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Completion of an Index Vector Set in Differential Algebra [1)]

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Abstract. In this paper, a local condition for completeness on an index vector set was given. Based on this, two algorithms were given: one is to compute the closure of a given finite set, and the other is to determine a minimum basis of the complementary space of a given finite set. These are important for determining integrability conditions of differential equations.

Key words prolongation, completion, closure

1. Introduction

It is difficult to deal with a large system of differential equations. In 1910, Riquier[**?**] generalized Cauchy-Kovalevskaya theorem on existence of solutions of a special system of differential equations. The advantages of Riquier's theorem was found in dealing with partial differential equations. Janet gave an unified method to simplify a system of linear partial differential equations to orthonomic form proposed by Riquier [**?**, **?**]. Furthermore, Ritt proposed differential algebra to deal with differential equations with algebraic point view [**?**]. Their theory was greatly developed by Herman, Kolchin, Wu, et al. [**?**, **?**, **?**, **?**], and recently, implemented by Schwarz, Reid, et al.[**?**, **?**, **?**, **?**]. The basic idea of their methods is differential algebraic reduction, which involves two important things, completion and integrability. The time used to determine integrability conditions, therefore the total time in reduction process is greatly affected by the size of completion and by determining it. By the well known corresponding between partial derivatives and index vectors, the problem can be attributed to the problem of completion of an index vector set[**?**]. In this paper, we try to improve the methods has been used to determine completion of an index vector set. We first give a local condition for completeness, and, based on this, propose the closed set which is a special kind of complete set. Then we give an algorithm to determine the closure of a given set, which is

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[2)] On leave from: Department of Mathematics Jinzhou Normal College, Jinzhou 121003, P.R. China the smallest closed set containing the given set. Finally we give an algorithm to construct a minimal basis of the complementary space of a given set.

Let N be the set of all non-negative integers. Consider operations on N^n. In addition to the usual addition $\alpha+\beta$, scalar product $j\alpha$, we also define the Hadamard product as follows

$$(\alpha_1, \alpha_2, \cdots, \alpha_n) \circ (\beta_1, \beta_2, \cdots, \beta_n) = (\alpha_1\beta_1, \alpha_2\beta_2, \cdots, \alpha_n\beta_n) \tag{1.1}$$

Denote by $\mathcal{F}$ the set of all finite subsets of N^n, and introduce an operator $\mathbf{b} : \mathcal{F} \to N^n$

$$\mathbf{b}_k(\Gamma) = \max\{\gamma_k | \gamma \in \Gamma\} \tag{1.2}$$

$k = 1, 2, \cdots, n$, which is called the upper bound operator. The set

$$B(\Gamma) = \{\gamma \in N^n | \gamma_k \le \mathbf{b}_k(\Gamma),\ k = 1, 2, \cdots, n\} \tag{1.3}$$

is called the bound of Γ.

Given a finite subset Γ of N^n, the set $M(\Gamma) = \{\gamma + \mu | \gamma \in \Gamma, \mu \in N^n\}$ is called the multiple space, $N(\Gamma) = N^n \setminus M(\Gamma)$ the complementary space. For a single vector $\gamma \in N^n$, we call

$$M(\gamma) = \{\gamma + \mu | \mu \in N^n\} \tag{1.4}$$

the prolongation of γ. In practice, it is often necessary to point out a prolongation direction. A prolongation direction is a special index vector δ whose components are 0 or 1. We denote by $\delta^{(*)}, \delta^{(0)}$ the directions with components all 1 and components 0 respectively, by $\delta^{(k)}$ the vector whose kth component is 1 and the others are all 0, $k = 1, 2, \cdots, n$. Using these notations the prolongation and the prolongation along a direction δ of γ can be expressed as

$$M(\gamma) = \{\gamma + \mu \circ \delta^{(*)} | \mu \in N^n\} \tag{1.5}$$

$$M(\gamma, \delta) = \{\gamma + \mu \circ \delta | \mu \in N^n\} \tag{1.6}$$

respectively. We say also that a prolongation δ is given for a finite set Γ if every element of Γ is given a prolongation direction $\delta(\gamma)$.

