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The Flat Central Configurations of Four Planet Motions[1)]

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Abstract. The flat central configurations of four planet motions is considered by using Wu Elimination (the Char-set method) in this paper. We obtain 12 collinear central configurations. Obtain a necessary condition for determining flat but non-collinear central configurations. And deduce that in general there are finitely many flat central configurations determined by their masses and their angular velocity.

Key Words: Central Configuration, Wu Elimination

1. Introduction

Wintner studied the configurations of planet motions in [1]. He conjectured that there are only finitely many central configurations with any number of particles of any masses (see [2]). For the case the particles are collinear, Moulton [3] obtained all solutions for arbitrary n particles with distinct masses, the number of the solutions is $n!/2$. This result was reverified by Smale and his followers using topological methods [4, 5]. Waldvogel [6] showed that for any n ($n > 4$), there exist a group of n particles which possess a non-flat central configuration. In [2], Professor Wu reduced the determination of the central configurations to a problem of polynomial equation-solving, and reverified the case of three particles by using the Char-set method (see [7]). In this paper, we determine the possible flat central configurations of four planets also by using the Wu Elimination (the Char-set method).

For the problem of determining flat central configurations of four bodies, from Wintner [1], it is known that for most sets of masses there exist a variety of (up to ten or more) central configurations; a square is a central configuration only in the case of the four equal masses; and for 4 given masses m_i, there exists at least one non-collinear flat central configuration only when the m_i satisfy certain inequalities. In this paper, we give all the possible solutions of the collinear case; for the non-collinear case, we obtain that there are only finitely many solutions determined by the masses and the angular velocity.

2. Notation and the Wintner Conjecture

Let $m_1, m_2, ..., m_n$ be masses of n particles moving under mutual Newtonian gravitational attractions, and $r_1, r_2, ..., r_n$ be the positions of these masses at a certain moment, with

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$r_i \neq r_j$ for $i \neq j$. A configuration formed of masses m_i at position $r_i, i = 1, 2, 3, ..., n$ is denoted by

$$\left[\begin{matrix} m_1 & m_2 & ... & m_n \\ r_1 & r_2 & ... & r_n \end{matrix} \right]$$

Definition 1. The center of n particles $m_1, ..., m_n$ at the positions $r_1, ..., r_n$, is defined to be the position $\sum_{i=1}^{n} m_i r_i / \sum_{i=1}^{n} m_i$.

Definition 2. A configuration is called a central configuration with respect to the masses $m_1, m_2, ..., m_n$, if there are initial velocities of the masses m_i such that under the Newtonian gravitational attractions the configurations formed by the masses during the motion will remain similar to the initial one.

From this definition, we know that a central configuration defines a class of configurations. As a direct consequence of Newtonian mechanics, for a central configuration the center of mass of the masses $m_1, m_2, ...m_n$ may be considered to be fixed during motion.

Definition 3. The inertial coordinate system associated to a central configuration is defined to be a cartesian coordinate system for which the origin is at the fixed center of mass of the masses $m_1, m_2, ..., m_n$.

In the following, we assume that $r_1, ..., r_n$ are the initial coordinates of the particles $m_1, ..., m_n$. By the definition of the center and the inertial coordinate system, we have

$$\sum_{i=1}^{n} m_i r_i = 0.$$

We set

$q_3(n) :=$ number of central configurations with given masses $m_1, ...m_n$;

$q_2(n) :=$ number of central configurations with given masses $m_1, ...m_n$ for which the masses are in the same plane;

$q_1(n) :=$ number of central configurations with given masses $m_1, ...m_n$ for which the masses are in the same line.

Wintner Conjecture: $q_3(n)$ and $q_2(n)$ are finite for all n and masses $m_1, ..., m_n$.

3. Reduction the Conjecture to Certain Polynomial-Solving

Let $pi, i = 1, 2, ..., n$ be polynomials and $PS = (p1, p2, ..., pn)$ be a polynomial set. We denote the zero set of PS by $zero(PS)$ and denote the restricted zero set of PS by $zero_{rc}(PS)$ for which the reality conditions are observed. We have to observe some reality conditions, for example, $r_{ij}, m_i(i, j = 1, 2, ..., n) > 0$.

Taking planar inertial coordinate system (x, y) with origin O at the center of masses. Let $r_i = (x_i, y_i), x_{ij} = x_i - x_j, y_{ij} = y_i - y_j, r_{ij} = \sqrt{(x_{ij}^2 + y_{ij}^2)}, (i, j = 1, 2, ..., n, i \neq j)$. By Newtonian mechanics, the Newtonian attraction of m_i at position r_i is equal to it's centrifugal force, hence we have a general system of equations (see [2])

$$\begin{aligned} \omega^2 x_i &= \textstyle\sum_{j \neq i} m_j \frac{x_{ij}}{r_{ij}^3}, \quad i = 1, 2, ..., n, \\ \omega^2 y_i &= \textstyle\sum_{j \neq i} m_j \frac{y_{ij}}{r_{ij}^3}, \quad i = 1, 2, ..., n \end{aligned}$$

where ω is an angular velocity and is a constant.

