Nearest Singular Polynomial 1)

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Abstract. The nearest singular polynomial with a multiple zero of multiplicity $k$ is considered based on minimization of quadratic form, where $k$ is any positive integer. Some recursive relations between the polynomials determining the multiple zeros for consecutive $k$’s are presented.

§1. Introduction

Some problems have been studied based on minimization of quadratic form [?] [?] [?] [?]. One of them is to find the nearest singular polynomial with a double zero to a given polynomials. In this paper, we will consider the case of nearest singular polynomial with a multiple zero of multiplicity $k$ for any positive integer $k$. This is not only a natural generalization of that studied in [?], but also some recursive relations related to determine nearest singular polynomial for consecutive $k$’s are derived.

In §2, we will give a suitable expression for the quadratic form in terms of the undetermined multiple zero $c$ and its conjugate, which leads to easily factor its derivatives with respect to $c$ and $\tau$. And the factored form of its first derivative is given in §3. In §4, some recursive relations of the factors of its first derivative for consecutive $k$’s are derived. In §5, we will determine which factor is useful for our purpose. Numerical examples are given in §6.

§2. Expression of quadratic form

The problem considered is the following. Given a monic polynomial $f$

$$f = x^m + \sum_{j=1}^{m} f_j x^{m-j}, f_j \in \mathbb{C},$$

find a monic polynomial $h(x)$ of the form

$$h = (x - c)^k (x^{m-k} + \sum_{j=1}^{m-k} \phi_j x^{m-k-j}), c, \phi_j \in \mathbb{C},$$

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such that
\[
\mathcal{N} := \| f - h \|^2
\]
is minimized, where
\[
\| p(x) \|^2 = \sum_{j=0}^{k} |p_j|^2
\]
for any
\[
p(x) = \sum_{j=0}^{k} p_j x^{k-j}, \quad p_j \in \mathbb{C}[x].
\]

Since
\[
f(x) - h(x) = \sum_{j=1}^{m} \left( f_j - \sum_{i=0}^{k} \binom{k}{i} (-1)^i c^i \phi_{j-i} \right) x^{m-j},
\]
where \( \phi_{-k+1} = \phi_{-k+2} = \cdots = \phi_{-1} = 0, \phi_0 = 1, \phi_{m-k+1} = \phi_{m-k+2} = \cdots = \phi_m = 0 \), then
\[
\mathcal{N} = \sum_{j=1}^{m} f_j - \sum_{i=0}^{k} \binom{k}{i} (-1)^i c^i \phi_{j-i}^2.
\] (1)

Let \( r := f - g - A\phi \), where
\[
\begin{align*}
f &= (f_1, \ldots, f_m)^T, \\
g &= (a_{21}, \ldots, a_{k1}, 0, \ldots, 0)^T, \\
\phi &= (\phi_1, \ldots, \phi_{m-k})^T, \\
A &= \left( a_{ij} \right)_{m \times (m-k)}, \\
a_{ij} &= \binom{k}{i-j} (-1)^{i-j} c^{i-j}.
\end{align*}
\]
Here as usual
\[
\binom{p}{q} = 0 \text{ if } q > p \text{ or } q < 0.
\]
It is easy to see that the \( j \)th component of \( r \) is the coefficient of \( x^{m-j} \) in \( f(x) - h(x) \). Thus
\[
\mathcal{N} = r^*r = (f - g - A\phi)^*(f - g - A\phi). \quad (2)
\]
For any \( c \), \( \mathcal{N} \) attains its minimum \( \mathcal{N}_m \) when \( \phi \) is the least square solution of equation
\[
f - g - A\phi = 0,
\]
i.e.
\[
\phi = A^+(f - g), \quad (3)
\]
where \( A^+ \) is the Penrose inverse of \( A \).

In the following, we assume \( \phi = A^+(f - g) \). Consequently
\[
\begin{align*}
r &= (I - AA^+)(f - g), \\
\mathcal{N}_m &= (f - g)^*(I - AA^+)(f - g). \quad (4)
\end{align*}
\]
Note that $\mathcal{N}_m$ depends on $c$ only. Our problem becomes finding $c$ such that $\mathcal{N}_m$ is minimized.