A basis of a subset Γ of N^n is a finite subset Γ_0 of Γ such that Γ_0 has been given a prolongation direction δ and satisfies the following relations

$$\begin{cases} \Gamma = \cup_{\alpha\in\Gamma_0} M(\alpha, \delta(\alpha)) \\ M(\alpha, \delta(\alpha)) \cap M(\beta, \delta(\beta)) = \{\}, \alpha \ne \beta, \alpha, \beta \in \Gamma_0 \end{cases} \tag{1.7}$$

where $\{\}$ stands for the empty set.

As the above statement, it is important to determine minimum bases of $M(\Gamma)$ and of $N(\Gamma)$ for a given finite subset Γ of N^n. In sections 2 and 3 we will discuss the completion, the closure and a kind of special prolongation direction, the so called multiplier direction, and algorithms to determine them. Closures and completions are all useful bases of $M(\Gamma)$. At the last section we give an algorithm to compute a minimum basis of $N(\Gamma)$.

2. Partition of index vectors

The concept of multiplier is very important in Janet theory[?]. In this section we propose multiplier direction, which is convenient to explain our idea, to replace that concept. Our theory is based on the partition of a index vector set.

For a finite subset Γ of N^n, the partition of Γ is given by the following algorithm:

Let $b_n = \mathbf{b}_n(\Gamma)$, then Γ is divided into $1 + b_n$ parts

$$\Gamma_0,\ \Gamma_1,\ \cdots,\ \Gamma_{b_n} \tag{2.1}$$

where

$$\Gamma_i = \{\gamma \in \Gamma | \gamma_n = i\} \quad 0 \le i \le b_n \tag{2.2}$$

Let $b_{n-1,j_n} = \mathbf{b}_{n-1}(\Gamma_{j_n})$, then Γ_{j_n} is divided into $1 + b_{n-1,j_n}$ parts

$$\Gamma_{0j_n},\ \Gamma_{1j_n},\ \cdots,\ \Gamma_{tj_n} \tag{2.3}$$

where $t = b_{n-1,j_n}$, and

$$\Gamma_{j_{n-1}j_n} = \{\gamma \in \Gamma_{j_n} | \gamma_{n-1} = j_{n-1}\},\ 0 \le j_{n-1} \le b_{n-1,j_n}, 0 \le j_n \le b_n \tag{2.4}$$

Continuing, we finally get that $b_{1j_2\cdots j_n} = \mathbf{b}_1(\Gamma_{j_2\cdots j_n})$, and

$$\Gamma_{j_1 j_2\cdots j_n},\ 0 \le j_1 \le b_{1j_2\cdots j_n}, \cdots, 0 \le j_n \le b_n \tag{2.5}$$

It is clear that $\Gamma_{j_1j_2\cdots j_n} = \{(j_1, j_2, \cdots, j_n)\}$ or $\{\}$, and $\Gamma = \cup_{j_1j_2\cdots j_n}\Gamma_{j_1j_2\cdots j_n}$. We call the above algorithm the partition of Γ.

Lemma 1. The partition of Γ has the following properties

1. $\alpha, \beta \in \Gamma_{j_k\cdots j_n} <=> \alpha_i = j_i = \beta_i,\ i = k, \cdots, n$;
2. If $\Gamma \subseteq \Gamma'$, then $\Gamma_{j_k\cdots j_n} \subseteq \Gamma'_{j_k\cdots j_n}$;
3. If $\Gamma = \Gamma' \cap \Gamma''$, then $\Gamma_{j_k\cdots j_n} = \Gamma'_{j_k\cdots j_n} \cap \Gamma''_{j_k\cdots j_n}$;

Definition 1. The multiplier direction of $\gamma \in \Gamma$ is a vector $\delta(\gamma) \in N^n$, such that

$$\delta_n(\gamma) = \begin{cases} 1 \text{ if } & \gamma_n = b_n, \\ 0 \text{ else} & \end{cases},\ \delta_k(\gamma) = \begin{cases} 1 \text{ if } & \gamma_k = b_{k\gamma_{k+1}\cdots\gamma_n}, \\ 0 \text{ else} & \end{cases} \tag{2.6}$$

where $k = 1, 2, \cdots, n-1$.

