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Nearest Singular Polynomials [1)]

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Abstract. The nearest singular polynomials to a given polynomial have been studied in [1], based on minimization of quadratic forms. An equivalent expression of the quadratic form is presented. It leads to a simple equation satisfied by the double zeros of the nearest singular polynomials. The amount of computational work is economized.

1. Introduction

An approach based on minimization of quadratic forms to study the problems related to polynomials with inexact coefficients has been mentioned or proposed by several authors[1,2,3]. In [1] the nearest singular polynomials to a given polynomial have been defined, and by the above approach, a parametric quadratic form with the double root of the perturbed polynomial as parameter is resulted.

In this paper we will prove that the related matrix is positive definite, and give another equivalent expression of the quadratic form, which is a natural generalization of the one given in [2]. The first derivative of this new expression can be factored into two nontrivial factors, and only one of them yields local minimum. Therefore for finding the double zeros of the nearest singular polynomials it is sufficient from this factor with simple expression, and more than half of the computation is saved. Furthermore a simple expression of the perturbed polynomial can be given.

In §2 we review briefly the approach given in [1]. The new expression of the quadratic form is proved in §3. The obvious consequences derived from it are given in §4, and numerical examples and discussions in §5.

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2. Karmarkar and Lakshman's Approach [1]

Given a monic polynomial f

$$f = x^m + \sum_{j=1}^{m} f_j x^{m-j} \tag{1}$$

with complex coefficients, a nearest singular polynomial to f is defined to be a monic polynomial h

$$h = (x-c)^2(x^{m-2} + \sum_{j=1}^{m-2} \phi_j x^{m-2-j}) \tag{2}$$

such that $\| f - h \|$ is minimized, where

$$\| g \|^2 = \sum_{j=0}^{k} |g_j|^2$$

for any

$$g = \sum_{j=0}^{k} g_j x^{k-j} \in \mathcal{C}[x]$$

Now

$$f - h = \sum_{j=1}^{m} (f_j - \phi_j + 2c\phi_{j-1} - c^2\phi_{j-2}) x^{m-j}$$

where

$$\phi_{-1} = 0, \phi_0 = 1, \phi_{m-1} = 0, \phi_m = 0,$$

and

$$\mathcal{N} := \| f - h \|^2 = \sum_{j=1}^{m} |f_j - \phi_j + 2c\phi_{j-1} - c^2\phi_{j-2}|^2$$

Our problem is to look for $c \in \mathcal{C}$ and $\phi_j' s \in \mathcal{C}$ that minimize $\mathcal{N}$.

Let

$$\phi = [\phi_1 \quad \phi_2 \quad \cdots \quad \phi_{m-2}]^T$$

$\mathcal{N}$ can be written as

$$\mathcal{N} = \phi^* Q_2^{(m-2)} \phi - \phi^* r - r^* \phi + s$$

where $*$ stands for conjugate transpose, $Q_2^{(m-2)} := (\quad q_{ij} \quad)$ is an Hermitian, Toeplitz and five diagonal $(m-2) \times (m-2)$ matrix, with

$$\begin{aligned} q_{11} &= 1 + 4c\bar{c} + (c\bar{c})^2 \\ q_{21} &= -2c(1 + c\bar{c}) \\ q_{31} &= c^2 \\ r &= [r_1 \quad r_2 \quad \cdots \quad r_{m-2}]^T \end{aligned}$$

with

$$\begin{array}{l} r_1 = f_1 - 2\overline{c}f_2 + \overline{c}^2 f_3 + 2c(1 + c\overline{c}) \\ r_2 = f_2 - 2\overline{c}f_3 + \overline{c}^2 f_4 - c^2 \\ r_k = f_k - 2\overline{c}f_{k+1} + \overline{c}^2 f_{k+2}, \quad k > 2 \end{array} \tag{3}$$

and

$$s = \sum_{j=1}^{m} |f_j|^2 + 2c\overline{f_1} + 2\overline{c}f_1 - c^2\overline{f_2} - \overline{c}^2 f_2 + 4c\overline{c} + (c\overline{c})^2. \tag{4}$$

In the next section we will prove that $Q_2^{(m-2)}$ is positive definite. Thus

$$\mathcal{N} = (Q_2^{(m-2)}\phi - r)^* Q_2^{(m-2)^{-1}} (Q_2^{(m-2)}\phi - r) - r^* Q_2^{(m-2)^{-1}} r + s$$

For fixed $c \in \mathcal{C}$, $\mathcal{N}$ attains minimum at ϕ satisfying

$$Q_2^{(m-2)}\phi - r = 0,$$

and

$$\mathcal{N}_m := min\mathcal{N} = -r^* Q_2^{(m-2)^{-1}} r + s \tag{5}$$

which is a real valued function of complex variable c.