Since the columns of $A$ are linearly independent, then

$$A^+ = (A^*A)^{-1}A^*.$$ 

Let $L$ be a $m \times m$ bi-diagonal Toeplitz matrix

$$L = \begin{bmatrix} 1 & -c & 1 & \cdots & 1 \\ -c & 1 & -c & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -c & \cdots & -c & 1 \end{bmatrix}. \quad (5)$$

It is easy to check that for any positive integer $k$, we have

$$L^k = (l_{ij}^{(k)}), \quad l_{ij}^{(k)} = \begin{pmatrix} k \\ i-j \end{pmatrix} (-1)^{i-j} c^{i-j},$$

$$L^{-k} = (l_{ij}^{(-k)}), \quad l_{ij}^{(-k)} = \begin{pmatrix} i-j+k-1 \\ k-1 \end{pmatrix} c^{i-j}.$$ 

The submatrix composed of the first $m-k$ columns of $L^k$ is $A$. Partition $L^k$ and $L^{-k}$ conformally

$$L^k = (A \ B), \quad L^{-k} = \begin{pmatrix} U \\ V \end{pmatrix}.$$ 

Let $W := U(I - V^*(VV^*)^{-1}V)$, then


So

$$W = (A^*A)^{-1}A^* = A^+.$$ 

Now $r$ and $\mathcal{N}_m$ become

$$r = (I - AA^+)(f - g) = V^*(VV^*)^{-1}V(f - g),$$

$$\mathcal{N}_m = (f - g)^*V^*(VV^*)^{-1}V(f - g). \quad (6)$$

It is easy to see that

$$V(f - g) = [\psi_1 \cdots \psi_k]^T := \psi, \quad \psi_i = \frac{1}{(k-1)!}(c^{i-1}f(c))^{(k-1)}.$$ 

$\psi$ can also be expressed as following:

$$\psi = \Omega J \eta,$$

where

$$\Omega = \begin{pmatrix} \Omega_{ij} \end{pmatrix}, \quad \Omega_{ij} = \frac{1}{(k-j)!} \begin{pmatrix} i-1 \\ j-1 \end{pmatrix} c^{i-j}, \quad J = \begin{pmatrix} \delta_{i,k+1-j} \end{pmatrix}$$
\[ \eta = \left[ f(c) \ f'(c) \ \cdots \ f^{(k-1)}(c) \right]. \]  

Therefore

\[ \mathcal{N}_m = \eta^* J \Omega^* (VV^*)^{-1} \Omega J \eta = \eta^* (J \Omega^{-1} V V^*(\Omega^{-1})^*)^{-1} \eta. \]  

Let

\[ \Lambda := J \Omega^{-1} V V^*(\Omega^{-1})^* J = \left( \lambda_{ij} \right), \]

then

\[ \lambda_{ij} = \frac{\partial^j c^i}{\partial c^1 \partial c^{j-1} q}, \ q = \sum_{j=0}^{m} \alpha^j. \]  

Hence

\[ \mathcal{N}_m = \eta^* \Lambda^{-1} \eta. \]  

It is true for any \( k \). We will write it with index \( k \) as

\[ \mathcal{N}^{(k)}_m = \eta^* \Lambda^{-1}^k \eta. \]  

(12) is the expression of quadratic form being used for the following discussion.

§3. First derivative

For any matrix or vector \( M = \left( m_{ij} \right) \), we will use the notation \( \frac{\partial M}{\partial c} = \left( \frac{\partial m_{ij}}{\partial c} \right) \).

**Theorem 1**

\[ \frac{\partial \mathcal{N}^{(k)}_m}{\partial c} = \frac{1}{(\det \Lambda_k)^2} P_{k+1} \bar{P}_k, \]  

where

\[ P_i := \det \begin{pmatrix} \Lambda_{i-1} & \eta_{h-1} \\ \omega_{i-1} & f(i-1) \end{pmatrix}, \ i = k, k + 1, \]

and \( \omega_{i-1}^* \) is the \( i \)th row of \( \Lambda_i \) deleting its last element.

