a positive equilibrium of system (1). The determinant of the Jacobian at $(u^*; v^*)$ being negative implies that this positive equilibrium is unstable.

5. Hofbauer-So-Takeuchi Conjecture for n=4

In this section, we give the proof of the main result of the present paper.

**THEOREM.** Hofbauer-So-Takeuchi conjecture is true for n=4.

**Proof.** From Lemma 2, it is clearly that the theorem is implied by the Property ($P$) of the Jacobian matrix of the corresponding eight ($2 \times 4$)-dimensional system (1).

The Jacobian matrix of system (1) for $n = 4$ simplified by the equilibrium equations (2) takes the form

$$m_{88} = \begin{pmatrix}
a_{11} & d_{12} & d_{13} & d_{14} & -bu_1 & 0 & 0 & 0 \\
d_{21} & a_{22} & d_{23} & d_{24} & 0 & -bu_2 & 0 & 0 \\
d_{31} & d_{32} & a_{33} & d_{34} & 0 & 0 & -bv_3 & 0 \\
d_{41} & d_{42} & d_{43} & a_{44} & 0 & 0 & 0 & -bu_4 \\
-cv_1 & 0 & 0 & 0 & a_{55} & e_{12} & e_{13} & e_{14} \\
0 & -cv_2 & 0 & 0 & e_{21} & a_{66} & e_{23} & e_{24} \\
0 & 0 & -cv_3 & 0 & e_{31} & e_{32} & a_{77} & e_{34} \\
0 & 0 & 0 & -cv_4 & e_{41} & e_{42} & e_{43} & a_{88}
\end{pmatrix}, \quad (8)$$

where $a_{11} = (-au_1^2 - d_{12}u_2 - d_{13}u_3 - d_{14}u_4)/u_1, a_{22} = (-d_{21}u_1 - au_2^2 - d_{23}u_3 - d_{24}u_4)/u_2, a_{33} = (-d_{31}u_1 - d_{32}u_2 - au_3^2 - d_{34}u_4)/u_3, a_{44} = (-d_{41}u_1 - d_{42}u_2 - d_{43}u_3 - au_4^2)/u_4, a_{55} = (-dv_1^2 - e_{12}v_2 - e_{13}v_3 - e_{14}v_4)/v_1, a_{66} = (-e_{21}v_1 - dv_2^2 - e_{23}v_3 - e_{24}v_4)/v_2, a_{77} = (-e_{31}v_1 - e_{32}v_2 - dv_3^2 - e_{34}v_4)/v_3$ and $a_{88} = (-e_{41}v_1 - e_{42}v_2 - e_{43}v_3 - dv_4^2)/v_4$.

Substituting $d_{ij} = d_{ji}, e_{ij} = e_{ij}$ ($i \leq j; i \neq j$) and $d = \frac{b}{a} + r$ ($r > 0$) into (8), multiplying
it by $a$, we obtain the following matrix

$$m_{8i} = \begin{pmatrix}
-a_{11} & d_{12}u_1 & d_{13}u_1 & d_{14}u_1 & -bu_1^2 & 0 & 0 & 0 \\
-d_{12}u_2 & a_{22} & d_{23}u_2 & d_{24}u_2 & 0 & -bu_2^2 & 0 & 0 \\
-d_{13}u_3 & d_{23}u_3 & a_{33} & d_{34}u_3 & 0 & 0 & -bu_3^2 & 0 \\
-d_{14}u_4 & d_{24}u_4 & d_{34}u_4 & a_{44} & 0 & 0 & 0 & -bu_4^2 \\
-cv_1^2 & 0 & 0 & 0 & a_{55} & e_{12}v_1a & e_{13}v_1a & e_{14}v_1a \\
0 & -cv_2^2 & 0 & 0 & e_{12}v_2a & a_{66} & e_{23}v_2a & e_{24}v_2a \\
0 & 0 & -cv_3^2 & a & e_{13}v_3a & e_{23}v_3a & a_{77} & e_{34}v_3a \\
0 & 0 & 0 & -cv_4^2 & e_{14}v_4a & e_{24}v_4a & e_{34}v_4a & a_{88}
\end{pmatrix}, \quad (9)
$$