Let $u_j = x_j + \mathbf{i}y_j, v_j = x_j - \mathbf{i}y_j (j = 1, 2, ..., n)$, where $\mathbf{i} = \sqrt{-1}$, and $m_0 = -\omega^2$, then the above equations become the following system of equations

$$\begin{array}{ll} (I) & m_0 u_i + \sum_{j\neq i} \frac{m_j}{(v_i - v_j) r_{ij}} = 0, \quad i = 1, 2, ..., n, \\ (II) & m_0 v_i + \sum_{j\neq i} \frac{m_j}{(u_i - u_j) r_{ij}} = 0, \quad i = 1, 2, ..., n, \end{array}$$

and

$$\begin{array}{lll} r_{ij} - (u_i - u_j)(v_i - v_j) = 0, & i \neq j, & i, j = 1, 2, ..., n, \\ r_{ij} = r_{ji}, & i \neq j, & i, j = 1, 2, ..., n. \end{array}$$

Thus, the determination of flat central configurations is reduced to equation-solving with variables u_i, v_j, r_{ij} as unknowns.

To simplify the notations we set $u_{ij} = u_i - u_j, v_{ij} = v_i - v_j, i < j$, and $u_{ij} = u_{1j} - u_{1i}, v_{ij} = v_{1j} - v_{1i}, i \neq j, i, j = 2, ..., n$.

4. The case of $n = 4$

The problem of the central configurations of three bodies was studied in [2] by using the Char-set method. In this section we study the problem of four bodies. In this case, the equations (I) are the following

$$\begin{array}{ll} q1 \equiv & m_0 u_1 + \frac{m_2}{(v_1 - v_2) r_{12}} + \frac{m_3}{(v_1 - v_3) r_{13}} + \frac{m_4}{(v_1 - v_4) r_{14}} = 0 \ , \\ q2 \equiv & m_0 u_2 - \frac{m_1}{(v_1 - v_2) r_{12}} + \frac{m_3}{(v_2 - v_3) r_{23}} + \frac{m_4}{(v_2 - v_4) r_{24}} = 0 \ , \\ q3 \equiv & m_0 u_3 - \frac{m_1}{(v_1 - v_3) r_{13}} - \frac{m_2}{(v_2 - v_3) r_{23}} + \frac{m_4}{(v_3 - v_4) r_{34}} = 0 \ , \\ q4 \equiv & m_0 u_4 - \frac{m_1}{(v_1 - v_4) r_{14}} - \frac{m_2}{(v_2 - v_4) r_{24}} - \frac{m_3}{(v_3 - v_4) r_{34}} = 0 \ . \end{array}$$

It is easy to see that this system is equivalent to the system which consist of $q0 \equiv m_1 u_1 + m_2 u_2 + m_3 u_3 + m_4 u_4 = 0$ and any three from q_1, q_2, q_3, q_4.

Note that q0 is identical with any one of the following:

$$\begin{array}{ll} q01 \equiv & (m_1 + m_2 + m_3 + m_4) u_1 - m_2 u_{12} - m_3 u_{13} - m_4 u_{14} = 0 \ , \\ q02 \equiv & (m_1 + m_2 + m_3 + m_4) u_2 + m_1 u_{12} - m_3 u_{23} - m_4 u_{24} = 0 \ , \\ q03 \equiv & (m_1 + m_2 + m_3 + m_4) u_3 + m_1 u_{13} + m_2 u_{23} - m_4 u_{34} = 0 \ , \\ q04 \equiv & (m_1 + m_2 + m_3 + m_4) u_4 + m_1 u_{14} + m_2 u_{24} + m_4 u_{34} = 0 \ . \end{array}$$

Clearing the fractions of $q1, q2, q3, q4$, we obtain $p1, p2, p3, p4$ respectively where