Lemma 2. If $\Gamma \subseteq \Gamma'$ and $\gamma \in \Gamma$, then

$$\delta_k(\gamma) = 0 \Rightarrow \delta'_k(\gamma) = 0 \tag{2.7}$$

where δ and δ' are the multiplier directions of Γ and Γ' respectively.

Next lemma describes an important property of multiplier direction, which gives us a chance to prolong Γ separately.

Lemma 3. For $\alpha, \beta \in \Gamma$, and $\beta \ne \alpha$, then

$$M(\alpha, \delta(\alpha)) \cap M(\beta, \delta(\beta)) = \{\} \tag{2.8}$$

Proof For $\gamma \in M(\alpha, \delta(\alpha)) \cap M(\beta, \delta(\beta))$, we let

$$\gamma = \alpha + \mu \circ \delta(\alpha) = \beta + \nu \circ \delta(\beta), \mu, \nu \in N^n \tag{2.9}$$

If $\alpha <_{IL} \beta$ ($<_{IL}$ is the inverse lexicographic order in N^n), we choose k such that $\alpha_k < \beta_k, \alpha_{k+1} = \beta_{k+1}, \cdots, \alpha_n = \beta_n$, which implies that $\delta_k(\alpha) = 0$. By (2.9), $\alpha_k = \beta_k + \nu_k \delta_k(\beta) \geq \beta_k$, this is a contradiction.

Let $M^*(\Gamma)$ be the disjoint union

$$\cup_{\gamma \in \Gamma} M(\gamma, \delta(\gamma)) \tag{2.10}$$

where δ is the multiplier direction of Γ.

Definition 2. A finite set Γ of index vectors is called complete if $M(\Gamma) = M^*(\Gamma)$.

Theorem 1. A finite set Γ of index vectors is complete if and only if

$$\forall \gamma \in \Gamma, \delta_k(\gamma) = 0 \ \Rightarrow \gamma + \delta^{(k)} \in M^*(\Gamma) \tag{2.11}$$

Proof The necessity is clear. We will prove the sufficiency. For any $\alpha \in M(\Gamma)$, we let

$$\alpha = \gamma + \mu, \ \mu \in N^n \ \gamma \in \Gamma \tag{2.12}$$

and choose γ in (2.12) as high as possible with the order $<_{IL}$.

If $\delta_k(\gamma) = 0$ and $\mu_k > 0$, then $\mu - \delta^{(k)} \in N^n$ and $\alpha = \gamma + \delta^{(k)} + (\mu - \delta^{(k)})$. By the condition, there exist β, ν such that

$$\gamma + \delta^{(k)} = \beta + \nu \circ \delta(\beta) \tag{2.13}$$

where $\beta \in \Gamma, \nu \in N^n$.

If $\nu_k \delta_k(\beta) \neq 0$, then $\nu_k \geq 1, \delta_k(\beta) = 1$. We have $\gamma = \beta + \nu' \circ \delta(\beta)$, where $\nu' = \nu - \delta^{(k)}$. But Lemma 3 implies $\gamma = \beta$, this is in cotradiction with $\delta_k(\gamma) = 0, \delta_k(\beta) = 1$. So $\nu_k \delta_k(\beta) = 0$, i.e.

$$\gamma_k + 1 = \beta_k, \ \gamma_l = \beta_l + \nu_l \delta_l(\beta), \quad l \neq k \tag{2.14}$$

.

We claim that $\gamma_{k+1} = \beta_{k+1}, \cdots, \gamma_n = \beta_n$, and therefore $\gamma <_{IL} \beta$. Otherwise there exists an integer $l (\geq k+1)$, such that $\gamma_l > \beta_l, \gamma_{l+1} = \beta_{l+1}, \cdots, \gamma_n = \beta_n$. This implies two facts: $\delta_l(\beta) = 0$ by the definition of multiplier direction, and $\delta_l(\beta) = 1$ by (2.14), they are contrary.