Karmarkar proposed to write $c = a + ib$, where a and b are real, and to consider $\mathcal{N}_m$ as a real rational function of real variables a and b.

The problem reduces to find real solutions (ξ, η) of the system

$$\frac{\partial \mathcal{N}_m}{\partial a} = 0, \frac{\partial \mathcal{N}_m}{\partial b} = 0 \tag{6}$$

for determined $c = \xi + i\eta$ and the corresponding nearest singular polynomials.

3. The Expression of the Quadratic Form

Let L be a $k \times k$ two diagonal Toeplitz matrix

$$L^{(k)} = \begin{bmatrix} 1 & & & & \\ -c & 1 & & & \\ & -c & 1 & & \\ & & & \ddots & \\ & & & -c & 1 \end{bmatrix}$$

and $Q_1^{(k)}$ be a $k \times k$ three diagonal Toeplitz, Hermitian and positive definite matrix[1]

$$Q_1^{(k)} = \begin{bmatrix} 1 + c\overline{c} & -\overline{c} & & & \\ -c & 1 + c\overline{c} & -\overline{c} & & \\ & \ddots & & & \\ & & & -c & 1 + c\overline{c} \end{bmatrix}$$

Denote the determinant of $Q_1^{(k)}$ by

$$q_1^{(k)} = \sum_{j=0}^{k} (c\bar{c})^j$$

and the determinant of $Q_2^{(k)}$ in §2 for $m = k+2$ by $q_2^{(k)}$.

Lemma 1. $Q_2^{(m-2)}$ is positive definite.

Proof: It is easy to see that

$$\begin{aligned} Q_2^{(m-2)} &= L^{(m-2)*^2} L^{(m-2)^2} + \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \\ M &= \begin{bmatrix} (c\bar{c})^2 & -2c\bar{c}^2 \\ -2c^2\bar{c} & 4c\bar{c} + (c\bar{c})^2 \end{bmatrix} \end{aligned}$$

$L^{(m-2)*^2} L^{(m-2)^2}$ is positive definite, and M is positive semidefinite. □

Lemma 2.

$$q_1^{(k)^2} = q_2^{(k)} - 2c\bar{c}q_2^{(k-1)} + (c\bar{c})^2 q_2^{(k-2)}$$

Proof: It is easy to see that

$$(Q_1^{(k)})^2 = Q_2^{(k)} - \begin{bmatrix} c\bar{c} & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & c\bar{c} \end{bmatrix}$$

The lemma follows from the above identity. □

Lemma 3.

$$q_2^{(k)} = q_1^{(k+1)} \frac{\partial^2 q_1^{(k+1)}}{\partial c \partial \bar{c}} - \frac{\partial q_1^{(k+1)}}{\partial c} \frac{\partial q_1^{(k+1)}}{\partial \bar{c}} \tag{7}$$

Proof: The lemma can be proved by induction using the recurrence relation in Lemma 2. □

Now we are ready to prove

Theorem 1.

$$\mathcal{N}_m = \frac{f(c)\overline{f(c)}}{q_1^{(m-1)}} + \frac{P_2\overline{P_2}}{q_1^{(m-1)} q_2^{(m-2)}}$$

where

$$P_2 := q_1^{(m-1)} f'(c) - \frac{\partial q_1^{(m-1)}}{\partial c} f(c) \tag{8}$$

Proof: Consider auxiliary linear system:

$$L^{(m)^{*2}} L^{(m)^2} \widehat{\phi} = \widehat{r} \tag{9}$$

where

$$\begin{aligned}\widehat{\phi} &= [\phi \ \ \phi_{m-1} \ \ \phi_m]^T \\ \widehat{r} &= [r \ \ r_{m-1} \ \ r_m]^T \\ r_{m-1} &= f_{m-1} - 2\overline{c} f_m + \beta_1 \\ r_m &= f_m + \beta_2\end{aligned}$$

β_1 and β_2 are parameters which will be specified later.

