MM Research Preprints,9–22
No. 18, Dec. 1999. Beijing

Vector Representations of Involutive Divisions [1)]

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Abstract. This paper has two parts. First, a vector representation of the involutive division proposed by Zharkov, Gerdt, etal was given. Using this representation, most of the properties of involutive divisions such as Noetherity, Artinity and Constructivity, can be greatly simplified. Second, Thomas and Janet divisions were generalized to two types of divisions, and new involutive divisions were found. Also the involutive completion algorithm given by Gerdt etal was improved.

1. Introduction

In [4], Gerdt and Blinkov proposed the involutive division in axiomatic form. This concept mainly comes from three classical approaches used to determine integrability conditions of partial differential equations (PDEs).

Based on Riquier's theorem [11], which is a generalization of the well-known Cauchy-Kowalevsky theorem on existence of solutions of partial differential equations, Janet proposed the involutivity conditions for orthonomic systems proposed by Riquier and designed an algorithm for their completion [8]. By his approach, independent variables are separated into two parts: multiplicative and non-multiplicative, and in order to find new integrability conditions one need only prolongs given equations along with non-multiplicative variables. The Riquier-Janet theory was developed by Ritt and Wu into the characteristic set method [12, 22]. Schwarz clarified the Riquier-Janet theory and used it to develop programs for dealing with determining equations of symmetries of PDEs [13, 14, 15].

Thomas used another separation method for independent variables into multiplicative and non-multiplicative ones, and generalized the Riquier-Janet theory to non-orthonomic algebraic PDEs [16]. Combining Thomas completion method with Ritt characteristic set approach, Wu proposed well-order principle and zero decomposition algorithms for nonlinear algebraic differential polynomial systems [19, 20, 21]. Wu's method was developed and used to solve polynomial systems [2, 3, 19, 20], to prove theorems in geometries[1], to simplify partial differential equations [9, 23], and to compute the well-behaved bases(equivalent to the Gröbner bases) of polynomial systems [22].

The third method proposed by Pommaret [10] allows to formulate the involutivity intrinsically, in a coordinates independent way. This approach was greatly developed by Zharkov and Blinkov [24]. They showed that sequential multiplication of the polynomial in the system

[1)] This work was supported in part by An outstanding Youth Grant from Chinese NSF, a National Key Projection on Basic Research, and by the Educational Committee of Liaoning Province, P.R. China

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by non-multiplicative variables, and reduction of these prolonged polynomials modulo others, by means of their multiplicative power products only, ends up, under certain conditions, with a Gröbner basis.

The above classical approaches are based on separations of independents. The common properties of them were extracted and used to give formal definition of the involutive division in [4, 5, 6]. Furthermore, Gerdt and his coworkers gave , for a general involutive division, algorithms on completion of polynomials and linear differential systems to an involutive base [7].

In oder to reveal further essential properties and to find new involutive divisionΦ , we introduce a vector representation of the involutive division in the present paper. Using this representation, most of the properties of the involutive division such as Noetherity, Artinity and constructivity, can be greatly simplified. Some new properties are also found. This paper is arranged as follows. In Section 2, we give a vector representation of the involutive division, called involutive direction, and descriptions of three classical directions. The basic properties such as Noetherity, Artinity and constructivity, are discussed in Section 3. A proof of the equivalence between the involutive direction and the involutive division can be found in Section 4. In Sections 5 and 6, We generalize Thomas and Janet directions to two types directions, and give an improvement for the algorithm to determine minimal involutive monomial basis, proposed in [5].

2. Basic Notions and Notations

For any real vector α, we denote by α_i the ith component of α, and for any two n-dimensional vectors α and β, we call the vector $(\alpha_1\beta_1, \cdots, \alpha_n\beta_n)$ Hadamard's product of α and β, denoted by $\alpha \circ \beta$. In this paper, we mainly consider exponent vectors, i.e. vectors whose components are all non-negative integers. The set of all such vectors is denoted by $\mathbf{N}^n$. Let $\Delta_n = \{\alpha \in \mathbf{N}^n | \alpha_i = 0 \text{ or } 1, i = 1, \cdots, n\}$. We introduce the following concepts.

Definition 1. δ is said to be a direction on $\mathbf{N}^n$ if for every finite nonempty subset $\Gamma$of $\mathbf{N}^n$, a map $\delta^\Gamma : \Gamma \to \Delta_n$ can be given.

For example, δ^* can be defined as follows: for any finite subset Γ of $\mathbf{N}^n, \delta^{*\Gamma}(\alpha) = (1, 1, \cdots, 1)$, $\forall \alpha \in \Gamma$. Then δ^* is a direction on $\mathbf{N}^n$. Clearly, the image $\delta^{*\Gamma}(\alpha)$ of α is independent of the set Γ which α belongs to, such directions are said to be global.

Definition 2. A direction δ on $\mathbf{N}^n$ is an called involutive direction if the following conditions are satisfied

(i) $\forall \alpha, \beta \in \Gamma$, if $\alpha + \mu \circ \delta^\Gamma(\alpha) = \beta + \nu \circ \delta^\Gamma(\beta), \mu, \nu \in \mathbf{N}^n$, then either $\beta - \alpha$ and $\delta^\Gamma(\alpha) - \delta^\Gamma(\beta)$ are both non-negative, or $\alpha - \beta$ and $\delta^\Gamma(\beta) - \delta^\Gamma(\alpha)$ are both non-negative.

(ii) If $\alpha \in \Sigma \subseteq \Gamma$, then $\delta^\Sigma(\alpha) - \delta^\Gamma(\alpha)$ is non-negative.

It is ease to examine δ^* is not involutive for $n \geq 2$. We now consider the three classical methods of variable separations [8, 10, 16]. In our terminology they are called Thomas direction, Janet direction and Pommaret direction respectively, which corresponds in turn to Thomas division, Janet division and Pommaret division respectively in [4]. For convenience, we define an operator $\mathbf{b}$ on $\mathcal{F}_n$, the set of all finite nonempty subset of $\mathbf{N}^n$, as follows

$$\mathbf{b} : \ \Gamma \mapsto \gamma \quad \forall \Gamma \in \mathcal{F}_n,$$

where $\gamma_i = \max\{\beta_i | \beta \in \Gamma\}, i = 1, 2, \cdots, n$. We call $\mathbf{b}$ the upper bound operator, and the set

$$B(\Gamma) = \{\beta \in \mathbf{N}^n | \beta_i \leq \mathbf{b}_i(\Gamma), i = 1, \cdots, n\}$$

the bound of Γ.

Example 1. Thomas direction can be defined as follows

$$\forall \alpha \in \Gamma, \delta_i^\Gamma(\alpha) = \begin{cases} 1 & \text{if } \alpha_i = \mathbf{b}_i(\Gamma) \\ 0 & \text{otherwise} \end{cases}$$

$i = 1, 2, \cdots, n$, where $\delta_i^\Gamma(\alpha), \mathbf{b}_i(\Gamma)$ represent the ith components of $\delta^\Gamma(\alpha)$ and $\mathbf{b}(\Gamma)$ respectively.