Substituting (2.13) into (2.12), we derive $\alpha = \beta + \nu \circ \delta(\beta) + (\mu - \delta^{(k)})$, which is in contradiction with the choice of γ. So the assumption that $\mu_k \neq 0$ implies $\delta_k(\gamma) = 1$. Hence $\alpha = \gamma + \mu \circ \delta(\gamma) \in M^*(\Gamma)$. The proof is completed.

Any complete set Γ' containing Γ and satisfying $M(\Gamma') = M(\Gamma)$ is called the completion of Γ. For the requirement of the prolongation theory of differential equations, we wish to get for a given finite set Γ of index vectors, a completion which is contained any other completion of Γ. But the following example tells us that thus completion does not exist at many times. The reason is that the intersection of two complete sets is not necessary to be complete.

Example. $\Gamma' = \{(3,2), (1,3), (1,4), (1,5), (2,5)\}$ and $\Gamma'' = \{(3,2), (2,3), (2,4), (2,5)\}$ are both complete, but the intersection $\Gamma = \Gamma' \cap \Gamma'' = \{(3,2), (2,5)\}$ is not.

To remedy the defect we introduce the concept of the closed set.

3. Closed sets

Definition 3. A finite set Γ of index vectors is called closed set if

$$\forall \gamma \in \Gamma,\ \delta_k(\gamma) = 0 \Rightarrow \gamma + \delta^{(k)} \in \Gamma \tag{3.1}$$

where δ is multiplier direction of Γ.

By theorem 1, a closed set is surely complete. Another well-property of closed sets is stated in the following theorem.

Theorem 2. The intersection of two closed sets is closed too.

Proof Let Γ', Γ'' be both closed sets, and $\Gamma = \Gamma' \cap \Gamma''$. For any $\gamma \in \Gamma$, if $\delta_k(\gamma) = 0$, then $\delta'_k(\gamma) = \delta''_k(\gamma) = 0$ by Lemma 2. So $\gamma + \delta^{(k)} \in \Gamma', \gamma + \delta^{(k)} \in \Gamma''$, and $\gamma + \delta^{(k)} \in \Gamma$. Γ is therefore a closed set.

For a finite set Γ, $\Gamma' = M(\Gamma) \cap B(\Gamma)$ is a closed set containing Γ. In fact, if $\gamma \in \Gamma'$ and $\delta'_k(\gamma) = 0$, then there exists a vector $\beta \in \Gamma'_{\gamma_{k+1}\cdots\gamma_n}$, such that $\beta_k > \gamma_k$. This implies $\gamma_k + 1 \leq \mathbf{b}_k(\Gamma)$. On the other hand, $\gamma \in M(\Gamma)$ implies $\gamma + \delta^{(k)} \in M(\Gamma)$. So $\gamma + \delta^{(k)} \in \Gamma'$, Γ' is closed.

Definition 4. The smallest closed set which contains Γ is called the the closure of Γ, denoted by $\bar{\Gamma}$.

The closure of Γ is unique, which is the intersection of all the closed sets containing Γ. It is not difficult to determine the closure of a finite set of index vectors. Here We give an algorithm.

[**Prolongation Algorithm**]

Step 1. Set $b_n = \mathbf{b}_n(\Gamma)$, and determine gradually the sets as follows

$$\begin{array}{l} \bar{\Gamma}_0 = \{\gamma \in \Gamma |\ \gamma_n = 0\} \\ \bar{\Gamma}_1 = \{\gamma \in \Gamma | \gamma_n = 1\} \cup \{\gamma + \delta^{(n)} | \gamma \in \bar{\Gamma}_0\} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \bar{\Gamma}_{b_n} = \{\gamma \in \Gamma | \gamma_n = b_n\} \cup \{\gamma + \delta^{(n)} | \gamma \in \bar{\Gamma}_{b_n - 1}\} \end{array} \tag{3.2}$$