The inverses of $L^{(m)*}$ and $L^{(m)}$ are obvious. Thus

$$\widehat{\phi} = L^{(m)^{-2}} L^{(m)*^{-2}} \widehat{r}$$

$$L^{(m)*^{-2}}\widehat{r} = \begin{bmatrix} f_1 + (m-1)\overline{c}^{m-2}\beta_1 + m\overline{c}^{m-1}\beta_2 + 2c \\ f_2 + (m-2)\overline{c}^{m-3}\beta_1 + (m-1)\overline{c}^{m-2}\beta_2 - c^2 \\ f_3 + (m-3)\overline{c}^{m-4}\beta_1 + (m-2)\overline{c}^{m-3}\beta_2 \\ \cdots \\ f_{m-2} + 2\overline{c}\beta_1 + 3\overline{c}^2\beta_2 \\ f_{m-1} + \beta_1 + 2\overline{c}\beta_2 \\ f_m + \beta_2 \end{bmatrix} \tag{10}$$

The last two components ϕ_{m-1} and ϕ_m of $\widehat{\phi}$ are:

$$\begin{aligned}\phi_{m-1} &= \textstyle\sum_{i=1}^{m-1}(m-i)c^{m-i-1}f_i + \beta_1\sum_{i=1}^{m-1}(m-i)^2 c^{m-i-1}\overline{c}^{m-i-1} \\ &\quad +\beta_2\textstyle\sum_{i=1}^{m-1}(m-i)(m-i+1)c^{m-i-1}\overline{c}^{m-i} + mc^{m-1} \\ &= f'(c) + \beta_1\frac{\partial^2 q_1^{(m-1)}}{\partial c\partial\overline{c}} + \beta_2(\overline{c}\frac{\partial^2 q_1^{(m-1)}}{\partial c\partial\overline{c}} + \frac{\partial q_1^{(m-1)}}{\partial c}) \\ \phi_m &= \textstyle\sum_{i=1}^{m}(m-i+1)c^{m-i}f_i + \beta_1\sum_{i=1}^{m-1}(m-i+1)(m-i)c^{m-i}\overline{c}^{m-i-1} \\ &\quad +\beta_2\textstyle\sum_{i=1}^{m}(m-i+1)^2 c^{m-i}\overline{c}^{m-i} + (m+1)c^m \\ &= (cf(c))' + \beta_1(c\frac{\partial^2 q_1^{(m-1)}}{\partial c\partial\overline{c}} + \frac{\partial q_1^{(m-1)}}{\partial\overline{c}}) + \beta_2(c\overline{c}\frac{\partial^2 q_1^{(m-1)}}{\partial c\partial\overline{c}} + 2c\frac{\partial q_1^{(m-1)}}{\partial c} + q_1^{(m-1)})\end{aligned}$$

And $\phi_{m-1} = \phi_m = 0$ iff

$$\begin{aligned}\beta_1 &= \frac{1}{q_2^{(m-2)}}(\frac{\partial^2 q_1^{(m-1)}}{\partial c\partial\overline{c}}\overline{c}f(c) + \frac{\partial q_1^{(m-1)}}{\partial c}f(c) - c\frac{\partial q_1^{(m-1)}}{\partial c}f'(c) - q_1^{(m-1)}f'(c)) \\ \beta_2 &= \frac{1}{q_2^{(m-2)}}(\frac{\partial q_1^{(m-1)}}{\partial\overline{c}}f'(c) - \frac{\partial^2 q_1^{(m-1)}}{\partial c\partial\overline{c}}f(c))\end{aligned} \tag{11}$$

In the following, we assume β_1 and β_2 take above values, therefore

$$\widehat{\phi} = \begin{bmatrix} \phi \\ 0 \\ 0 \end{bmatrix}$$

Since

$$L^{(m)*^2}L^{(m)^2} = \begin{bmatrix} Q_2^{(m-2)} & * \\ * & * \end{bmatrix}$$

then

$$\begin{aligned} \widehat{r}^*\widehat{\phi} &= r^*\phi \\ r^*Q_2^{(m-2)^{-1}}r &= r^*\phi = \widehat{r}^*\widehat{\phi} = \widehat{r}^*L^{(m)^{-2}}L^{(m)*^{-2}}\widehat{r} \\ &= (L^{(m)*^{-2}}\widehat{r})^*(L^{(m)*^{-2}}\widehat{r}) \\ &= s - \frac{f(c)\overline{f(c)}}{q_1^{(m-1)}} - \frac{P_2\overline{P_2}}{q_1^{(m-1)}q_2^{(m-2)}} \end{aligned}$$

by (10) and (11). □

4. The Equation satisfied by c

Instead of the system (5), we consider

$$\frac{\partial \mathcal{N}_m}{\partial c} = 0, \frac{\partial \mathcal{N}_m}{\partial \bar{c}} = 0 \tag{12}$$