Example 2. Janet direction can be defined as follows

$$\forall \alpha \in \Gamma, \quad \begin{array}{l} \delta_1^\Gamma(\alpha) = \begin{cases} 1 & \text{if } \alpha_1 = \mathbf{b}_1(\Gamma) \\ 0 & \text{otherwise} \end{cases} \\ \delta_i^\Gamma(\alpha) = \begin{cases} 1 & \text{if } \alpha_i = \mathbf{b}_i(\Gamma_{\alpha_1 \cdots \alpha_{i-1}}) \\ 0 & \text{otherwise} \end{cases} \end{array}$$

where $\Gamma_{\alpha_1 \cdots \alpha_{i-1}} = \{\beta \in \Gamma | \beta_j = \alpha_j, j = 1, \cdots, i-1\}, \quad i = 2, \cdots, n.$

Example 3. Pommaret direction is defined as

$$\forall \alpha \in \Gamma, \delta_i^\Gamma(\alpha) = \begin{cases} 1 & \text{if } i \geq L(\alpha) \\ 0 & \text{otherwise} \end{cases} \qquad i = 1, \cdots, n$$

where $L(\alpha)$ represents the number k such that the kth component of α is the last non-zero component of α if $\alpha \neq 0$. $k = 0$ if $\alpha = 0$.

For Thomas and Janet directions, we have

$$\alpha + \mu \circ \delta^\Gamma(\alpha) = \beta + \nu \circ \delta^\Gamma(\beta) \Longrightarrow \alpha = \beta \tag{2.1}$$

where $\alpha, \beta \in \Gamma$. Otherwise , we may suppose, without loss of generality, $\alpha_1 = \beta_1, \cdots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i$. According to the definition of Thomas and Janet directions, we have $\delta_i^\Gamma(\beta) = 0$, whence $\alpha_i + \mu_i \delta_i^\Gamma(\alpha) = \beta_i \geq \alpha_i$ by (2.1), contradiction. The condition (i) of Definition 2 is satisfied. As for condition (ii), note the fact: if $\alpha \in \Sigma \subseteq \Gamma$ then $\Sigma_{j_1 \cdots j_{i-1}} \subseteq \Gamma_{j_1 \cdots j_{i-1}}$. Therefore $\mathbf{b}_i(\Sigma) \leq \mathbf{b}_i(\Gamma), \mathbf{b}_i(\Sigma_{j_1 \cdots j_{i-1}}) \leq \mathbf{b}_i(\Gamma_{j_1 \cdots j_{i-1}}), j = 2, \cdots, n$. Hence $\delta_i^\Gamma(\alpha) = 1$ implies $\delta_i^\Sigma(\alpha) = 1$, i.e. $\delta^\Sigma - \delta^\Gamma$ is non-negative. So Thomas and Janet directions are both involutive.

Pommaret direction is clearly global, so the condition (ii) of Definition 2 is satisfied. For any two vectors $\alpha, \beta \in \Gamma$, we suppose $k = L(\alpha) \leq L(\beta) = j$. (2.1) implies $\alpha_1 = \beta_1, \cdots, \alpha_{k-1} = \beta_{k-1}, \alpha_k + \mu_k = \beta_k, \cdots, \alpha_j + \mu_j = \beta_j, \alpha_{j+1} = \beta_{j+1} = 0, \cdots, \alpha_n = \beta_n = 0$ by the definition of Pommaret direction. Hence $\beta - \alpha$ and $\delta^\Gamma(\alpha) - \delta^\Gamma(\beta)$ are both negative. Pommaret direction is also involutive.

Definition 3. A set Γ is called autoreduced with respect to a direction δ, or δ- autoreduced if (2.1) implies $\alpha = \beta$ for every pair of $\alpha, \beta \in \Gamma$.

Any finite set is autoreduced with respect to Thomas and Janet directions.

3. Noetherian, Artinian, and Constructive Directions

From now on, δ represents an involutive direction, $<_{lex}$ represents lexicographical order.

3.1. Noetherian directions

For $\alpha \in \Gamma$, we call $\alpha + \mu \circ \delta^{\Gamma}(\alpha)$ a prolongation of α. The set of all prolongations of α is denoted by $P_{\delta}^{\Gamma}(\alpha)$. Let $P_{\delta}(\alpha) = \cup_{\alpha \in \Gamma} P_{\delta}^{\Gamma}(\alpha)$.

Definition 4. A finite subset Γ of $\mathbf{N}^n$ is called complete with respect to δ, if $P_{\delta}(\Gamma) = P^*(\Gamma)$, where $P^*(\Gamma) = P_{\delta^*}(\Gamma)$.

Clearly $\Gamma' = P_{\delta}(\Gamma) \cap B(\Gamma)$ is complete with respect to Thomas and Janet directions, and $\Gamma' \supseteq \Gamma$. For any direction δ and any finite set Γ, if there exists a finite set Γ' such that

(a) Γ' is complete with respect to δ, and $\Gamma' \supseteq \Gamma$;

(b) $P_{\delta}(\Gamma') = P^*(\Gamma)$.

then Γ is said to be *finitely generated* with respect to δ, Γ' is called a completion of Γ. If every finite set is finitely generated with respect to δ, then δ is said to be *Noetherian.*

Thomas and Janet directions are Noetherian, but Pommaret direction is not.

3.2. Artinian directions

For the convenience, we often omit the Γ and δ in $\delta^{\Gamma}(\alpha)$ and $P_{\delta}(\Gamma)$ respectively. Denote by $\delta_i(\alpha)$ the ith component of $\delta(\alpha)$, and by $\delta^{(i)}$ the vector whose ith component is 1 and the other components are 0. Index i is called multiplier(or non-multiplier) if $\delta_i(\alpha) = 1$(or $\delta_i(\alpha) = 0$). If i is a non-multiplier of α, we call $\alpha + \delta^{(i)}$ a non- multiplicative prologation of α.

Definition 5. $\alpha, \beta \in \Gamma$, β is said to be a pseudo-divisor of α if there exists an index i and a vector $\mu \in \mathbf{N}^n$, such that

$$\alpha + \delta^{(i)} = \beta + \mu \circ \delta(\beta), \;\; \delta_i(\alpha) = 0. \tag{3.1}$$

Lemma 1. If (3.1) holds, then

(T) $alpha + \delta^{(i)} = \beta$, i.e. $\alpha_i + 1 = \beta_i, \alpha_j = \beta_j, j \neq i$, are valid for Thomas direction;

(J) $\alpha_1 = \beta_1, \cdots, \alpha_{i-1} = \beta_{i-1}, \alpha_i + 1 = \beta_i$, are valid for Janet direction;

(P) $L(\alpha) > L(\beta)$, or $L(\alpha) = L(\beta)$ and $\alpha_1 = \beta_1, \cdots, \alpha_{i-1} = \beta_{i-1}, \alpha_i + 1 = \beta_i$, are valid for Pommaret direction

Consider a sequence

$$^{(1)}\beta, \;\; ^{(2)}\beta, \;\; \cdots, \;\; ^{(k)}\beta, \;\; \cdots \tag{3.2}$$

where the $^{(i)}\beta \in \Gamma$. If every $^{(i+1)}\beta$ is a pseudo-divisor of $^{(i)}\beta$ for $i = 1, 2, \cdots$, then (3.2) is said to be a pseudo-divisor sequence of Γ. If for every finite set Γ, every pseudo-divisor sequence is finite, then δ is said to be Artinian. Since Γ is a finite set, the property that 'every pseudo-divisor sequence is finite' is equivalent to the property that 'every pseudo-divisor sequence consists of distinct elements'.