Step 2. Suppose that all $\bar{\Gamma}_{j_{k+1}\cdots j_n}$'s have been determined, set $b_{kj_{k+1}\cdots j_n} = \mathbf{b}_k(\bar{\Gamma}_{j_{k+1}\cdots j_n})$, and determine gradually the sets as follows

$$\begin{array}{l} \bar{\Gamma}_{0j_{k+1}\cdots j_n} = \{\gamma \in \bar{\Gamma}_{j_{k+1}\ldots j_n} | \gamma_k = 0\} \\ \bar{\Gamma}_{1j_{k+1}\cdots j_n} = \{\gamma \in \bar{\Gamma}_{j_{k+1}\cdots j_n} | \gamma_k = 1\} \cup \{\gamma + \delta^{(k)} | \gamma \in \bar{\Gamma}_{0j_{k+1}\cdots j_n}\} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \bar{\Gamma}_{tj_{k+1}\cdots j_n} = \{\gamma \in \bar{\Gamma}_{j_{k+1}\cdots j_n} | \gamma_k = t\} \cup \{\gamma + \delta^{(k)} | \gamma \in \bar{\Gamma}_{t-1,j_{k+1}\cdots j_n}\} \end{array} \tag{3.3}$$

where $t = b_{kj_{k+1}\cdots j_n}$, $(k = n-1, \cdots, 2, 1)$.

Step 3. Let $\bar{\Gamma} = \cup_{j_1 j_2 \cdots j_n} \bar{\Gamma}_{j_1 j_2 \cdots j_n}$.

$\bar{\Gamma}$ is surely the closure of Γ. To prove this we note that

$$(\bar{\Gamma})_{j_k \cdots j_n} = \bar{\Gamma}_{j_k \cdots j_n},\ k = 1, 2, \cdots, n \tag{3.4}$$

Denote by $\bar{\delta}$ the multiplier direction of $\bar{\Gamma}$. For any $\gamma \in \bar{\Gamma}$, if $\bar{\delta}_k(\gamma) = 0$, then $\gamma_k < \mathbf{b}_k(\bar{\Gamma}_{\gamma_{k+1}\cdots\gamma_n})$. So $\gamma + \delta^{(k)} \in \bar{\Gamma}_{\gamma_k+1,\gamma_{k+1}\cdots\gamma_n} \subseteq \bar{\Gamma}$ by the algorithm. $\bar{\Gamma}$ is a closed set.

Suppose that Γ' is any closed set containing Γ. It is clear that $\mathbf{b}_n(\Gamma') \geq \mathbf{b}_n(\Gamma)$, and $\bar{\Gamma}_0 \subseteq \Gamma'_0, \{\gamma \in \Gamma | \gamma_n = 1\} \subseteq \Gamma'_1$. These imply that $\bar{\Gamma}_1 \subseteq \Gamma'_1$ since Γ' is closed. In this way, the relations $\bar{\Gamma}_{j_n} \subseteq \Gamma'_{j_n}(0 \leq j_n \leq b_n)$ can be proved in turn. Furthermore, the relations $\bar{\Gamma}_{j_k \cdots j_n} \subseteq \Gamma'_{j_k \cdots j_n}$ $(k = 1, 2, \cdots, n)$ can be proved by induction. These relations imply $\bar{\Gamma} \subseteq \Gamma'$. Hence $\bar{\Gamma}$ is indeed the closure of Γ.

There are several advantages on the closure even though it is not necessary to be minimum completion(with the least number of elements). In addition to the ease to be determined (its multiplier direction can be also determined simultaneously as in Definition 1), the closure can be easily contracted to a smaller completion.