Obviously, $\frac{\partial \mathcal{N}_m}{\partial c}$, $\frac{\partial \mathcal{N}_m}{\partial \bar{c}}$ are complex conjugate to each other. One of them is sufficient to determine c. The trace T and determinant D of the Hessian matrix

$$\begin{bmatrix} \frac{\partial^2 \mathcal{N}_m}{\partial a^2} & \frac{\partial^2 \mathcal{N}_m}{\partial a \partial b} \\ \frac{\partial^2 \mathcal{N}_m}{\partial a \partial b} & \frac{\partial^2 \mathcal{N}_m}{\partial b^2} \end{bmatrix}$$

are

$$T = 4\frac{\partial^2 \mathcal{N}_m}{\partial c \partial \bar{c}}$$

and

$$D = 4\left(\left(\frac{\partial^2 \mathcal{N}_m}{\partial c \partial \bar{c}}\right)^2 - \frac{\partial^2 \mathcal{N}_m}{\partial c^2}\frac{\partial^2 \mathcal{N}_m}{\partial \bar{c}^2}\right)$$

By direct computation, we have

$$\frac{\partial \mathcal{N}_m}{\partial c} = \frac{1}{q_1^{(m-1)}q_2^{(m-2)^2}}\overline{P_2}P_3 \tag{13}$$

where

$$P_3 := q_2^{(m-2)}P_2' - \frac{\partial q_2^{(m-2)}}{\partial c}P_2 \tag{14}$$

Furthermore

$$\frac{\partial^2 \mathcal{N}_m}{\partial c \partial \bar{c}} = \frac{1}{q_1^{(m-1)}q_2^{(m-2)^3}}\left(P_3\overline{P_3} - \left(q_2^{(m-2)}\frac{\partial^2 q_2^{(m-2)}}{\partial c \partial \bar{c}} - \frac{\partial q_2^{(m-2)}}{\partial c}\frac{\partial q_2^{(m-2)}}{\partial \bar{c}}\right)P_2\overline{P_2}\right) \tag{15}$$

and when $P_2 = 0$

$$D = \frac{4P_2'\overline{P_2'}}{q_1^{(m-1)^4} q_2^{(m-2)^2}} (q_1^{(m-1)^2} P_2'\overline{P_2'} - q_2^{(m-2)^2} f(c)\overline{f(c)}) \tag{16}$$

Theorem 2. $\mathcal{N}_m$ attains its local minimum at c satisfying $P_2 = 0$ if

$$q_1^{(m-1)^2} P_2'\overline{P_2'} - q_2^{(m-2)^2} f(c)\overline{f(c)} > 0 \tag{17}$$

And the minimum value of $\mathcal{N}_m$ is

$$\frac{f(c)\overline{f(c)}}{q_1^{(m-1)}}$$

and the perturbed polynomial is

$$h = f(x) - \frac{f(c)}{q_1^{(m-1)}} \sum_{j=0}^{m-1} (\bar{c}x)^j$$

Proof: The last part comes from that ϕ is unique for fixed c, and

$$\begin{aligned} \|f-h\|^2 &= \|\tfrac{f(c)}{q_1^{(m-1)}} \textstyle\sum_{j=0}^{m-1} (\bar{c}x)^j\|^2 \\ &= \tfrac{f(c)\overline{f(c)}}{q_1^{(m-1)^2}} \textstyle\sum_{j=0}^{m-1} (c\bar{c})^j \;=\; \tfrac{f(c)\overline{f(c)}}{q_1^{(m-1)}} \end{aligned}$$

The other parts are obvious. □

Theorem 3 The zero set of $P_2(P_3)$ contains all zeros of f with multiplicity $\geq 2(\geq 3)$.

Proof: By the expressions of P_2 and P_3. □

Thus when f has a double root, then $h = f$.

Lemma 4.

$$q_2^{(k-2)} = \sum_{j=0}^{2k-4} \alpha_j (c\bar{c})^j$$

where $\alpha_j = \alpha_{2k-4-j} = \frac{1}{6}(j+1)(j+2)(j+3) > 0, \forall j \leq k-2$.