For Thomas and Janet directions, β is a pseudo-divisor of α implies $\alpha <_{lex} \beta$. For Pommaret direction, β is a pseudo-divisor of α implies $L(\alpha) > L(\beta)$, or $L(\alpha) = L(\beta)$ and $\alpha <_{lex} \beta$. So the three directions are all Artinian directions since relations $<_{lex}$ and $>$ are all transitive.

Theorem 1. Let δ be an Artinian direction. A finite set Γ is complete whith respect to δ if and only if

$$\delta_i(\gamma) = 0 \Rightarrow \gamma + \delta^{(i)} \in P_\delta(\Gamma), \tag{3.3}$$

for all $\gamma \in \Gamma$.

Proof. The necessity is clear. For the sufficiency, we suppose $P^*(\Gamma) \setminus P_\delta(\Gamma) \neq \phi$, and consider any element γ of it. Set $\gamma = {}^{(1)}\!\beta + {}^{(1)}\!\nu, {}^{(1)}\!\beta \in \Gamma$. Since $\gamma \notin P_\delta(\Gamma)$, there is an index i_1, such that ${}^{(1)}\!\nu_{i_1} \neq 0$, and $\delta_{i_1}({}^{(1)}\!\beta) = 0$. So ${}^{(1)}\!\beta + {}^{(1)}\!\nu = {}^{(1)}\!\beta + \delta^{(i_1)} + ({}^{(1)}\!\nu - \delta^{(i_1)})$. By (3.3), ${}^{(1)}\!\beta + \delta^{(i_1)} \in P_\delta(\Gamma)$, i.e. there is a ${}^{(2)}\!\beta \in \Gamma$, such that ${}^{(1)}\!\beta + \delta^{(i_1)} = {}^{(2)}\!\beta + {}^{(2)}\!\mu \circ \delta({}^{(2)}\!\beta)$, whence ${}^{(2)}\!\beta$ is a pseudo-divisor of ${}^{(1)}\!\beta$. We have ${}^{(1)}\!\beta + {}^{(1)}\!\nu = {}^{(2)}\!\beta + {}^{(2)}\!\nu \notin P_\delta(\Gamma)$. By this method we may get an infinite pseudo-divisor sequence of Γ. This is in contradiction with Artinian property of δ.

For a given Γ, if (3.3) holds for all $\gamma \in \Gamma$, we say Γ is locally complete. In [23], the authors gave another proof for Janet direction only. A special kind of complete set, called closed set, was introduced. An algorithm to compute the minmal closed set Γ' such that $\Gamma' \supseteq \Gamma$ and $P_\delta(\Gamma') = P^*(\Gamma)$, which is called closure of Γ, was given for a finite set Γ.

3.3. Constructive directions

Definition 6. A non-multiplicative prolongation $\alpha + \delta^{(i)}$ of $\Gamma \in \mathcal{F}_\setminus$ is said to be critical if the following conditions are satisfied:

(c) $\alpha + \delta^{(i)} \notin P_\delta(\Gamma)$;

(d) If $\alpha + \delta^{(i)} = \beta + \delta^{(j)} + \gamma$, where $\beta + \delta^{(j)}$ is also a nonmulti plicative prolongation of Γ and $\gamma \neq 0$, then $/\beta + \delta^{(j)} \in P_\delta(\Gamma)$

A direction δ is said to be constructive if, for every finite set Γ, no critical prolongation $\alpha + \delta^{(i)}$ of Γ can be expressed as

$$\alpha + \delta^{(i)} = \beta + \mu \circ \delta(\beta) + \nu \circ \delta^{\Gamma'}(\beta') \tag{3.4}$$

where $\beta \in \Gamma, \beta' = \beta + \mu \circ \delta(\beta), \Gamma' = \Gamma \cup \{\beta'\}$.

Thomas, Janet and Pommaret directions are all constructive. For Thomas direction, we suppose that $\alpha + \delta^{(i)}$ is a critical prolongation and (3.4) holds. If $\delta_j^{\Gamma}(\beta) = 0$, then $\beta'_j = \beta_j < \mathbf{b}_j(\Gamma) \leq \mathbf{b}_j(\Gamma')$. Hence $\delta_j^{\Gamma'}(\beta') = 0$. Setting $\nu' = \nu \circ \delta^{\Gamma'}(\beta')$, we have $\alpha + \delta^{(i)} = \beta + (\mu + \mu') \circ \delta^{\Gamma}(\beta) \in P_\delta(\Gamma)$, which is in contradiction with (c) of Definition 6. For Pommaret direction, $\delta_j(\beta) = 0$ implies $j < L(\beta) \leq L(\beta')$. So $\delta_j^{\Gamma'}(\beta') = 0$, and the remaining part is similar to the proof for Thomas direction.

As for Janet direction, choose α as high as possible with respect to $<_{lex}$, such that $\alpha + \delta^{(i)}$ is a critical prolongation, and (3.4) holds. We claim that $\alpha <_{lex} \beta$. Otherwise one may suppose $\alpha_1 = \beta_1, \cdots, \alpha_{k-1} = \beta_{k-1}, \alpha_k > \beta_k$. If $k \geq i$, then $\alpha_1 = \beta_1 = \beta', \cdots, \alpha_{i-1} = \beta_{i-1} = \beta'_{i-1}, \alpha_i \geq \beta_i$ by (3.4). Since $\delta_i(\alpha) = 0$, $\alpha_i < \mathbf{b}_i(\Gamma_{\alpha_1 \cdots \alpha_{i-1}}) = \mathbf{b}_i(\Gamma_{\beta_1 \cdots \beta_{i-1}})$, whence $\delta_i(\beta) = 0$. Hence $\beta'_i = \beta_i < \mathbf{b}_i(\Gamma_{\beta_1 \cdots \beta_{i-1}}) \leq \mathbf{b}_i(\Gamma'_{\beta'_1 \cdots \beta'_{i-1}})$, and $\delta_i^{\Gamma'}(\beta') = 0$. By (3.4), $\alpha_i + 1 = \beta_i$, contradiction. If $k < i$, then by (3.4), $\alpha_1 = \beta_1 = \beta', \cdots, \alpha_{i-1} = \beta_{k-1} = \beta'_{k-1}, \alpha_k > \beta_k$. This implies $\delta_k(\beta) = 0, \beta'_k = \beta_k, \delta_k^{\Gamma'}(\beta') = 0$, and $\alpha_k = \beta_k$, which is a contradiction.

Since $\alpha + \delta^{(i)} \notin P_\delta(\Gamma)$, we have $\nu_j \delta_j^{\Gamma'}(\beta') \neq 0$ and $\delta_j(\beta) = 0$. Hence $\beta' \neq \beta$. (3.4) can be rewritten as

$$\alpha + \delta^{(i)} = \beta + \delta^{(j)} + \gamma \tag{3.5}$$

where $\gamma = \mu \circ \delta(\beta) + (\nu \circ \delta^{\Gamma'}(\beta') - \delta^{(j)}) \neq 0$. By (d) of Definition 6, $\beta + \delta^{(j)} \in P_\delta(\Gamma)$. Setting $\beta + \delta^{(j)} = {}^{(1)}\alpha + \sigma \circ \delta({}^{(1)}\alpha), {}^{(1)}\alpha \in \Gamma$, by Lemma 1 we have $\beta <_{lex} {}^{(1)}\alpha$, and

$$\alpha + \delta^{(i)} = {}^{(1)}\alpha + {}^{(1)}\gamma,\ \alpha <_{lex} {}^{(1)}\alpha \tag{3.6}$$

