Reduced form of Yang-Mills equation of $SU(2)$ on $R^{4,0}$

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Abstract. An explicit expression of a differential linear transformation is obtained in this letter. The Yang-Mills equation of $SU(2)$ on Euclidean four-dimensional flat space becomes a reduced form via this differential linear transformation. And then, some solutions of the Yang-Mills equation could be obtained very easily.

Key Words: Yang-Mills equation, Differential linear transformation, Wu elimination

1. Notations

Let $R^{4,0}$ be the Euclidean four-dimensional flat space and $x = (x^1, x^2, x^3, x^4)$ be the coordinates. Let $gl(2)$ be the Lie algebra of group $SU(2)$. The anti-Hermitian representations of a basis of $gl(2)$ are taken as follows

$$
X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (1.1)
$$

in which $i = \sqrt{-1}$.

The 12 unknown functions $A^a_i(x)$, $a = 1, 2, 3$, $i = 1, 2, 3, 4$ are denoted by

$$
A^1_i(x) = A_i(x), \quad A^2_i(x) = B_i(x), \quad i = 1, 2, 3, 4, \quad (1.2)
$$

The connections are given by

$$
G_j = A_j(x)X_1 + B_j(x)X_2 + C_j(x)X_3, \quad j = 1, 2, 3, 4. \quad (1.3)
$$

That is,

$$
G_j = \begin{pmatrix} A_j(x)i & B_j(x) + C_j(x)i \\ -B_j(x) + C_j(x)i & -A_j(x)i \end{pmatrix},
$$
The curvatures are given by
\[ F_{jk} = \partial_k G_j - \partial_j G_k + G_j G_k - G_k G_j, \quad j, k = 1, 2, 3, 4, \] (1.4)
in which the notations \( \partial_j = \partial/\partial x^j, \quad j = 1, 2, 3, 4. \)

The Yang-Mills equation of \( SU(2) \) on \( R^{4,0} \) consists of 12 equations \[ \sum_{k=1}^{4} (\partial_k F_{jk} - G_k F_{jk} + F_{jk} G_k) = 0, \quad j = 1, 2, 3, 4. \] (1.5)

2. Differential linear transformation

We introduce

**Definition**: a transformation \( S : u(x) \mapsto v(x), v = C(u) \) between functions \( u(x) \) and \( v(x) \) is called a **differential linear transformation** if \( C(u) \) consists of linear differential operators only.

In [6], Maxwell equation is studied by using differential linear transformation. The goal of this letter is to find a differential linear transformation such that the Yang-Mills equation of \( SU(2) \) on \( R^{4,0} \) becomes a reduced form. Now we set

\[ A_i(x) = \sum_{j=1}^{4} s_{ij} \partial_j u(x), \]
\[ B_i(x) = \sum_{j=1}^{4} s_{ij} \partial_j v(x), \quad i = 1, 2, 3, 4, \] (1.6)
\[ C_i(x) = \sum_{j=1}^{4} s_{ij} \partial_j w(x), \]
in which \( u(x), v(x), w(x) \) are three new functions and \( s_{ij}, i, j = 1, 2, 3, 4 \) are unknown constants. We denote them by a matrix

\[ S = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}. \]

3. Reduced form of Yang-Mills equation

For \( R^{4,0} \), we substitut \( A_i(x), B_i(x), C_i(x), \quad i = 1, 2, 3, 4 \) which are given by formula (1.6) in Yang-Mills equation (1.5) and obtain certain product terms such as \( \partial_j u(x) \partial_k u(x) \partial_l u(x), \quad j, k, l = 1, 2, 3, 4. \) The coefficients of these terms are polynomials
of unknown constants $s_{ij}$, $i, j = 1, 2, 3, 4$. We set these polynomials equal to zeros and then obtain a system of polynomial equations. By using Wu elimination [1,2], we get a solution of this system

$$S_{40} = s \begin{pmatrix} 1 & -i & -1 & i \\ -i & 1 & i & -1 \\ 1 & i & 1 & -i \\ -i & -1 & i & 1 \end{pmatrix}, \quad (1.7)$$

in which $s$ is a parameter.

The main conclusion of this letter is

**Theorem.** By the differential linear transformation given by formula (1.6) and (1.7), the Yang-Mills equation (1.5) turns out to be a reduced form

$$YM_{j1} = L_{j1}D_{40}(u) + P_{j1}(w)D_{40}(v) + Q_{j1}(v)D_{40}(w) = 0,$$

$$YM_{j2} = L_{j2}D_{40}(v) + P_{j2}(u)D_{40}(w) + Q_{j2}(w)D_{40}(u) = 0,$$

$$YM_{j3} = L_{j3}D_{40}(w) + P_{j3}(v)D_{40}(u) + Q_{j3}(u)D_{40}(v) = 0,$$

$$j = 1, 2, 3, 4,$$

in which

$$D_{40} = \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2$$

is the Laplace operator and

$$L_{ji}, P_{ji}, Q_{ji}, j = 1, 2, 3, 4, \; i = 1, 2, 3$$

are certain linear differential operators.

For example, for $j = 1$ we have

$$YM_{11} = +s(\partial_3 + \partial_4 + \mathbf{i}\partial_5)D_{40}(u)$$

$$+2s^2(\partial_2w + \partial_1w - \mathbf{i}\partial_1w + \mathbf{i}\partial_5w)D_{40}(v)$$

$$-2s^2(\partial_2v + \partial_1v - \mathbf{i}\partial_1v + \mathbf{i}\partial_5v)D_{40}(w),$$

$$YM_{12} = +s(\partial_4 + \partial_2 + \mathbf{i}\partial_5)D_{40}(v)$$

$$+2s^2(-\partial_2w - \partial_4w - \mathbf{i}\partial_3w + \mathbf{i}\partial_1w)D_{40}(u)$$

$$+2s^2(\partial_2u + \partial_4u + \mathbf{i}\partial_3u - \mathbf{i}\partial_1u)D_{40}(w),$$

$$YM_{13} = +s(\partial_3 - \mathbf{i}\partial_4 - \mathbf{i}\partial_2)D_{40}(w)$$

$$+2s^2(-\partial_1v + \partial_3v - \mathbf{i}\partial_2v - \mathbf{i}\partial_4v)D_{40}(u)$$

$$+2s^2(\partial_1u - \partial_3u + \mathbf{i}\partial_2u + \mathbf{i}\partial_4u)D_{40}(v).$$
Proof. Formula (1.8) can be obtained via direct computation.

It is trivial that any solution $u(x), v(x), w(x)$ of

$$D_{40}(u) = 0, \; D_{40}(v) = 0, \; D_{40}(w) = 0 \quad (1.9)$$

provide a solution of Yang-Mills equation of $SU(2)$ on $R^{40}$. Thus we have

Proposition. By using differential linear transformation given by (1.6) and (1.7), a lot of solutions of Yang-Mills equation of $SU(2)$ on $R^{40}$ can be obtained from the reduced form of Yang-Mills equation.

The reduced form of Yang-Mills equation of $SU(2)$ on other four-dimensional flat space is studied in [4,5].

References


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