Proof: It can be proved by Lemma 3 and the expression of $q_1^{(k)}$. □

Lemma 5.

$$q_2^{(k)} \frac{\partial^2 q_2^{(k)}}{\partial c \partial \bar{c}} - \frac{\partial q_2^{(k)}}{\partial c} \frac{\partial q_2^{(k)}}{\partial \bar{c}} > 0.$$

Proof: By lemma 4

$$\begin{aligned} q_2^{(k)} \tfrac{\partial^2 q_2^{(k)}}{\partial c \partial \bar{c}} - \tfrac{\partial q_2^{(k)}}{\partial c} \tfrac{\partial q_2^{(k)}}{\partial \bar{c}} &= \textstyle\sum_{j=0}^{2k} \alpha_j (c\bar{c})^j \sum_{j=1}^{2k} j^2 \alpha_j (c\bar{c})^{j-1} - c\bar{c}(\sum_{j=1}^{2k} j\alpha_j (c\bar{c})^{j-1})^2 \\ &= \textstyle\sum_{j=1}^{2k} j^2 \alpha_j (c\bar{c})^{j-1} + c\bar{c}(\sum_{j=1}^{2k} \alpha_j (c\bar{c})^{j-1} \sum_{j=1}^{2k} j^2 \alpha_j (c\bar{c})^{j-1} - (\sum_{j=1}^{2k} j\alpha_j (c\bar{c})^{j-1})^2) \end{aligned}$$

The first term is obviously positive, and the second term is positive by Minkowsky inequality. □

Theorem 4. For c satisfying $P_3 = 0$, the trace is non-positive.

Proof: By (15) and Lemma 5. □

It is obvious that the solution of $P_3 = 0$ is useless for our present purpose. But we would like to note that $P_3 = 0$ is useful when c is a triple zero of h.

In summary, it is sufficient to solve $P_2 = 0$, and among the solutions satisfying (17), compare the values $f(c)\bar{f}(c)/q_1^{(m-1)}$.

Note that the nearest singular polynomial may not be unique, see examples below.

5. Numerical Examples

Ex.1 $f = x^5 - x$.

There are 4 nearest singular polynomials due to geometry of the zeros of f. The zeros of one of them are given below

zeros of h	zeros of f
0.5806857529(double)	0, 1
-1.050883646	-1
-0.07472166958+0.9804509313 i	i
-0.07472166958-0.9804509313 i	-i

The other 3 can be obtained by rotation with an angle $\pi/2$, π and $3\pi/2$ respectively.

$$\mathcal{N}_m = 0.1763296119.$$

Ex.2 $f = x^5 + 1$

There are 5 nearest singular polynomials

zeros of h	zeros of f
0.7676270658 (double)	$e^{\pm\pi i/5}$
-0.907566588	-1
$-0.2166162160 \pm 0.8808010592i$	$e^{\pm 3\pi i/5}$

The other four can be obtained by rotation.

$$\mathcal{N}_m = 0.7092712403.$$

Note that in this example $c = 0$ is a solution, which is a saddle point.

We have simplified the equation satisfied by the double zeros c, but how to find all c efficiently is another problem. Fortunately the following bound is helpful.

Let c_{opt} denote an optimal c. The value of $\mathcal{N}_m$ at c_{opt} is less than that at 0, i. e.

$$\frac{f(c_{opt})\overline{f(c_{opt})}}{\sum_{j=0}^{m-1}|c_{opt}|^{2j}} \leq |f_{m-1}|^2 + |f_m|^2$$

Since c_{opt} is a zero of h, then

$$\begin{aligned} |c_{opt}| &\leq 1 + \max_j |f_j - \tfrac{f(c_{opt})\overline{c_{opt}}^j}{\sum_{j=0}^{m-1}|c_{opt}|^{2j}}| \\ &\leq 1 + \max_j |f_j| + \tfrac{|f(c_{opt})|}{(\sum_{j=0}^{m-1}|c_{opt}|^{2j})^{\frac{1}{2}}} \\ &\leq 1 + \max_j |f_j| + (|f_{m-1}|^2 + |f_m|^2)^{\frac{1}{2}} \end{aligned}$$

The techniques used here can be applied to the nearest singular polynomial $h(x)$ with a zero of multiplicity > 2, but there are several relations needed to prove.

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