$$\begin{array}{ll} p1 \equiv & v_{12} r_{12} v_{13} r_{13} v_{14} r_{14} m_0 u_1 + m_2 v_{13} r_{13} v_{14} r_{14} \\ & + m_3 v_{12} r_{12} v_{14} r_{14} + m_4 v_{12} r_{12} v_{13} r_{13} = 0 \ , \\ p2 \equiv & v_{12} r_{12} v_{23} r_{23} v_{24} r_{24} m_0 u_2 - m_1 v_{23} r_{23} v_{24} r_{24} \\ & + m_3 v_{12} r_{12} v_{24} r_{24} + m_4 v_{12} r_{12} v_{23} r_{23} = 0 \ , \\ p3 \equiv & v_{13} r_{13} v_{23} r_{23} v_{34} r_{34} m_0 u_3 - m_1 v_{23} r_{23} v_{34} r_{34} \\ & - m_2 v_{13} r_{13} v_{34} r_{34} + m_4 v_{13} r_{13} v_{23} r_{23} = 0 \ , \\ p4 \equiv & v_{14} r_{14} v_{24} r_{24} v_{34} r_{34} m_0 u_4 - m_1 v_{24} r_{24} v_{34} r_{34} \\ & - m_2 v_{14} r_{14} v_{34} r_{34} - m_3 v_{14} r_{14} v_{24} r_{24} = 0 \ . \end{array}$$

Let

$$\begin{aligned} p9 \equiv &\quad r_{12}^2 - u_{12}v_{12} = 0 \ , \\ p10 \equiv &\quad r_{13}^2 - u_{13}v_{13} = 0 \ , \\ p11 \equiv &\quad r_{14}^2 - u_{14}v_{14} = 0 \ , \\ p12 \equiv &\quad r_{23}^2 - u_{23}v_{23} = 0 \ , \\ p13 \equiv &\quad r_{24}^2 - u_{24}v_{24} = 0 \ , \\ p14 \equiv &\quad r_{34}^2 - u_{34}v_{34} = 0 \ . \end{aligned}$$

For simplification, we introduce the notations $a = m_1+m_2+m_3+m_4$ and $f_{ij} = a+m_0r_{13}^3, \ \ i < j, \ \ i, j = 1, 2, 3, 4.$

Now we consider the system of polynomials

$$\{q01, p1, p2, p4, p9, p10, p11, p12, p13, p14\}.$$

After eliminating variables $u_1, u_{12}, u_{13}, u_{14}$ of $p1$ step by step via pseudo-devision by polynomials $[q01, p9, p10, p11]$, we obtain a polynomial

$$p15 \equiv m_2r_{13}r_{14}v_{13}v_{14}f_{12} + m_3r_{12}r_{14}v_{12}v_{14}f_{13} + m_4r_{12}r_{13}v_{12}v_{13}f_{14} = 0 \ .$$

For polynomials $p2$,($p4$), after eliminating the variables $u_2, u_{12}, u_{23}, u_{24}$, $(u_4, u_{14}, u_{24}, u_{34}$, respectively) by polynomials $[q02, p9, p12, p13]$($[q04, p11, p13, p14]$, respectively) and replacing $v_{ij} (i, j \neq 1)$ with $(v_{1j} - v_{1i})$ in both of the polynomials we obtain

$$\begin{aligned} q16 \equiv &\quad m_1r_{23}r_{24}(v_{13} - v_{12})(v_{14} - v_{12})f_{12} - m_3r_{12}r_{24}v_{12}(v_{14} - v_{12})f_{23} \\ &\quad -m_4r_{12}r_{23}v_{12}(v_{13} - v_{12})f_{24} = 0 \ , \\ q18 \equiv &\quad m_1r_{24}r_{34}(v_{14} - v_{12})(v_{14} - v_{13})f_{14} + m_2r_{14}r_{34}v_{14}(v_{14} - v_{13})f_{24} \\ &\quad +m_3r_{14}r_{24}v_{14}(v_{14} - v_{12})f_{34} = 0 \ . \end{aligned}$$

For the system (II), we have the similar results

$$\begin{aligned} qq0 \equiv &\quad m_1v_1 + m_2v_2 + m_3v_3 + m_4v_4 = 0 \ , \\ q5 \equiv &\quad m_0v_1 + \tfrac{m_2}{(u_1-u_2)r_{12}} + \tfrac{m_3}{(u_1-u_3)r_{13}} + \tfrac{m_4}{(u_1-u_4)r_{14}} = 0 \ , \\ q6 \equiv &\quad m_0v_2 - \tfrac{m_1}{(u_1-u_2)r_{12}} + \tfrac{m_3}{(u_2-u_3)r_{23}} + \tfrac{m_4}{(u_2-u_4)r_{24}} = 0 \ , \\ q7 \equiv &\quad m_0v_3 - \tfrac{m_1}{(u_1-u_3)r_{13}} - \tfrac{m_2}{(u_2-u_3)r_{23}} + \tfrac{m_4}{(u_3-u_4)r_{34}} = 0 \ , \\ q8 \equiv &\quad m_0v_4 - \tfrac{m_1}{(u_1-u_4)r_{14}} - \tfrac{m_2}{(u_2-u_4)r_{24}} - \tfrac{m_3}{(u_3-u_4)r_{34}} = 0 \ . \end{aligned}$$