where ${}^{(1)}\gamma = \sigma + \delta({}^{(1)}\alpha) + \gamma \neq 0$. Since $\alpha + \delta^{(i)} \notin P_\delta(\Gamma)$, we have ${}^{(1)}\gamma_{i_1} \neq 0$, such that $\delta_{i_1}({}^{(1)}\alpha) = 0$. Rewrite (3.6) as $\alpha + \delta^{(i)} = {}^{(1)}\alpha + \delta^{(i_1)} + ({}^{(1)}\gamma - \delta^{(i_1)})$. By the choice of α, ${}^{(1)}\gamma - \delta^{(i_1)} \neq 0$, whence ${}^{(1)}\alpha + \delta^{(i_1)} \in P_\delta(\Gamma)$. Setting ${}^{(1)}\alpha + \delta^{(i_1)} = {}^{(2)}\alpha + \tau \circ \delta({}^{(2)}\alpha), {}^{(2)}\alpha \in \Gamma$. By Lemma 1 we have ${}^{(1)}\alpha <_{lex} {}^{(2)}\alpha$, and

$$\alpha + \delta^{(i)} = {}^{(2)}\alpha + {}^{(2)}\gamma,\quad \alpha <_{lex} {}^{(2)}\alpha \tag{3.7}$$

where ${}^{(2)}\gamma = \tau \circ \delta({}^{(2)}\alpha) + ({}^{(1)}\gamma - \delta^{(i_1)}) \neq 0$. In this way, we would get an infinite sequence

$$\alpha <_{lex} \beta <_{lex} {}^{(1)}\alpha <_{lex} {}^{(2)}\alpha <_{lex} \cdots$$

of Γ, which is in contradiction with the finiteness of Γ.

Theorem 2. Let δ be Artinian and constructive. Any completion Γ' of a finite set Γ contains all critical prolongations of Γ.

Proof. Suppose that there exists a critical prolongation $\alpha+\delta^{(i)}$ of Γ, which does not belong to Γ'. Since $\alpha + \delta^{(i)} \in P^*(\Gamma) = P_\delta(\Gamma')$, there is a $\beta' \in \Gamma'$, such that $\alpha + \delta^{(i)} = \beta' + \nu \circ \delta^{\Gamma'}(\beta')$. Set $\nu' = \nu \circ \delta^{\Gamma'}(\beta'), \Gamma'' = \Gamma \cup \{\beta'\}$. By (ii) of Definition 2, $\delta^{\Gamma''}(\beta') - \delta^{\Gamma'}(\beta')$ is non-negative. Hence $\nu' = \nu' \circ \delta^{\Gamma''}(\beta')$, and

$$\alpha + \delta^{(i)} = \beta' + \nu',\quad \nu' \neq 0. \tag{3.8}$$

We claim that $\beta' \in P_\delta(\Gamma)$. Since $\beta' \in \Gamma' \subseteq P^*(\Gamma') = P^*(\Gamma)$, it can be rewritten as

$$\beta' = {}^{(1)}\beta + {}^{(1)}\gamma,\ {}^{(1)}\beta \in \Gamma. \tag{3.9}$$

If there is a ${}^{(1)}\gamma_j \neq 0$, such that $\delta_j^\Gamma({}^{(1)}\beta) = 0$, substituting $\beta' = {}^{(1)}\beta + \delta^{(j)} + ({}^{(1)}\gamma - \delta^{(j)})$ into (3.8) we have $\alpha + \delta^{(i)} = {}^{(1)}\beta + \delta^{(j)} + ({}^{(1)}\gamma - \delta^{(j)}) + \nu'$. Then ${}^{(1)}\beta + \delta^{(j)} \in P_\delta(\Gamma)$ since $\alpha + \delta^{(i)}$ is critical and $\nu' \neq 0$. Let ${}^{(1)}\beta + \delta^{(j)} = {}^{(2)}\beta + \sigma \circ \delta^\Gamma({}^{(2)}\beta)$, ${}^{(2)}\beta \in \Gamma$. Then ${}^{(2)}\beta$ is a pseudo-divisor of ${}^{(1)}\beta$, and

$$\beta' = {}^{(2)}\beta + {}^{(2)}\gamma \tag{3.10}$$

where ${}^{(2)}\gamma = \sigma \circ \delta^\Gamma({}^{(2)}\beta) + ({}^{(1)}\gamma - \delta^{(j)})$. For $\beta' = {}^{(2)}\beta + {}^{(2)}\gamma$, we discuss about ${}^{(2)}\gamma$ as about ${}^{(1)}\gamma$ for $\beta' = {}^{(1)}\beta + {}^{(1)}\gamma$, and so on. Because δ is Artinian, we can achieve that, by a finite number of steps, there exits a $\beta \in \Gamma$ such that $\beta' = \beta + \mu$, and $\mu_k \neq 0$ implies $\delta_k^\Gamma(\beta) = 1, k = 1, \cdots, n$. So $\beta' = \beta + \mu \circ \delta^\Gamma(\beta) \in P_\delta(\Gamma)$. Substituting $\beta' = \beta + \mu \circ \delta(\beta)$ into (3.8), we have $\alpha + \delta^{(i)} = \beta + \mu \circ \delta(\beta) + \nu' \circ \delta^{\Gamma''}(\beta')$, which is in contradiction with constructivity of δ.

4. Equivalence between Involutive Direction and Involutive Division

In [4], an axiomatic definition on involutive division was given. Let $\mathbf{M}$ be the set of all n-variables monomials.

Definition 7. An involutive division L on $\mathbf{M}$ is given, if for any finite monomial set $U \subset \mathbf{M}$ and any $u \in U$ there is a submonoid $L(u, U)$ of $\mathbf{M}$ satisfying the conditions:

(a) If $w \in L(u, U)$ and $v|w$, then $v \in L(u, U)$;

(b) If $u, v \in U$ and $uL(u, U) \cap vL(u, U) \neq 0$, then $u \in vL(v, U)$ or $v \in uL(u, U)$;

(c) If $v \in U$ and $v \in uL(u, U)$, then $L(v, U) \subseteq L(u, U)$;

(d) If $V \subseteq U$, then $L(u, U) \subseteq L(u, V)$ for all $u \in V$.

For convenience, we denote by x^α the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. It is clear that the map $\varphi : \alpha \mapsto x^\alpha$ is an isomorphism between monoids $(\mathbf{N}^n, +)$ and $(\mathbf{M}, \cdot)$. We will prove

Theorem 3. The involutive direction in Definition 2 and the involutive division in Definition 7 are equivalent under the map φ.

Proof. Let δ be an involutive direction on $\mathbf{N}^n$. For any element u of a finite subset U of $\mathbf{M}$, let $\Gamma = \varphi^{-1}(U)$, and $u = x^\alpha$. We need to prove that the set

$$L(u, U) = \{x^\beta | \beta = \mu \circ \delta^\Gamma(\alpha), \mu \in \mathbf{N}^n\} \tag{4.1}$$

is a submonoid of $(\mathbf{M}, \cdot)$, and (a), (b), (c), (d) hold. Hence L is an involutive division on $\mathbf{M}$.