**Proof:** Let \( \xi_k := \Lambda_k^{-1} \eta_k \), i.e., \( \eta_k = \Lambda_k \xi_k \). Now

\[ \frac{\partial \Lambda_k \xi_k}{\partial c} + \Lambda_k \frac{\partial \xi_k}{\partial c} - \frac{\partial \eta_k}{\partial c} = 0, \]

\[ \Lambda_k \frac{\partial \xi_k}{\partial c} = -\frac{\partial \Lambda_k}{\partial c} \Lambda_k^{-1} \eta_k + \frac{\partial \eta_k}{\partial c}. \]

For \( i < k \) the \( i \)th row of \( \frac{\partial \Lambda_k}{\partial c} \) is the \((i + 1)\)th row of \( \Lambda_k \). And the \( k \)th row of \( \frac{\partial \Lambda_k}{\partial c} \) is the
\[(k + 1)\text{th row of } \Lambda_{k+1} \text{ deleting its last element and be denoted by } \omega_k^* . \text{ Therefore}
\]
\[
\Lambda_k \frac{\partial \xi_k}{\partial c} = - \begin{bmatrix} 0 & 1 & & \cdots & 1 \\ \omega_k^* \Lambda_k^{-1} & & & \end{bmatrix} \eta_k + \frac{\partial \eta_k}{\partial c}
\]
\[
= - \begin{bmatrix} f'(c) & \cdots & f^{(k-1)}(c) & \omega_k^* \Lambda_k^{-1} \end{bmatrix}^T + \begin{bmatrix} f'(c) & \cdots & f^{(k)}(c) \end{bmatrix}^T
\]
\[
= \begin{bmatrix} 0 & \cdots & 0 & f^{(k)}(c) - \omega_k^* \Lambda_k^{-1} \eta_k \end{bmatrix},
\]
\[
\frac{\partial \Lambda_{k+1}^*}{\partial c} = \frac{\partial}{\partial c}(\eta_k \xi_k) = \eta_k^* \Lambda_k^{-1} \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} f^{(k)}(c) - \omega_k^* \Lambda_k^{-1} \eta_k]^T
\]
\[
= (f^{(k)}(c) - \omega_k^* \Lambda_k^{-1} \eta_k)^{\det \Lambda_{k+1}^{-1}} (f^{(k-1)}(c) - \omega_k^* \Lambda_{k-1}^{-1} \eta_{k-1})^x
\]
\[
= \frac{1}{(\det \Lambda_k)^2} \det \begin{bmatrix} \Lambda_k & \eta_k \\ \omega_k^* & f^{(k)}(c) \end{bmatrix} \left( \det \begin{bmatrix} \Lambda_{k-1} & \eta_{k-1} \\ \omega_{k-1}^* & f^{(k-1)}(c) \end{bmatrix} \right)^x
\]
\[
= \frac{1}{(\det \Lambda_k)^2} P_{k+1} P_k.
\]

So we have (13). \(\Box\)

§4 Recursive relations

In the following, we will prove some recursive relations between \(\Lambda_k\) and \(P_k\).

Theorem 2.

\[
\det \Lambda_k \frac{\partial^2 \det \Lambda_k}{\partial c \partial \xi_k} = \frac{\partial}{\partial c} \det \Lambda_k \frac{\partial}{\partial \xi_k} \det \Lambda_k = \det \Lambda_{k-1} \det \Lambda_{k+1}.
\]  (15)

Proof: We have

\[
det \Lambda_{k+1} = \det \begin{bmatrix} \Lambda_{k-1} & u_1 & u_2 \\ u_1^* & \alpha & \xi \\ u_2^* & \beta & \eta \end{bmatrix} = \det \Lambda_{k-1} \begin{bmatrix} \alpha & \xi \\ \beta & \eta \end{bmatrix} - \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} \Lambda_{k-1}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]
\[
= \det \Lambda_{k-1} \left( \alpha - u_1^* \Lambda_{k-1}^{-1} u_1 \right) \left( \eta - u_2^* \Lambda_{k-1}^{-1} u_2 \right)
\]
\[
- \det \Lambda_{k-1} \left( \beta - u_2^* \Lambda_{k-1}^{-1} u_1 \right) \left( \xi - u_1^* \Lambda_{k-1}^{-1} u_2 \right),
\]
\[
det \Lambda_k = \det \begin{bmatrix} \Lambda_{k-1} & u_1 \\ u_1^* & \alpha \end{bmatrix} = \det \Lambda_{k-1} \left( \alpha - u_1^* \Lambda_{k-1}^{-1} u_1 \right).
\]