Clearing the fractions, we have the following polynomial equations

$$\begin{aligned} p5 \equiv &\quad u_{12}r_{12}u_{13}r_{13}u_{14}r_{14}m_0v_1 + m_2u_{13}r_{13}u_{14}r_{14} + m_3u_{12}r_{12}u_{14}r_{14} \\ &\quad +m_4u_{12}r_{12}u_{13}r_{13} = 0 \ , \\ p6 \equiv &\quad u_{12}r_{12}u_{23}r_{23}u_{24}r_{24}m_0v_2 - m_1u_{23}r_{23}u_{24}r_{24} + m_3u_{12}r_{12}u_{24}r_{24}+ \\ &\quad m_4u_{12}r_{12}u_{23}r_{23} = 0 \ , \\ p7 \equiv &\quad u_{13}r_{13}u_{23}r_{23}u_{34}r_{34}m_0v_3 - m_1u_{23}r_{23}u_{34}r_{34} - m_2u_{13}r_{13}u_{34}r_{34} \\ &\quad +m_4u_{13}r_{13}u_{23}r_{23} = 0 \ , \\ p8 \equiv &\quad u_{14}r_{14}u_{24}r_{24}u_{34}r_{34}m_0v_4 - m_1u_{24}r_{24}u_{34}r_{34} - m_2u_{14}r_{14}u_{34}r_{34} \\ &\quad -m_3u_{14}r_{14}u_{24}r_{24} = 0 \ . \end{aligned}$$

Similarly to the equations $p1, p2, p3, p4$, we can obtain polynomial equations $p19, p20, p21$ from $p5, p6, p8$ respectively where

$$\begin{aligned} p19 \equiv\ & m_2 r_{13} r_{14} u_{13} u_{14} f_{12} + m_3 r_{12} r_{14} u_{12} u_{14} f_{13} \\ & + m_4 r_{12} r_{13} u_{12} u_{13} f_{14} = 0 \ , \\ p20 \equiv\ & m_1 r_{23} r_{24} (u_{13} - u_{12})(u_{14} - u_{12}) f_{12} - m_3 r_{12} r_{24} u_{12} (u_{14} - u_{12}) f_{23} \\ & - m_4 r_{12} r_{23} u_{12} (u_{13} - u_{12}) f_{24} = 0 \ , \\ p21 \equiv\ & m_1 r_{24} r_{34} (u_{14} - u_{12})(u_{14} - u_{13}) f_{14} + m_2 r_{14} r_{34} u_{14} (u_{14} - u_{13}) f_{24} \\ & + m_3 r_{14} r_{24} u_{14} (u_{14} - u_{12}) f_{34} = 0 \ . \end{aligned}$$

And we have $qq01 \equiv (m_1 + m_2 + m_3 + m_4) v_1 - m_2 v_{12} - m_3 v_{13} - m_4 v_{14} = 0$. Since while doing the pseudo-dividing from $p1, p2, p4, p5, p6, p8$ to $p15, p16, p18, p19$, $p20, p21$, all the initials of the middle polynomials are not zero, we have

$$\begin{aligned} zero\{p1, p2, ..., p14\} = \ & zero\{q01, p1, p2, p4, p9, p10, p11, p12, p13, p14\} \\ & \cap zero\{qq01, p5, p6, p8, p9, p10, p11, p12, p13, p14\} \ , \end{aligned}$$

and

$$\begin{aligned} zero_{rc}\{q01, p1, \ & p2, p4, p9, p10, p11, p12, p13, p14\} = \\ & zero_{rc}\{q01, p15, p16, p18, p9, p10, p11, p12, p13, p14\} \ , \\ zero_{rc}\{qq01, p5, \ & p6, p8, p9, p10, p11, p12, p13, p14\} = \\ & zero_{rc}\{qq01, p19, p20, p21, p9, p10, p11, p12, p13, p14\} \ . \end{aligned}$$

To solve system $\{q01, p15, p16, p18, p9, p10, p11, p12, p13, p14\}$, we set the following order

$$\begin{aligned} m_1 \prec m_2 \prec m_3 \prec m_4 \prec r_{12} \prec r_{13} \prec r_{14} \prec r_{23} \prec r_{24} \prec r_{34} \prec m_0 \prec f_{12} \prec f_{13} \\ \prec f_{14} \prec f_{23} \prec f_{24} \prec f_{34} \prec v_{14} \prec v_{13} \prec v_{12} \prec u_1 \end{aligned}$$