Taking $\mu = 0$, then $\beta = 0, x^\beta = 1 \in L(u, U)$. If $v, w \in L(u, U)$, let $v = x^\beta, w = x^\gamma$, where $\beta = \mu \circ \delta^\Gamma(\alpha), \gamma = \nu \circ \delta^\Gamma(\alpha)$. Then $\beta + \gamma = (\mu + \nu) \circ \delta^\Gamma(\alpha)$, whence $vw = x^{\beta+\gamma} \in L(u, U)$. Hence $L(u, U)$ is a submonoid of $(\mathbf{M}, \cdot)$.

If $w \in L(u, U), v|w$, set $w = x^\gamma, \gamma = \nu \circ \delta^\Gamma(\alpha)$. Then $\delta_k^\Gamma(\alpha) = 0$ implies $\beta_k = 0$. Hence $\beta = \beta \circ \delta^\Gamma(\alpha), v = x^\beta \in L(u, U)$, and (a) holds.

For $u, v \in U$, set $u = x^\alpha, v = x^\beta, \alpha, \beta \in \Gamma$. If $uL(u, U) \cap vL(v, U) \neq \phi$, supposing $w = x^\gamma \in uL(u, U) \cap vL(v, U)$, then $\gamma = \alpha + \mu \circ \delta^\Gamma(\alpha) = \beta + \nu \circ \delta^\Gamma(\beta), \mu, \nu \in \mathbf{N}^n$. By (ii) of Definition 2, we may suppose, without loss of generality, that $\delta^\Gamma(\alpha) - \delta^\Gamma(\beta)$ and $\beta - \alpha$ are both non-negative. Let $\nu' = \nu \circ \delta^\Gamma(\beta)$. Then $\nu \circ \delta^\Gamma(\beta) = \nu' \circ \delta^\Gamma(\alpha)$. Hence $\alpha + (\mu - \nu') \circ \delta^\Gamma(\alpha) = \beta$. Let $\mu' = (\mu - \nu') \circ \delta^\Gamma(\alpha)$. Then $\mu' \in \mathbf{N}^n$, and $\beta = \alpha + \mu' \circ \delta^\Gamma(\alpha)$. We have $v \in uL(u, U)$, which implies (b).

For $u, v \in U$ let $u = x^\alpha, v = x^\beta, \alpha, \beta \in \Gamma$. If $v \in uL(u, U)$, then $\beta = \alpha + \mu \circ \delta^\Gamma(\alpha)$, i.e. $\beta + 0 \circ \delta^\Gamma(\beta) = \alpha + \mu \circ \delta^\Gamma(\alpha)$. By (ii) of Definition 2, $\delta^\Gamma(\alpha) - \delta^\Gamma(\beta)$ is non-negative since $\beta - \alpha$ is non-negative. For $w \in L(v, U)$, let $w = x^\gamma$. Then $\gamma = \nu \circ \delta^\Gamma(\beta) = \nu' \circ \delta^\Gamma(\alpha)$ (ν', taken as above). Hence $w \in L(u, U)$. We have $L(v, U) \subseteq L(u, U)$, which implies (c).

For $u \in V \subseteq U$, let $u = x^\alpha, \Sigma = \varphi^{-1}(V)$. We have $\alpha \in \Sigma \subseteq \Gamma$, and $\delta^\Sigma(\alpha) - \delta^\Gamma(\alpha)$ is non-negative by (ii) of Definition 2. Similar to the above proof, we derive $L(u, U) \subseteq L(u, V)$, which implies (d).

Conversely, given a involutive division L on $\mathbf{M}$, we will construct a involutive dirction δ on $\mathbf{N}^n$. For any element α of any finite subset Γ of $\mathbf{N}^n$, setting $U = \{x^\beta | \beta \in \Gamma\}, u = x^\alpha$, we define $\delta^\Gamma(\alpha)$ as follows:

$$\delta_k^\Gamma(\alpha) = \begin{cases} 1 & \text{if } x_k \text{ effectively appears in some elements of } L(u, U); \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

$k = 1, 2, \cdots, n$. Clearly δ^Γ is a map from Γ to Δ_n. Now we begin to prove that conditions (i), (ii) of Definition 2 are satisfied.

Let $\alpha, \beta \in \Gamma$, and $\alpha + \mu \circ \delta^\Gamma(\alpha) = \beta + \nu \circ \delta^\Gamma(\beta)$. Let $u = x^\alpha, v = x^\beta$. Then $u, v \in U$. We claim that $\bar{u} = x^{\mu \circ \delta^\Gamma(\alpha)} \in L(u, U)$. In fact, if $\delta_k^\Gamma(\alpha) = 1, x_k$ effectively appears in some element, say u', of $L(v, U)$, whence $x_k \in L(u, U)$ by (a) of Definition 7. Since $L(u, U)$ is a monoid, we have $\bar{u} = x_1^{\mu_1 \delta_1^\Gamma(\alpha)} \cdots x_k^{\mu_k \delta_k^\Gamma(\alpha)} \cdots x_n^{\mu_n \delta_n^\Gamma(\alpha)} \in L(u, U)$. Similarly $\bar{v} = x^{\nu \circ \delta^\Gamma(\beta)} \in L(v, U)$. So $u\bar{u} = v\bar{v} \in uL(u, U) \cap vL(v, U)$. By (b) of Definition 7, either $u \in vL(v, U)$, or $v \in uL(u, U)$. In the case $v \in uL(u, U)$, $\beta - \alpha$ is non-negative and $L(v, U) \subseteq L(u, U)$. Hence $\delta^\Gamma(\alpha) - \delta^\Gamma(\beta)$ is non-negative. And likewise for the case $u \in vL(v, U)$. Then (i) is satisfied.

For $\alpha \in \Sigma \subseteq \Gamma$. Let $V = \{x^\beta | \beta \in \Sigma\}$. Then $u = x^\alpha \in V \subseteq U$. By (d) of Definition 7, $L(u, U) \subseteq L(u, V)$. Hence $\delta^\Sigma(\alpha) - \delta^\Gamma(\alpha)$ is non-negative. Then (ii) is satisfied.

Table 1. corresponding relations between involutive directions and involutive divisions

$(\mathbf{N}^n, +) \xrightarrow{\varphi} (\mathbf{M}, \cdot): \ \alpha \mapsto u = x^\alpha$
$\Gamma \longrightarrow U, \ \ \delta^\Gamma \longrightarrow L_U$
$\delta_i(\alpha) = 1$, or $0 \longrightarrow ux_i = u \times x_i$ or not
$\alpha + \mu \circ \delta(\alpha) \longrightarrow u \times v_1, \ v_1 = x^{\mu \circ \delta(\alpha)}$
$\{\mu \circ \delta(\alpha) \mid \mu \in \mathbf{N}^n\} \longrightarrow L(u, U), \ \ P_\delta(\Gamma) \longrightarrow C_L(U), \ \ P^*(\Gamma) \longrightarrow C(U)$
δ is Artinian $\longrightarrow L$ is continuous, δ is Artinian and constructive $\longrightarrow L$ is constructive

In [5], Gerdt etal proposed two new involutive divisions, Division I and Division II, different from that of Thomas, Janet and Pommaret. We list them as examples and show how to represent them in using involutive directions.

Example 4. Division I. Let U be a finite monomial set. The variable x_i is non-multiplicative for $u \in U$ if there is a $v \in U$, such that

$$x_{i_1}^{d_1} \cdots x_{i_m}^{d_m} u = \mathrm{lcm}(u, v), \ 1 \le m \le [n/2], \ d_j > 0 (1 \le j \le m) \tag{4.3}$$

and $x_i \in \{x_{i_1}, \cdots, x_{i_m}\}$. In other words, for $\alpha \in \Gamma$, $\delta_i^\Gamma(\alpha) = 0$ iff there exits $\beta \in \Gamma$, such that $\beta_i > \alpha_i$ and the number of positive components of $\beta - \alpha$ ranges from 1 to $[n/2]$. In the next section, we will generalize this concept.