Since in \(\det \Lambda_k\), the partial derivative of the \(i\)th row with respect to \(c\) is the \((i + 1)\)th row for \(i < k\), and the \(k\)th row of \(\frac{\partial \det \Lambda_k}{\partial c}\) is the \((k + 1)\)th row of \(\det \Lambda_{k+1}\) deleting its last element
and similarly for column with respect to \( c \), then
\[
\frac{\partial}{\partial c} \det \Lambda_k = \det \begin{pmatrix} 
\Lambda_{k-1} & u_1 \\
u_2 & \beta 
\end{pmatrix} = \det \Lambda_{k-1} \left( \beta - u_2' \Lambda_{k-1}^{-1} u_1 \right),
\]
\[
\frac{\partial}{\partial \bar{c}} \det \Lambda_k = \det \begin{pmatrix} 
\Lambda_{k-1} & u_2 \\
u_1' & \xi 
\end{pmatrix} = \det \Lambda_{k-1} \left( \xi - u_1' \Lambda_{k-1}^{-1} u_2 \right),
\]
\[
\frac{\partial^2}{\partial c \partial \bar{c}} \det \Lambda_k = \det \begin{pmatrix} 
\Lambda_{k-1} & u_2' \\
u_2' & \eta 
\end{pmatrix} = \det \Lambda_{k-1} \left( \eta - u_2' \Lambda_{k-1}^{-1} u_2 \right).
\]

Thus, we get (15). \( \square \)

In an analogous fashion, we derive Theorem 3 and 4.

**Theorem 3.**
\[
P_k \frac{\partial^2}{\partial c \partial \bar{c}} P_k - \frac{\partial}{\partial c} P_k \frac{\partial}{\partial \bar{c}} P_k = P_{k-1} P_{k+1}.
\tag{16}
\]

**Theorem 4.**
\[
\det \Lambda_k \frac{\partial}{\partial c} P_k - P_k \frac{\partial}{\partial \bar{c}} \det \Lambda_k = \det \Lambda_{k-1} P_{k+1}.
\tag{17}
\]

§5 Useful factor

Let \( c = a + ib \), where \( a \) and \( b \) are real, and consider \( N_m^{(k)} \) as a real rational function of real variables \( a \) and \( b \). The problem is to find real solutions of the system
\[
\frac{\partial N_m^{(k)}}{\partial a} = 0, \quad \frac{\partial N_m^{(k)}}{\partial b} = 0.
\tag{18}
\]

Because
\[
\frac{\partial N_m^{(k)}}{\partial c} = \frac{1}{2} \left( \frac{\partial N_m^{(k)}}{\partial a} - i \frac{\partial N_m^{(k)}}{\partial b} \right), \quad \frac{\partial N_m^{(k)}}{\partial \bar{c}} = \frac{1}{2} \left( \frac{\partial N_m^{(k)}}{\partial a} + i \frac{\partial N_m^{(k)}}{\partial b} \right).
\tag{19}
\]

It is sufficient to consider
\[
\frac{\partial N_m^{(k)}}{\partial c} = 0,
\tag{20}
\]
for determining \( c \). The trace \( T \) and determinant \( D \) of the Hessian matrix can be expressed as
\[
T = 4 \frac{\partial^2 N_m^{(k)}}{\partial c \partial \bar{c}}, \quad D = 4 \left( \frac{\partial^2 N_m^{(k)}}{\partial c^2} - \frac{\partial^2 N_m^{(k)}}{\partial c \partial \bar{c}} \right)^2.
\tag{21}
\]

By Theorem 1, we consider \( P_k = 0 \) and \( P_{k+1} = 0 \) separately.