By using Wu elimination (Char-set method), we decompose the system $\{q01, q15, q16, q18\}$ and obtain five systems of polynomial equations as follows:
(1). $ZS1 = zero\{z11, z12, z13, z14, z15, z16\}$, in which

$$\begin{aligned} z11 \equiv\ & a u_1 - m_2 u_{12} - m_3 u_{13} - m_4 u_{14} = 0 \ , \\ z12 \equiv\ & f_{12} = 0 \ , \\ z13 \equiv\ & m_3 r_{14} v_{14} f_{13} + m_4 r_{13} v_{13} f_{14} = 0 \ , \\ z14 \equiv\ & f_{24} = 0 \ , \\ z15 \equiv\ & f_{23} = 0 \ , \\ z16 \equiv\ & m_4 r_{13} m_1 r_{34} f_{14} + m_4 r_{13} m_3 r_{14} f_{34} + m_3 r_{14} f_{13} m_1 r_{34} = 0 \ . \end{aligned}$$

(2). $ZS2 = zero\{z21, z22, z23, z24, z25\}$, in which $z21 = z11, z22 = z12, z23 = z13$ and

$$\begin{aligned} z24 \equiv\ & -v_{12} m_3 r_{24} f_{23} - v_{12} f_{24} r_{23} m_4 + m_3 r_{24} f_{23} v_{14} + v_{13} f_{24} r_{23} m_4 = 0 \ , \\ z25 \equiv\ & r_{13} r_{23} m_4 m_2 r_{14} r_{34} f_{24} + r_{13} r_{23} m_4 m_1 r_{24} r_{34} f_{14} + r_{13} m_3 r_{24} f_{23} m_2 r_{14} r_{34} \\ & + r_{13} r_{23} m_4 m_3 r_{14} r_{24} f_{34} + m_3 r_{14} f_{13} r_{23} m_1 r_{24} r_{34} = 0 \ . \end{aligned}$$

(3). $ZS3 = zero\{z31, z32, z33, z34, z35, z36\}$, in which $z31 = z11, z32 = z14$ and

$$\begin{aligned} z33 \equiv\ & v_{12} m_1 r_{23} f_{12} + v_{12} f_{23} r_{12} m_3 - m_1 r_{23} f_{12} v_{13} = 0 \ , \\ z34 \equiv\ & -m_1 r_{34} f_{14} v_{14} + m_1 r_{34} f_{14} v_{13} - m_3 r_{14} v_{14} f_{34} = 0 \ , \\ z35 \equiv\ & m_1 r_{34} f_{14} m_3 r_{12} r_{14} f_{13} + m_3 r_{14} f_{34} m_4 r_{12} r_{13} f_{14} + m_3 r_{14}^2 f_{34} r_{13} m_2 f_{12} \\ & + m_1 r_{34} f_{14}^2 m_4 r_{12} r_{13} + m_1 r_{34} f_{14} r_{13} m_2 r_{14} f_{12} = 0 \ , \\ z36 \equiv\ & r_{34} m_2 r_{13} r_{14} m_1 r_{23} f_{12} + r_{34} m_2 r_{13} r_{14} m_3 r_{12} f_{23} + r_{34} r_{12} m_3 f_{13} m_1 r_{23} r_{14} \\ & + m_4 r_{12} r_{23} r_{13} m_3 r_{14} f_{34} + m_4 r_{12} r_{23} r_{13} m_1 r_{34} f_{14} = 0 \ . \end{aligned}$$

(4). $ZS4 = zero\{z41, z42, z43, z44, z45, z46\}$, in which $z41 = z11, z42 = z12$ and

$$
\begin{aligned}
z43 \equiv\ & f_{13} = 0 \ , \\
z44 \equiv\ & f_{14} = 0 \ , \\
z45 \equiv\ & m_2m_4r_{23}r_{34}f_{24} + m_2m_3r_{24}r_{34}f_{23} + m_3m_4r_{23}r_{24}f_{34} = 0 \ , \\
z46 \equiv\ & m_3r_{24}f_{34}v_{12} + m_2r_{34}f_{24}v_{13} - (m_2r_{34}f_{24} + m_3r_{24}f_{34})v_{14} = 0 \ .
\end{aligned}
$$