where $a_{11} = -au_1^2 - d_{12}u_2 - d_{13}u_3 - d_{14}u_4$, $a_{22} = -d_{12}u_1 - au_2^2 - d_{23}u_3 - d_{24}u_4$, $a_{33} = -d_{13}u_1 - d_{23}u_2 - au_3^2 - d_{34}u_4$, $a_{44} = -d_{14}u_1 - d_{24}u_2 - d_{34}u_3 - au_4^2$, $a_{55} = -(\frac{bc}{a} + r)v_1^2a - e_{12}v_1a - e_{13}v_2a - e_{14}v_3a$, $a_{66} = -e_{12}v_2a - (\frac{bc}{a} + r)v_2^2a - e_{23}v_3a - e_{24}v_4a$, $a_{77} = -e_{13}v_1a - e_{23}v_2a - (\frac{bc}{a} + r)v_3^2a - e_{34}v_4a$ and $a_{88} = -e_{14}v_1a - e_{24}v_2a - e_{34}v_3a - (\frac{bc}{a} + r)v_4^2a$.

Clearly, each element of (9) is a polynomial in $u_i$, $v_i$, $a$, $b$, $c$, $r$, $d_{ij}$ and $e_{ij}$ ($i \leq j; i \neq j$).

Since the Property (P) is the same for both matrices (8) and (9), it is sufficient to show that matrix (9) has Property (P).

The leading principal minors of the eight by eight matrix (9) are denoted by $m_{8i}$ ($i = 1, ..., 8$).

All the terms in $(-1)^i m_{8i}$ ($i = 1, 2, 3, 4$) are positive. By using the PD-Algorithm, we check $m_{8i}$ for $i = 5, 6, 7, 8$:

$$PD[(-1)^5m_{85}] = \{f_1\}, \quad PD[m_{86}] = \{f_1\}, \quad PD[(-1)^7m_{87}] = \{f_2, f_3, f_4\},$$

$$PD[m_{88}] = \{f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}\}.$$

We write the final polynomial set $RS = \{f_1, ..., f_{11}\}$ in the appendix. It is checked there that $f_4, f_8, f_9, f_{11}$ are positive semidefinite and others are positive definite. In the manipulation some positive terms are removed; therefore we have shown that all $(-1)^i m_{8i}$ are positive definite, i.e., the leading principal minors of (9) alternate in sign. By Lemma 2, system (1) has a globally stable equilibrium point, i.e., system (1) is globally stable.

This completes the proof of the theorem.

References

Appendix

The eleven polynomials appear in Section 5 are as follows:

\begin{align*}
  f_1 &= u_3^2 - u_4^2 > 0, \\
  f_2 &= (u_1^2 + u_2^2)(v_1^2 + v_2^2) + (u_1 v_1 + u_2 v_2)(u_2 v_1 - u_1 v_2)^2 > 0, \\
  f_3 &= (u_1^2 + u_2^2)(u_2 v_1 + u_1 v_2)(u_2 v_1 - u_1 v_2)^2 + u_1^2 v_1^2 > 0, \\
  f_4 &= (u_1 v_1 + u_2 v_2)(u_2 v_1 - u_1 v_2)^2 + (u_3 v_1 + u_1 v_3)(u_3 v_1 - u_1 v_3)^2 + (u_3 v_2 + u_2 v_3)(u_3 v_2 - u_2 v_3)^2 > 0, \\
  f_5 &= (u_1^2 + u_2^2)(v_1^2 + v_2^2) + 2(u_1 v_1 + u_2 v_2)(u_2 v_1 - u_1 v_2)^2 > 0, \\
  f_6 &= 2(u_1^2 + u_2^2)(v_1^2 + v_2^2) + (u_2 v_1 - u_1 v_2)^2 > 0,
\end{align*}