**Theorem 5.** \( N_m^{(k)} \) attains its local minimum at \( c \) satisfying \( P_k = 0 \) if
\[
(\det \Lambda_{k-1})^4 P_{k+1} \overline{P_{k+1}} - (\det \Lambda_k)^4 P_{k-1} \overline{P_{k-1}} > 0.
\tag{22}
\]
Proof: If $P_k = 0$, then we have:
\[
\frac{\partial P_k}{\partial c} = \frac{\det \Lambda_{k-1} P_{k+1}}{\det \Lambda_k} \frac{\partial T_k}{\partial c} = -P_{k-1} P_{k+1} \frac{\partial P_k}{\partial c} = - \frac{\det \Lambda_k}{\det \Lambda_{k-1}} P_{k-1},
\]
\[
T = 4 \frac{P_{k+1}}{(\det \Lambda_k)^2} \frac{\partial T_k}{\partial c} = 4 \frac{P_{k+1}}{(\det \Lambda_k)^2} \frac{\det \Lambda_{k-1} P_{k+1}}{\det \Lambda_k} = 4 \frac{\det \Lambda_{k-1}}{(\det \Lambda_k)^3} P_{k+1} P_{k+1} \geq 0,
\]
\[
D = 4 \left( \left( \frac{\det \Lambda_{k-1}}{(\det \Lambda_k)^3} P_{k+1} P_{k+1} \right)^2 - \frac{P_{k-1} P_{k-1} P_{k+1} P_{k+1}}{(\det \Lambda_k)^4} \right)
\]
\[
= 4 \frac{P_{k+1} P_{k+1}}{(\det \Lambda_{k-1})^2 (\det \Lambda_k)^3} \left( (\det \Lambda_{k-1})^4 P_{k+1} P_{k+1} - (\det \Lambda_k)^4 P_{k-1} P_{k-1} \right).
\]
So $\mathcal{N}^{(k)}_m$ attains its local minimum at $c$ satisfying $P_k = 0$ if $D > 0$, i.e.,
\[
(\det \Lambda_{k-1})^4 P_{k+1} P_{k+1} - (\det \Lambda_k)^4 P_{k-1} P_{k-1} > 0.
\]
\[
\Box
\]
For $P_{k+1} = 0$ we have the following conclusion.

**Theorem 6.** For $c$ satisfying $P_{k+1} = 0$, the trace is non-positive.

**Proof:** If $P_{k+1} = 0$ then
\[
\frac{\partial^2 \mathcal{N}^{(k)}_m}{\partial c \partial \bar{c}} = \frac{T_k}{(\det \Lambda_k)^2} \frac{\partial P_{k+1}}{\partial \bar{c}}.
\]
\[
\frac{\partial P_{k+1}}{\partial c} \frac{\partial \det \Lambda_{k-1}}{\partial c} = \frac{\partial \det \Lambda_k}{\partial c} \frac{\partial P_k}{\partial c} + \det \Lambda_k \frac{\partial^2 P_k}{\partial c \partial \bar{c}} - \frac{\partial P_k}{\partial c} \frac{\partial \det \Lambda_k}{\partial \bar{c}} - P_k \frac{\partial^2 \det \Lambda_k}{\partial c \partial \bar{c}}
\]
\[
\frac{\partial \det \Lambda_k}{\partial \bar{c}} \frac{\partial P_k}{\partial \bar{c}} - P_k \frac{\partial^2 \det \Lambda_k}{\partial c \partial \bar{c}} = \frac{\partial \det \Lambda_k}{\partial \bar{c}} \frac{\partial P_k}{\partial \bar{c}} - \frac{\partial P_k}{\partial c} \frac{\partial \det \Lambda_k}{\partial \bar{c}} = \frac{\det \Lambda_k}{P_k} \frac{\partial \det \Lambda_k}{\partial \bar{c}} P_{k-1} P_{k+1} = 0
\]
So we have
\[
T = 4 \frac{\partial^2 \mathcal{N}^{(k)}_m}{\partial c \partial \bar{c}} = -4 \frac{(\det \Lambda_k)^3}{P_k} P_{k+1} \leq 0.
\]
\[
\Box
\]
We conclude that it is sufficient to consider those zeros of $P_k = 0$ which are not zeros of $P_{k+1} = 0$. As for zeros of $P_{k+1} = 0$, they are candidates of $c$ with multiplicity higher than $k$. For the common zeros of $P_k = 0$ and $P_{k+1} = 0$, by repeatedly using Theorem 3 and Theorem 4, they are zeros of $P_j = 0$ for all $m \geq j > k$.