(5). $ZS5 = zero\{z51, z52, z53, z54\}$, in which $z51 = z11$ and

$$
\begin{aligned}
z52 \equiv\ & -m_2r_{13}r_{14}v_{13}v_{14}f_{12} - m_3r_{12}r_{14}v_{12}v_{14}f_{13} - m_4r_{12}r_{13}v_{12}v_{13}f_{14} = 0 \ , \\
z53 \equiv\ & -v_{14}^2m_3r_{12}r_{14}f_{13}m_1r_{24}r_{34}f_{14} - v_{14}^2m_3r_{12}r_{14}^2f_{13}m_2r_{34}f_{24} \\
& -v_{14}^2m_3^2r_{12}r_{14}^2f_{13}r_{24}f_{34} + v_{14}m_3r_{12}r_{14}f_{13}m_1r_{24}r_{34}f_{14}v_{13} \\
& -v_{14}m_2r_{13}r_{14}^2v_{13}f_{12}m_3r_{24}f_{34} - v_{14}m_4r_{12}r_{13}v_{13}f_{14}^2m_1r_{24}r_{34} \\
& -v_{14}m_2r_{13}r_{14}v_{13}f_{12}m_1r_{24}r_{34}f_{14} + v_{14}m_3r_{12}r_{14}^2f_{13}m_2r_{34}f_{24}v_{13} \\
& -v_{14}m_4r_{12}r_{13}v_{13}f_{14}m_3r_{14}r_{24}f_{34} - v_{14}m_4r_{12}r_{13}v_{13}f_{14}m_2r_{14}r_{34}f_{24} \\
& +m_4r_{12}r_{13}v_{13}^2f_{14}m_2r_{14}r_{34}f_{24} + m_4r_{12}r_{13}v_{13}^2f_{14}^2m_1r_{24}r_{34} \\
& +m_2r_{13}r_{14}v_{13}^2f_{12}m_1r_{24}r_{34}f_{14} = 0 \ , \\
z54 \equiv\ & m_2r_{13}r_{14}r_{34}m_1r_{23}r_{24}f_{12} + m_2r_{13}r_{14}r_{34}m_4r_{12}r_{23}f_{24} \\
& +m_2r_{13}r_{14}r_{34}m_3r_{12}r_{24}f_{23} + r_{13}m_4r_{12}r_{23}m_3r_{24}f_{34}r_{14} \\
& +r_{13}m_4r_{12}r_{23}m_1r_{24}r_{34}f_{14} + r_{34}m_3r_{12}f_{13}m_1r_{23}r_{24}r_{14} = 0 \ .
\end{aligned}
$$

Similarly decomposing the system $\{qq01, p19, p20, p21\}$ we also obtain five systems of polynomial equations. They may be got from above via replacing the v_1, u_{ij} with u_1, v_{ij}, respectively.

Combining these two groups of systems of polynomial equations and adding the polynomials $p9, p10, p11, p12, p13, p14$, we get 15 systems of polynomial equations. We solve these systems by using the Char-set method and find that all the possible solutions of each system are divided into three cases according to how many $f_{ij} = 0$ as following:
Case 1, Only one f_{ij} equals to zero. It deduces to the possible solutions of collinear case;
Case 2, Two f_{ij} equal to zero. Then it is easy to see that there are at least three f_{ij} are zero. Considering the reality conditions, it has no solutions.
Case 3, There is not any f_{ij} equal to zero. It leads to the possible solutions of non-collinear case.

Most of the 15 systems deduce to the case 2 which get the solutions either all $r_{ij} = 0$ or three of the four points are both collinear and co-circle.

4.1. Collinear case

In the case 1, we have the following system of polynomial equations

$$EQS1 = \{z21, z22, z23, z24, z25, y11, y12, y13, p9, p10, p11, p12, p13, p14\},$$

in which $z21, z22, z23, z24, z25$ are given in the system $ZS2$ and $p9, p10, \cdots, p14$ are given at the page 4 and $y11, y12, y13$ are given by the $z21, z24, z25$ via replacing the v_1, u_{ij} with $u_1, v_{ij},\ i, j = 1, 2, 3, 4.$ respectively.

Note that the total degree of some polynomials in the system $EQS1$ is 7. So, it is very complex to solve $EQ1$. However, we can get the characteristic sets of $EQS1$ by using Wu Elimination. We set the order as following

$$m_1 \prec m_2 \prec m_3 \prec m_4 \prec r_{12} \prec r_{13} \prec r_{23} \prec r_{24} \prec r_{34} \prec m_0 \prec f_{12} \prec f_{13}$$
$$\prec f_{14} \prec f_{23} \prec f_{24} \prec f_{34} \prec u_{14} \prec u_{13} \prec u_{12} \prec v_{14} \prec v_{13} \prec v_{12} \prec u_1 \prec v_1 \ .$$