It can be checked that the above polynomials can be written as follows:

\begin{align*}
  f_1 &= u_3^2 - u_4^2 > 0, \\
  f_2 &= u_1^4 + u_2^4 - u_1^2 u_2^2 v_1^2 - u_1 u_2^3 v_1 v_2 - u_1^2 u_2 v_1^3 + 2u_1^2 u_2^2 v_1^2 + u_1^3 v_1^3, \\
  f_3 &= u_2 u_3^3 v_1^4 + u_1 u_2^3 v_1^4 + 2u_1^2 u_2^3 v_1^2 v_2 - u_1 u_2^2 u_3^3 v_1^2 v_2 - u_1^2 u_2 u_3^3 v_1^2 v_2 - u_1^2 u_2 u_3^3 v_1 v_2^2 + u_1^3 u_2 u_3^3 v_1^2 v_2 + u_1^3 u_2^2 u_3^3 v_1^2 v_2 + u_2 u_3^3 v_1 v_2^3 + u_2^2 u_3 v_1^3 v_2 + u_3^2 v_1^3 v_2 + u_3 u_2^2 v_1^3 v_2 + u_3^3 u_2 v_1^3 v_2 + u_3^4 v_1^3 v_2, \\
  f_4 &= u_2^2 v_1^2 + u_2 u_3^3 v_1 v_2 - u_1 u_2^2 v_1^3 v_2 - u_2^2 u_3 v_1^3 v_2 - u_2^3 u_3^2 v_1^2 v_2 + u_2^3 u_3^3 v_1^2 v_2 + u_2 u_3^3 v_1 v_2^3 - u_2^2 u_3^3 v_1 v_2^3 - u_2 u_3^4 v_1^2 v_2 + u_3^4 v_1^2 v_2 + u_3^5 v_1^2 v_2 + u_3^6 v_1^2 v_2 + u_3^7 v_1^2 v_2, \\
  f_5 &= u_2 u_3^3 v_1^4 + 2u_1^2 u_2^3 v_1^2 v_2 - u_1 u_2^2 u_3^3 v_1^2 v_2 - u_1^2 u_2 u_3^3 v_1^2 v_2 - u_1^2 u_2 u_3^3 v_1 v_2^2 + u_1^3 u_2 u_3^3 v_1^2 v_2 + u_1^3 u_2^2 u_3^3 v_1^2 v_2 + u_2 u_3^3 v_1 v_2^3 - u_2 u_3^3 v_1 v_2^3 - u_2 u_3^4 v_1^2 v_2 + u_3^4 v_1^2 v_2 + u_3^5 v_1^2 v_2 + u_3^6 v_1^2 v_2 + u_3^7 v_1^2 v_2, \\
  f_6 &= 2(u_1^2 + u_2^2)(v_1^2 + v_2^2) + (u_2 v_1 - u_1 v_2)^2 > 0.
\end{align*}
\[ f_7 = (u^3_1 + u^3_2 + u^3_3)(v^3_1 + v^3_2 + v^3_3) + (u_2v_1 + u_1v_2)(u_2v_1 - u_1v_2)^2 + (u_3v_1 + u_1v_3)(u_3v_1 - u_1v_3)^2 + (u_3v_2 + u_2v_3)(u_3v_2 - u_2v_3)^2 > 0, \]

\[ f_8 = \sum_{i \neq j, i,j=1}^{4}(u_i v_j + u_j v_i)(u_i v_j - u_j v_i)^2 \geq 0, \]

\[ f_9 = (u^3_1 + u^3_2)(u_3v_2 + u_2v_3)(u_3v_2 - u_2v_3)^2 + (u^3_2 + u^3_3)(u_1v_4 + u_4v_1)(u_1v_4 - u_4v_1)^2 \geq 0, \]

\[ f_{10} = (v^3_3 + v^3_4)(u_1v_2 + u_2v_1)(u_1v_2 - u_2v_1)^2 + (v^3_1 + v^3_2)(u_3v_4 + u_4v_3)(u_3v_4 - u_4v_3)^2 + (u^3_1 + u^3_2 + u^3_3 + u^3_4)(v^3_1 + v^3_2)(v^3_3 + v^3_4) \geq 0. \]

\[ f_{11} = (u^3_3 + u^3_4)(v^3_3 + v^3_4)(u_1v_2 + u_2v_1)(u_1v_2 - u_2v_1)^2 + (u^3_1 + u^3_2)(v^3_1 + v^3_2)(u_3v_4 + u_4v_3)(u_3v_4 - u_4v_3)^2 \geq 0. \]