Example 5. Division II. For monomial $u = x_1^{d_1} \cdots x_n^{d_n}$, the variable x_i is multiplicative if $d_i = d_{\max}(u)$, where $d_{\max} = \max\{d_1, \cdots, d_n\}$. In other words, for $\alpha \in \Gamma$, $\delta_i^\Gamma(\alpha) = 1$ iff α_i is the maximal component of α.

5. Generalization of Thomas and Janet Directions

We denote by $N_p(\alpha)$ the number of positive components of α. Let l be an integer not less than $[n/2]$. For any finite subset Γ of $\mathbf{N}^n$, set

$$B_l^\Gamma(\alpha) = \{\beta \in \Gamma | 1 \le N_p(\beta - \alpha) \le l\}. \tag{5.1}$$

Example 6. A direction δ on $\mathbf{N}^n$ is defined as

$$\delta_i^\Gamma(\alpha) = 1 \text{ iff } \beta_i \le \alpha_i, \text{ for all } \beta \in B_l^\Gamma(\alpha). \tag{5.2}$$

In other words, $\delta_i^\Gamma(\alpha) = 0$ iff there is a $\beta \in B_l^\Gamma(\alpha)$, s.t. $\beta_i > \alpha_i$.

Clearly, when $l = [n/2]$, δ is just the direction corresponding to division I. Thomas direction corresponds to the case $l = n$. A direction defined in Example 6 is called Thomas type direction, denoted by T_l. T_l is Noetherian since $P^*(\Gamma) \cap B(\Gamma)$ is a completion of Γ. We will discuss the involutivity, Artinity and constructivity of T_l.

T_l is involutive. Set $\delta = T_l$. For $\alpha, \beta \in \Gamma$, let

$$\alpha + \mu \circ \delta(\alpha) = \beta + \nu \circ \delta(\beta) \tag{5.3}$$

where $\mu, \nu \in \mathbf{N}^n$. If there are indices i and j, such that $\alpha_i > \beta_i$, and $\alpha_j < \beta_j$, then either $\alpha \in B_l^\Gamma(\beta)$ or $\beta \in B_l^\Gamma(\alpha)$ since $l \geq [n/2]$. But $\alpha \in B_l^\Gamma(\beta), \alpha_i > \beta_i$ implies $\delta_i(\beta) = 0$, whence $\alpha_i + \mu_i\delta_i(\alpha) = \beta_i \geq \alpha_i$, contradiction. And likewise for $\beta \in B_l^\Gamma(\alpha)$. We proved either $\alpha - \beta$ or $\beta - \alpha$ is non-negative. Let us suppose that $\alpha - \beta$ is non-negative. If $\delta_k(\beta) = 0$, by the definition, there exists a vector $\gamma \in B_l^\Gamma(\beta)$, such that $\gamma_k > \beta_k$. But by (5.3) $\alpha_k \leq \beta_k$, $\gamma_k > \alpha_k, \gamma \in B_l^\Gamma(\alpha)$ since $\alpha - \beta$ is non-negative, whence $\delta_k(\alpha) = 0$, contradiction. Hence $\delta(\beta) - \delta(\alpha)$ is non-negative, the condition (i) of Definition 2 is satisfied. The condition (ii) is clearly satisfied since $B_l^\Sigma(\alpha) \subseteq B_l^\Gamma(\alpha)$ for $\alpha \in \Sigma \subseteq \Gamma$. Therefore δ is involutive.

T_l is constructive. For any critical prolongation $\alpha + \delta^{(i)}$ of Γ, let us suppose

$$\alpha + \delta^{(i)} = \beta + \mu \circ \delta^\Gamma(\beta) + \nu \circ \delta^{\Gamma'}(\beta') \tag{5.4}$$

holds for some $\beta \in \Gamma$, where $\beta' = \beta + \mu \circ \delta^\Gamma(\beta), \Gamma' = \Gamma \cup \{\beta'\}$. If $\delta_k^\Gamma(\beta) = 0$, then $\beta'_k = \beta_k$, and there exists a vector $\gamma \in B_l^\Gamma(\beta)$, such that $\gamma_k > \beta_k = \beta'_k$. This implies $\gamma \in B_l^{\Gamma'}(\beta')$, and $\delta_k^{\Gamma'}(\beta') = 0$. Setting $\nu' = \nu \circ \delta^{\Gamma'}(\beta')$, we have $\alpha + \delta^{(i)} = \beta + (\mu + \nu') \circ \delta^\Gamma(\beta) \in P_\delta(\Gamma)$, a contradiction.

T_l is Artinian. To prove that we fist give a lemma. For $\beta \in \Gamma$, $\beta \oplus \gamma$ means that there exists $\mu \in \mathbf{N}^n$, such that $\gamma = \mu \circ \delta^\Gamma(\beta)$.

Lemma 2. Let δ be a direction defined in Example 6. If

$$\alpha + \sigma = \beta \oplus \gamma, \ \sigma \circ \gamma = 0, \ \gamma \neq 0 \tag{5.5}$$

where $\alpha, \beta \in \Gamma, \ \sigma, \gamma \in \mathbf{N}^n$, then $N_p(\gamma) > l$.

It is easy to prove lemma 2, since $N_p(\gamma) \leq l$ implies $\alpha \in B_l^\Gamma(\beta)$, which would be in contradiction with $\beta \oplus \gamma$.

Proof of Artinian property of T_l: Given any pseudo-divisor sequence S of Γ

$$^{(1)}\alpha, \ ^{(2)}\alpha, \ \cdots, \ ^{(k)}\alpha, \ \cdots \tag{5.6}$$

where

$$^{(k)}\alpha + \delta^{(i_k)} = {}^{(k+1)}\alpha \oplus {}^{(k+1)}\gamma, \quad \delta_{i_k}(^{(k)}\alpha) = 0 \tag{5.7}$$

$k = 1, 2, \cdots$. If all $^{(k+1)}\gamma$ are zero, then $^{(k)}\beta <_{lex} {}^{(k+1)}\beta$ for all k. Hence $^{(i)}\beta \neq {}^{(j)}\beta$ for $i \neq j$. Now we suppose that there is at least one $^{(k+1)}\gamma \neq 0$. Extract from S such a subsequence S'

$$^{(1)}\beta, \ ^{(2)}\beta \ \ldots \ ^{(j)}\beta \ \ldots \tag{5.8}$$

where ${}^{(j)}\beta = {}^{(k_j+1)}\alpha \in S'$ satisfying ${}^{(k_j+1)}\gamma \neq 0$. We will prove ${}^{(i)}\beta \neq {}^{(j)}\beta$ for $i \neq j$.