We obtain 12 characteristic sets of $EQS1$ They correspond to the 12 possible collinear central configurations. For example, we have the following one of the 12 characteristic sets:

$$\begin{aligned}
cs1 \equiv\ & r_{13} + r_{12} - r_{23} = 0 \ , \\
cs2 \equiv\ & r_{13} + r_{24} + r_{14} - r_{23} = 0 \ , \\
cs3 \equiv\ & r_{13} + r_{14} - r_{34} = 0 \ , \\
cs4 \equiv\ & f_{12} = 0 \ , \\
cs5 \equiv\ & r_{14}^2 m_3 f_{13} - f_{14} r_{13}^2 m_4 = 0 \ , \\
cs6 \equiv\ & f_{23} r_{14} r_{24} m_3 + m_3 r_{13} f_{23} r_{24} - m_4 r_{23}^2 f_{24} - r_{23} f_{23} r_{24} m_3 = 0 \ , \\
cs7 \equiv\ & r_{13} r_{23} m_4 m_2 r_{14} r_{34} f_{24} + r_{13} r_{23} m_4 m_1 r_{24} r_{34} f_{14} + r_{13} m_3 r_{24} f_{23} m_2 r_{14} r_{34} \\
& + r_{13} r_{23} m_4 m_3 r_{14} r_{24} f_{34} + m_3 r_{14} f_{13} r_{23} m_1 r_{24} r_{34} = 0 \ , \\
cs8 \equiv\ & r_{14} u_{13} + u_{14} r_{13} = 0 \ , \\
cs9 \equiv\ & r_{23} u_{14} + r_{14} u_{13} - r_{14} u_{12} + r_{13} u_{13} - r_{13} u_{12} - u_{13} r_{23} = 0 \ , \\
cs10 \equiv\ & r_{14}^2 - u_{14} v_{14} = 0 \ , \\
cs11 \equiv\ & v_{13} r_{14} + r_{13} v_{14} = 0 \ , \\
cs12 \equiv\ & v_{14} r_{23} + v_{13} r_{14} - r_{14} v_{12} + v_{13} r_{13} - r_{13} v_{12} - v_{13} r_{23} = 0 \ , \\
cs13 \equiv\ & a u_1 - m_2 u_{12} - m_3 u_{13} - m_4 u_{14} = 0 \ , \\
cs14 \equiv\ & a v_1 - m_2 v_{12} - m_3 v_{13} - m_4 v_{14} = 0 \ .
\end{aligned}$$

Hence, we have the following

Theorem 1. There are 12 collinear central configurations for four planets.

4.2. Flat but non-collinear case

In case 3, we have the following system of polynomial equations:

$$EQS2 = \{z51, z52, z53, z54, y21, y22, y23, y24, p9, p10, p11, p12, p13, p14\}$$

in which $z51, z52, z53, z54$ are given in the system $ZS5$ and $p9, p10, \cdots, p14$ as above in the system $EQS1$ and $y21, y22, y23, y24$ are given by the $z51, z52, z53, z54$ via replacing v_1, u_{ij} with u_1, v_{ij} respectively.

We set the following order

$$m_1 \prec m_2 \prec m_3 \prec m_4 \prec r_{12} \prec r_{13} \prec r_{23} \prec r_{24} \prec r_{34} \prec m_0 \prec g_{12} \prec g_{13}$$
$$\prec g_{14} \prec g_{23} \prec g_{24} \prec g_{34} \prec u_{14} \prec u_{13} \prec u_{12} \prec v_{14} \prec v_{13} \prec v_{12} \prec u_1 \prec v_1$$

and by using Wu Elimination we obtain the following character set which has possible real solutions.