§5. Numerical Examples

**Example 1:** $f = x^5 - x$.

For $k = 2$, we have get in [?] that there are 4 nearest singular polynomials due to the geometry of the zeros of $f$, one of them is given below:
\[
h \approx x^5 + 0.03895547966x^4 + 0.06708530296x^3 + 0.1155277233x^2 \\
-0.8010494959x + 0.3426130279.
\]
The zeros of $h$ are:
The other 3 can be obtained by rotation with an angle $\pi/2$, $\pi$ and $3\pi/2$ respectively.

For $k = 3$, there are 4 nearest singular polynomials, one of them is given below:

$$h \approx x^5 + (-0.1510160305 - 0.1510160305i)x^4 - 0.311275054ix^3 + (0.3858582928 - 0.3858582928i)x^2 - 0.5746952045x + 0.09187090467 + 0.09187090467i$$

with the roots given below:

<table>
<thead>
<tr>
<th>zeros of $h$</th>
<th>zeros of $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5806857529 ( double )</td>
<td>0, 1</td>
</tr>
<tr>
<td>-1.050883646</td>
<td>-1</td>
</tr>
<tr>
<td>-0.07472166958+0.9804509313i</td>
<td>i</td>
</tr>
<tr>
<td>-0.07472166958-0.9804509313i</td>
<td>-i</td>
</tr>
</tbody>
</table>

$N_m = 0.1763296120$.

c = 0 is the common zero of $P_4$ and $P_5$, for $k = 4, 5$, the nearest singular polynomial is:

$$h = x^5,$$

and

$N_m = 1$.

Example 2:

$$f = (x - 0.89 - 0.03i)(x - 0.88 + 0.02i)(x - 0.87)(x - 1)$$

$$= x^4 + (-3.64 - 0.011)x^3 + (4.9637 + 0.0273i)x^2 + (-3.005606 - 0.024782i)x$$

$$+ 0.681906 + 0.007482i.$$ 

For $k = 2$, we have get in [?] that the nearest singular polynomial of $f$ is unique:

$$h \approx x^4 + (-3.639999897 - 0.0100012076i)x^3 + (4.963700115 + 0.02729986094i)x^2$$

$$+ (-3.005605870 - 0.02478216008i)x + 0.6819061456 + 0.007481815756i,$$

with the roots given below:

<table>
<thead>
<tr>
<th>zeros of $h$</th>
<th>zeros of $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8768135619 - 0.01006779565i ( double )</td>
<td>0.88 - 0.02i, 0.87</td>
</tr>
<tr>
<td>0.8866786823 + 0.02982563187i</td>
<td>0.89 + 0.03i</td>
</tr>
<tr>
<td>0.9996940915 + 0.0003100788900i</td>
<td>1</td>
</tr>
</tbody>
</table>
\[ N_m = 0.1552760144 \times 10^{-12}. \]

For \( k = 3 \), the nearest singular polynomial with a zero of multiplicity 3 is also unique:

\[ h \approx x^4 + (-3.639968566 - 0.01002119406i)x^3 + (4.963698969 + 0.02730077333i)x^2 \\
+(-3.005622319 - 0.0247706306i)x + 0.6819004541 + 0.007485892787i. \]

The zeros of \( h \) are:

\[
\begin{array}{c|c}
\text{zeros of } h & \text{zeros of } f \\
\hline
0.8817735725 + 0.002337412033i (\text{ triple }) & 0.89 + 0.03i, 0.88 - 0.02i, 0.87 \\
0.9946478484 + 0.003008957959i, & 1 \\
\end{array}
\]

\[ N_m = 0.3311925673 \times 10^{-8}. \]

For \( k = 4 \), the nearest singular polynomial with a zero of multiplicity 4 is:

\[ h \approx x^4 + (-3.637528548 - 0.009999075252i)x^3 + (4.961817732 + 0.02727894127i)x^2 \\
+(-3.008080147 - 0.02480691942i)x + 0.6838580852 + 0.007519618582i. \]

The zeros of \( h \) are:

\[
\begin{array}{c|c}
\text{zeros of } h & \text{zeros of } f \\
\hline
0.9093821369 + 0.002499768813i & 0.89 + 0.03i, 0.88 - 0.02i, 0.87, 1 \\
\end{array}
\]

\[ N_m = 0.00425554008. \]

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References