Consider ${}^{(1)}\beta$ and ${}^{(2)}\beta$. Let

$$\begin{array}{l} {}^{(k_1)}\alpha + \delta^{(i_{k_1})} = {}^{(k_1+1)}\alpha \oplus {}^{(k_1+1)}\gamma,\ {}^{(k_1+1)}\alpha + \delta^{(i_{k_1+1})} = {}^{(k_1+2)}\alpha,\ \cdots \\ {}^{(k_2-1)}\alpha + \delta^{(i_{k_2-1})} = {}^{(k_2)}\alpha,\ {}^{(k_2)}\alpha + \delta^{(i_{k_2})} = {}^{(k_2+1)}\alpha \oplus {}^{(k_2+1)}\gamma. \end{array} \tag{5.9}$$

Then

$${}^{(k_1+1)}\alpha + \delta^{(i_{k_1+1})} \cdots + \delta^{(i_{k_2})} = {}^{(k_2+1)}\alpha \oplus {}^{(k_2+1)}\gamma. \tag{5.10}$$

Consider $\delta^{(i_{k_2})}$ and ${}^{(k_2+1)}\gamma$. If $\delta^{(i_{k_2})} \circ {}^{(k_1+1)}\gamma \neq 0$, we may eliminate $\delta^{(i_{k_2})}$ from both sides of (5.10),

$${}^{(k_1+1)}\alpha + \delta^{(i_{k_1+1})} \cdots + \delta^{(i_{k_2-1})} = {}^{(k_2+1)}\alpha \oplus \gamma'.$$

If $\delta^{(i_{k_2})} \circ {}^{(k_1+1)}\gamma = 0$, we set $\gamma' = {}^{(k_2+1)}\alpha$. Then consider $\delta^{(i_{k_2-1})}$ and γ', and so on. Let $\delta^{(i_h)}$ be the last one eliminated. Then $k_1 + 1 \leq h \leq k_2$, (5.10) is reduced to

$${}^{(h)}\alpha + \sigma' = {}^{(k_2+1)}\alpha \oplus \gamma,\ \delta^{(i_h)} \circ {}^{(k_2+1)}\gamma \neq 0, \tag{5.11}$$

or simply

$${}^{(1)}\beta + \sigma = {}^{(2)}\beta \oplus \gamma,\ \sigma \circ \gamma = 0. \tag{5.12}$$

We claim that $\gamma \neq 0$, and $\sigma \circ \delta({}^{(1)}\beta) = 0$.

If $\gamma = 0$, then $h < k_2, {}^{(h)}\alpha + \sigma' = {}^{(k_2+1)}\alpha, {}^{(h+1)}\alpha + \sigma' = {}^{(k_2+1)}\alpha + \delta^{(i_h)}$, and $\delta^{(i_h)} \circ \sigma' = 0$, whence ${}^{(h+1)}\alpha_{i_h} > {}^{(k_2+1)}\alpha_{i_h}, {}^{(h+1)}\alpha \in B_l({}^{(k_2+1)}\alpha), \delta_{i_h}({}^{(k_2+1)}\alpha) = 0$. This is in contraction with $\delta^{(i_h)} \circ {}^{(k_2+1)}\gamma \neq 0$, and ${}^{(k_2+1)}\alpha \oplus {}^{(k_2+1)}\gamma$. By Lemma 2, $N_p(\gamma) > l$. If $\sigma \neq 0$, then $N_p(\sigma) \leq l, {}^{(2)}\beta \in B_l({}^{(1)}\beta)$. So $\sigma \circ \delta({}^{(1)}\beta) = 0$.

We may treat any pair of ${}^{(j+1)}\beta$ and ${}^{(j+2)}\beta, j = 1, \cdots$ similar to ${}^{(1)}\beta$ and ${}^{(2)}\beta$. We have

$${}^{(1)}\beta + {}^{(1)}\sigma = {}^{(2)}\beta \oplus {}^{(2)}\tau,\ \cdots,\ {}^{(j+1)}\beta + {}^{(j+1)}\sigma = {}^{(j+2)}\beta \oplus {}^{(j+2)}\tau \tag{5.13}$$

where ${}^{(k)}\sigma \circ {}^{(k)}\tau = 0,\ {}^{(k-1)}\sigma \circ {}^{(k)}\tau = 0,\ N_p({}^{(k)}\tau) > l,\ k = 2, 3, \cdots$. Next we prove that for any $j > 1$, there are vectors ${}^{(j)}\zeta, {}^{(j)}\xi \in \mathbf{N}^n$, such that

$${}^{(1)}\beta + {}^{(j)}\zeta = {}^{(j)}\beta + {}^{(j)}\xi,\ N_p({}^{(j)}\xi) > l,\ {}^{(j)}\zeta \circ {}^{(j)}\xi = 0. \tag{5.14}$$

It is obvious for $j = 2$. Suppose, by induction, that (5.14) holds for j. Consider the case $j + 1$. By (5.13),

$${}^{(1)}\beta + {}^{(j)}\sigma + {}^{(j)}\zeta = ({}^{(j+1)}\beta \oplus {}^{(j+1)}\tau) + {}^{(j)}\xi,\ {}^{(j)}\sigma \circ {}^{(j+1)}\zeta = 0,\ {}^{(j)}\zeta \circ {}^{(j)}\xi = 0. \tag{5.15}$$

Let ${}^{(j)}\zeta = \lambda + {}^{(j)}\zeta', {}^{(j+1)}\tau = \lambda + {}^{(j+1)}\tau'$, such that ${}^{(j)}\zeta' \circ {}^{(j+1)}\tau' = 0$. Since $N_p({}^{(j)}\zeta) \leq l < N_p({}^{(j+1)}\tau)$, ${}^{(j+1)}\tau' \neq 0$. Similarly, we may reduce ${}^{(j)}\sigma$ and ${}^{(j)}\xi$ to ${}^{(j)}\sigma'$ and ${}^{(j)}\xi'$, such that ${}^{(j)}\sigma' \circ {}^{(j)}\xi' = 0, {}^{(j)}\xi' \neq 0$, whence (5.15) can be rewritten as

$${}^{(1)}\beta + {}^{(j+1)}\zeta = {}^{(j+1)}\beta + {}^{(j+1)}\xi \tag{5.16}$$

where ${}^{(j+1)}\zeta = {}^{(j)}\sigma' + {}^{(j)}\zeta',\ {}^{(j+1)}\xi = {}^{(j+1)}\tau' + {}^{(j)}\xi' \neq 0,\ {}^{(j+1)}\zeta \circ {}^{(j+1)}\xi = 0$. If $N_p({}^{(j+1)}\xi) \leq l$, then ${}^{(1)}\beta \in B_l({}^{(j+1)}\beta)$. For ${}^{(j+1)}\tau'_k \neq 0, \delta_k({}^{(j+1)}\beta) = 1$, since ${}^{(j+1)}\beta \oplus {}^{(j+1)}\tau'$ and ${}^{(1)}\beta_k > {}^{(j+1)}\beta_k$, we have $\delta_k({}^{(j+1)}\beta) = 0$, a contradiction. The assertion has been proven. And likewise for any pair of ${}^{(i)}\beta$ and ${}^{(j)}\beta, (j > i)$. So S' consists of distinct terms.

Table 2. Example for Thomas type directions

Exponent vectors	Directions		
	$T_4 = T$	T_3	T_2=II
(2,2,2,1)	(1,1,1,1)	(1,1,1,1)	(1,1,1,1)
(1,1,0,0)	(0,0,0,0)	(1,1,0,0)	(1,1,0,0)
(0,1,0,0)	(0,0,0,0)	(0,1,0,0)	(0,1,1,1)
(1,1,1,1)	(0,0,0,1)	(0,0,0,1)	(1,1,1,1)

If there were two identical terms in sequence S, one would construct easily a sequence T, such that the subsequence T'(extract from T as S' from S) contains two identical terms, which is contradiction with what we proved above. So the sequence S consists of distinct terms, and T_l is Artinian.