[**Contraction Algorithm**]

For every $\bar{\Gamma}_{j_2 \cdots j_n}$ derived from the Prolongation Algorithm, if

1. there are at least two elements in $\bar{\Gamma}_{j_2 \cdots j_n}$;
2. $\alpha = (b_{1 j_2 \cdots j_n}, j_2, \cdots, j_n) \bar{\in} \Gamma$.

are both satisfied, then let

$$\bar{\Gamma}_{j_2 \cdots j_n} = \bar{\Gamma}_{j_2 \cdots j_n} \setminus \{\alpha\} \tag{3.5}$$

Repeat step (3.5) until at least one of the conditions is broken, then denote

$$\Gamma^* = \cup_{j_2 \cdots j_n} \bar{\Gamma}_{j_2 \cdots j_n} \tag{3.6}$$

Γ^* is also a completion of Γ, called modified completion. In general, modified completion is of smaller size than other completions we have known up to now. For example, $\Gamma = \{(0,1,1),(3,1,1),(3,0,3),(2,2,3)\}$, then

$$\bar{\Gamma} = \left\{ \begin{array}{lllll} (0,1,1) & (0,1,2) & (0,1,3) & (0,2,3) & (3,0,3) \\ (1,1,1) & (1,1,2) & (1,1,3) & (1,2,3) & \\ (2,1,1) & (2,1,2) & (2,1,3) & (2,2,3) & \\ (3,1,1) & (3,1,2) & (3,1,3) & (3,2,3) & \end{array} \right\}$$

$$\Gamma^* = \bar{\Gamma} \setminus \{(1,1,2),(2,1,2),(3,1,2);(1,1,3),(2,1,3),(3,1,3);(3,2,3)\}$$

According to Schwarz's algorithm[8,9], the completion of Γ is $\bar{\Gamma} \setminus \{(3,2,3)\}$; Wu's completion[10] is $\bar{\Gamma} \cup \{(0,2,1),(1,2,1),(2,2,1),(3,2,1),(0,2,2),(1,2,2),(2,2,2),(3,2,2)\}$.

4. Minimum basis of $N(\Gamma)$

According to Wu's theory [**?**], $B(\Gamma) \setminus M(\Gamma)$ is a base of $N(\Gamma)$ with a special prolongation direction. It is easy to be determined , but is not minimum. In this section we give an algorithm to determine a minimum basis of $N(\Gamma)$. Such base, corresponding to the independent parametric derivatives, can be used to measure the size of the set of solutions of differential equations with the normal form, as Schwarz, Reid did [**?**, **?**].

Lemma 4. For two prolongation directions δ and δ', $M(\alpha, \delta) \cap M(\beta, \delta')$ to be not empty, it is necessary and sufficient that

$$\alpha_k < \beta_k \Rightarrow \delta_k = 1; \quad \beta_k < \alpha_k \Rightarrow \delta'_k = 1, \quad k = 1, 2, \cdots, n$$

Furthermore, if $M(\alpha,\delta)\cap M(\beta,\delta')\neq\{\}$, then $M(\alpha,\delta)\cap M(\beta,\delta') = M(\sigma,\delta'')$, where $\sigma=\mathbf{b}(\{\alpha,\beta\})$, $\delta''=\delta\circ\delta'$.

[**Algorithm to determine minimum bases of** $N(\Gamma)$]

Step 1. Set $Q=M(\Gamma)=\cup_{\alpha\in\Gamma}M(\alpha,\delta^{(*)}), P=B(\Gamma)\setminus Q, R=\{\}$.

Step 2. If $P\neq\{\}$, take the lowest element α of P according to order $<_{IL}$, and define the prolongation direction $\delta(\alpha)$ as follows

$$\delta_1(\alpha)=\begin{cases}1 \text{ if } M(\alpha,\delta^{(1)})\cap Q=\{\}\\ 0 \text{ else}\end{cases}$$

$$\delta_j(\alpha)=\begin{cases}1 \text{ if } M(\alpha,(\delta_1(\alpha),\cdots,\delta_{j-1}(\alpha),1,0,\cdots,0))\cap Q=\{\},\\ 0 \text{ else}\end{cases} \tag{4.1}$$

where $j=2,\cdots,n$.

Step 3. Set $R=R\cup\{\alpha\}, Q=Q\cup M(\alpha,\delta(\alpha)), P=B(\Gamma)\setminus Q$, Go to step 2.

Lemma 5. $\forall\alpha\in B(\Gamma)\cap N(\Gamma)$, there exists an index vector $\sigma\in R$, such that $\alpha\in M(\sigma,\delta(\sigma))$. Furthermore, if $\alpha_k=\mathbf{b}_k(\Gamma)$, then $\delta_k(\sigma)=1$.