$$
\begin{aligned}
cs21 \equiv\ & r_{12}^4 r_{34}^2 - r_{23}^2 r_{12}^2 r_{34}^2 + r_{23}^2 r_{12}^2 r_{13}^2 - r_{12}^2 r_{24}^2 r_{34}^2 - r_{13}^2 r_{12}^2 r_{34}^2 + r_{14}^2 r_{12}^2 r_{24}^2 \\
& + r_{12}^2 r_{34}^4 - r_{12}^2 r_{34}^2 r_{14}^2 - r_{12}^2 r_{13}^2 r_{24}^2 - r_{14}^2 r_{23}^2 r_{12}^2 + r_{14}^4 r_{23}^2 + r_{14}^2 r_{23}^4 \\
& + r_{13}^2 r_{34}^2 r_{14}^2 - r_{14}^2 r_{34}^2 r_{23}^2 + r_{23}^2 r_{24}^2 r_{34}^2 - r_{14}^2 r_{23}^2 r_{24}^2 + r_{13}^2 r_{24}^4 + r_{13}^4 r_{24}^2 \\
& - r_{13}^2 r_{23}^2 r_{24}^2 - r_{13}^2 r_{14}^2 r_{24}^2 - r_{13}^2 r_{14}^2 r_{23}^2 - r_{13}^2 r_{24}^2 r_{34}^2 = 0\ , \\
cs22 \equiv\ & 2 r_{12}^2 r_{13}^2 r_{14}^2 g_{13} - 2 g_{12} r_{13}^2 r_{14}^2 r_{23}^2 + r_{13}^2 g_{12} r_{14}^2 r_{12}^2 + r_{13}^4 g_{12} r_{14}^2 \\
& - r_{12}^2 g_{13} r_{34}^4 + 2 r_{12}^2 r_{13}^2 g_{13} r_{34}^2 + 2 r_{12}^2 g_{13} r_{34}^2 r_{14}^2 + g_{12} r_{13}^2 r_{12}^2 r_{34}^2 \\
& - g_{12} r_{13}^4 r_{12}^2 + r_{13}^2 g_{12} r_{14}^2 r_{34}^2 - r_{13}^2 g_{12} r_{14}^4 - g_{12} r_{13}^2 r_{24}^2 r_{34}^2 \\
& + r_{13}^4 g_{12} r_{24}^2 + r_{13}^2 g_{12} r_{24}^2 r_{14}^2 - r_{12}^2 r_{13}^4 g_{13} - r_{14}^4 r_{12}^2 g_{13} = 0\ , \\
cs23 \equiv\ & 2 r_{12}^2 g_{14} r_{13}^2 - r_{12}^2 g_{13} r_{34}^2 + r_{12}^2 r_{13}^2 g_{12} + r_{14}^2 g_{12} r_{13}^2 \\
& - g_{12} r_{13}^2 r_{24}^2 + r_{12}^2 g_{13} r_{13}^2 + r_{14}^2 r_{12}^2 g_{13} = 0\ , \\
cs24 \equiv\ & r_{13}^2 r_{12}^2 g_{14}^2 - g_{13} r_{14}^2 g_{23} r_{12}^2 - g_{23} r_{12}^2 g_{14} r_{13}^2 - r_{14}^2 g_{23} g_{12} r_{13}^2 \\
& - r_{14}^2 r_{12}^2 g_{13}^2 - r_{14}^2 g_{13} g_{12} r_{13}^2 - r_{14}^2 r_{12}^2 g_{13} g_{12} - r_{14}^2 r_{13}^2 g_{12}^2 = 0\ , \\
cs25 \equiv\ & r_{13}^2 g_{12} g_{14} + r_{13}^2 g_{14}^2 + r_{13}^2 g_{14} g_{24} - r_{14}^2 g_{13} g_{23} - r_{14}^2 g_{13}^2 - r_{14}^2 g_{13} g_{12} = 0\ , \\
cs26 \equiv\ & g_{34} + g_{24} + g_{23} + g_{14} + g_{13} + g_{12} = 0\ , \\
cs27 \equiv\ & r_{14}^2 r_{13}^2 g_{12}^2 u_{13} u_{14} - u_{14} r_{13}^2 r_{12}^2 g_{14}^2 u_{13} - u_{14}^2 r_{13}^2 r_{12}^2 g_{14} g_{13} \\
& - r_{14}^2 u_{13}^2 r_{12}^2 g_{13} g_{14} - r_{14}^2 u_{13} r_{12}^2 g_{13}^2 u_{14} = 0\ , \\
cs28 \equiv\ & u_{14} r_{13}^2 r_{12}^2 g_{14} + r_{14}^2 u_{13} r_{12}^2 g_{13} + r_{14}^2 r_{13}^2 u_{12} g_{12} = 0\ , \\
cs29 \equiv\ & r_{14}^2 - u_{14} v_{14} = 0\ , \\
cs30 \equiv\ & r_{13}^2 - u_{13} v_{13} = 0\ , \\
cs31 \equiv\ & r_{12}^2 - u_{12} v_{12} = 0\ , \\
cs32 \equiv\ & a u_1 - m_2 u_{12} - m_3 u_{13} - m_4 u_{14} = 0\ , \\
cs33 \equiv\ & a v_1 - m_2 v_{12} - m_3 v_{13} - m_4 v_{14} = 0\ ,
\end{aligned}
$$

in which the notations $g_{12}, g_{13}, g_{14}, g_{23}, g_{24}, g_{34}$ are introduced by the following equations

$$
\begin{aligned}
h1 \equiv\ & m_1 m_2 f_{12} - r_{12} g_{12} = 0\ , \\
h2 \equiv\ & m_1 m_3 f_{13} - r_{13} g_{13} = 0\ , \\
h3 \equiv\ & m_1 m_4 f_{14} - r_{14} g_{14} = 0\ , \\
h4 \equiv\ & m_2 m_3 f_{23} - r_{23} g_{23} = 0\ , \\
h5 \equiv\ & m_2 m_4 f_{24} - r_{24} g_{24} = 0\ , \\
h6 \equiv\ & m_3 m_4 f_{34} - r_{34} g_{34} = 0\ .
\end{aligned}
$$

Hence, from this characteristic set we know that all the solutions are uniquely determined by m_i, m_0, or m_i, ω. So there are finitely many solutions of the system $EQS2$ which are determined by m_i, m_0, or m_i, ω. Thus we have

Theorem 2. In general, the number of the central configuration in planet motions of 4 bodies is finite.

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