Proof The firt part of the lemma is clearly right. For $\alpha\in B(\Gamma)\cap N(\Gamma)$, let $\alpha\in M(\sigma,\delta(\sigma))$. To prove the second part , we suppose that $\alpha_k=\mathbf{b}_k(\Gamma)$ and $\delta_k(\sigma)=0$, which implies $\alpha_k=\sigma_k$. We let R_0 represent the set $\{\beta\in R|\beta<_{IL}\sigma\}$, and define a new prolongation direction for σ as follows

$$\delta'_k(\sigma)=1,\ \delta'_l(\sigma)=\delta_l(\sigma), l\neq k \tag{4.2}$$

By the algorithm above,

$$M(\sigma,\delta(\sigma))\cap\cup_{\beta\in R_0}M(\beta,\delta(\beta))=\{\} \tag{4.3}$$

$$M(\sigma,\delta'(\sigma))\cap\cup_{\beta\in R_0}M(\beta,\delta(\beta))\neq\{\} \tag{4.4}$$

and (4.4) implies that there exist vectors $\beta\in R_0,\mu,\nu\in N^n$, such that $\sigma+\mu\circ\delta'(\sigma)=\beta+\nu\circ\delta(\beta)$. i.e

$$\begin{cases}\sigma_l+\mu_l\delta_l(\sigma)=\beta_l+\nu_l\delta_l(\beta) & l\neq k\\ \sigma_k+\mu_k=\beta_k+\nu_k\delta_k(\beta)\end{cases} \tag{4.5}$$

But $\sigma_k=\mathbf{b}_k(\Gamma)\geq\beta_k,\nu_k\delta_k(\beta)\geq\mu_k$, gives $\nu_k\geq\mu_k$. Discussing two cases: $\delta_k(\beta)=0$ and $\delta_k(\beta)=1$, we derive $\sigma_k=\beta_k+(\nu_k-\mu_k)\delta_k(\beta)$. Combining with (4.5) we have

$$\begin{aligned}&\sigma_l+\mu_l\delta_l(\sigma)=\beta_l+\nu_l\delta_l(\beta),\quad l\neq k\\ &\sigma_k+\mu_k\delta_k(\sigma)=\beta_k+(\nu_k-\mu_k)\delta_k(\beta),\ (\text{since } \delta_k(\sigma)=0)\end{aligned} \tag{4.6}$$

These are contrary to (4.3). So $\delta_k(\sigma)=1$.

Theorem 2. The set R with the prolongation direction δ obtained by the above algorithm is a minimum basis of $N(\Gamma)$. Furthermore, N^n has the decomposition as follows

$$N^n=\cup_{\beta\in R}M(\beta,\delta(\beta))\oplus\cup_{\alpha\in C(\Gamma)}M(\alpha,\delta^*(\alpha)) \tag{4.7}$$

where $\oplus$ is disjoint union, $C(\Gamma)$ is any completion of Γ, δ^* is multiplier direction of $C(\Gamma)$.

Proof It is sufficient to prove that

$$N(\Gamma) = \cup_{\beta \in R} M(\beta, \delta(\beta)) \tag{4.8}$$

by the algorithm.

For any $\gamma \in N(\Gamma)$, set $\beta_k = \min\{\gamma_k, \mathbf{b}_k(\Gamma)\}$, $k = 1, 2, \cdots, n$, then $\beta \in B(\Gamma) \cap N(\Gamma)$. By Lemma 5, there exists $\sigma \in R$, such that $\beta \in M(\sigma, \delta(\sigma))$, i.e $\beta = \sigma + \mu \circ \delta(\sigma)$, $\mu \in N^n$, and $\delta_k(\sigma) = 1$ when $\beta_k = \mathbf{b}_k(\Gamma)$. So $\gamma = \sigma + \mu' \circ \delta(\sigma) \in M(\sigma, \delta(\sigma))$, where $\mu' = (\gamma - \beta) + \mu$.

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