To generalize the concept of Janet direction on $\mathbf{N}^n$, we introduce ordered dissection of a positive integer.

Definition 8. An ordered dissection with length l of a positive integer n is a vector $(n_1, n_2, \cdots, n_l)$, where the n_i are positive integers, such that

$$n = n_1 + n_2 + \cdots + n_l. \tag{5.17}$$

Set $s_0 = 0, s_k = n_1 + \cdots + n_k,\ k = 1, \cdots, l$.

Example 7. For any ordered dissection $(n_1, n_2, \cdots, n_l)$ of n, we define a direction δ on $\mathbf{N}^n$: for every finite subset Γ of $\mathbf{N}^n, \forall \beta \in \Gamma$,

$$\begin{array}{l} \delta_i^\Gamma(\beta) = 1 \text{ iff } \beta_i = \mathbf{b}_i(\Gamma), \text{ if } i \le s_1; \\ \delta_i^\Gamma(\beta) = 1 \text{ iff } \beta_i = \mathbf{b}_i(\Gamma_{\beta_1 \cdots \beta_{s_k}}), \text{ if } s_k < i \le s_{k+1}, \end{array} \tag{5.18}$$

$k = 1, \cdots, l-1$, where $\Gamma_{j_1 \cdots j_m}$ is the same as in Example 2.

We will address simply that δ is Noetherian, Artinian and constructive. The direction such defined is called Janet type direction(correspondingly Janet type division), denoted by $J_{(n_1, \cdots, n_l)}$.

At first, δ is involutive. $\forall \alpha, \beta \in \Gamma$, if $\alpha + \mu \circ \delta^\Gamma(\alpha) = \beta + \nu \circ \delta^\Gamma(\beta)$, then $\alpha = \beta$. Otherwise, we suppose, without loss of generality, $\alpha_1 = \beta_1, \cdots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i$. If $i \le s_1$, then $\delta_i^\Gamma(\beta) = 0$, whence $\beta_i = \alpha_i + \mu_i \delta_i^\Gamma(\alpha) \ge \alpha_i$, a contradiction. If $s_k < i \le s_{k+1}$, then $\Gamma_{\beta_1 \cdots \beta_{s_k}} = \Gamma_{\alpha_1 \cdots \alpha_{s_k}}$, whence $\delta_i^\Gamma(\beta) = 0$, a contradiction. So the condition (i) of Definition 2 is satisfied. The condition (ii) is clearly satisfied. So δ is involutive. δ is Noetherian, since $B(\Gamma) \cap P^*(\Gamma)$ is a completion of Γ for any finite set Γ.

Secondly, we prove that δ is Artinian. The following lemma is important.

Lemma 3. $\forall \alpha, \beta \in \Gamma$, if $s_{k-1} < i \le s_k$, and

$$\delta_i^\Gamma(\alpha) = 0,\ \alpha + \delta^{(i)} = \beta + \nu \circ \delta^\Gamma(\beta) \tag{5.19}$$

then $\beta_i = \alpha_i + 1,\ \beta_j = \alpha_j$, for $j \ne i$ and $j \le s_k$.

Proof. For $j \le i-1$, we claim $\alpha_j = \beta_j$, which implies $\alpha \in \Gamma_{\beta_1\cdots\beta_{s_{k-1}}}$ since $s_{k-1} < i \le s_k$. Otherwise, suppose that $\alpha_1 = \beta_1, \cdots, \alpha_{j-1} = \beta_{j-1}, \alpha_j > \beta_j$. Then $\delta_j^\Gamma(\beta) = 0$(similar to above argument). By (5.19), $\alpha_j = \beta_j$, a contradiction. Similarly, $\alpha_j = \beta_j$ for $i < j \le s_k$ since $\alpha \in \Gamma_{\beta_1\cdots\beta_{s_{k-1}}}$. As for i, if $\nu_i\delta_i^\Gamma(\beta) \ne 0$, then $\delta_i(\beta) = 1$, $\beta_i \le \alpha_i < \mathbf{b}_i(\Gamma_{\alpha_1\cdots\alpha_{s_{k-1}}}) = \mathbf{b}_i(\Gamma_{\beta_1\cdots\beta_{s_{k-1}}})$. We derived $\delta_i^\Gamma(\beta) = 0$, a contradiction. So $\alpha_i + 1 = \beta_i$, and the proof is completed.

By Lemma 3, β is a pseudo-divisor of α implies $\alpha <_{lex} \beta$. Hence δ is Artinian. The proof of constructivity of δ is similar to the proof for Janet direction, we omit it here.

Table 3. Example for Janet type directions

Exponent vectors	Directions			
	$J_{(3)} = T$	$J_{(2,1)}$	$J_{(1,2)}$	$J_{(1,1,1)} = J$
(1,1,1)	(0,0,0)	(0,0,1)	(0,0,1)	(0,0,1)
(1,2,1)	(0,1,0)	(0,1,1)	(0,1,1)	(0,1,1)
(2,1,0)	(0,0,0)	(0,0,0)	(0,1,0)	(0,1,0)
(2,1,1)	(0,0,0)	(0,0,1)	(0,1,1)	(0,1,1)
(3,0,2)	(1,0,1)	(1,0,1)	(1,0,1)	(1,0,1)
(3,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)

6. An Improved Completion Algorithm

In [4, 5], the authors gave an algorithm to determine a minimal involutive completion for a given finite set of monomials. In their algorithm, a given finite set of monomials was enlarged to its completion by adding one monomial each step. By Theorem 2, we can enlarge a given non-complete finite set, by adding all critical prolongations in each step,to a minimal completion of the given set. It is easy to see that deciding whether ux_j is a critical is is as simpler as deciding whether ux_j is the lowest element with respect to a given ordering. Based on these considerations, we give an improved completion algorithm.

[Improved Completion Algorithm]

Imput : U, a finite monomial set
Output : $\tilde{U}$, a minimal complition of U
begin
  $\tilde{U} := U$
  while exist $u \in \tilde{U}$ and $x \in NM_L(u, \tilde{U})$,
  such that $u \cdot x$ has no involutive divisors in $\tilde{U}$ **do**
  choose all critical prolongations of $\tilde{U}$, say, $\bar{U}$
  $\tilde{U} := \tilde{U} \cup \bar{U}$
  end
end

For example, consider Thomas division and a set $U = \{xy, y^2, z\}$. By their algorithm, taking the lexicographical ordering with $z < y < x$:

$$\begin{aligned}U &= \{xy, y^2, z\} \to \{xy, y^2, z, yz\} \to \{xy, y^2, z, yz, y^2z\} \to \{xy, y^2, z, yz, y^2z, xz\} \\ &\to \{xy, y^2, z, yz, y^2z, xz, xyz\} \to \{xy, y^2, z, yz, y^2z, xz, xyz, xy^2\} \\ &\to \{xy, y^2, z, yz, y^2z, xz, xyz, xy^2, xy^2z\} = \tilde{U}_T\end{aligned}$$

By the improved algorithm:

$$\begin{aligned}U &= \{xy, y^2, z\} \to \{xy, y^2, z, xy^2, xz, yz\} \to \{xy, y^2, z, xy^2, xz, yz, xyz, y^2z\} \\ &\to \{xy, y^2, z, yz, y^2z, xz, xyz, xy^2, xy^2z\} = \tilde{U}_T\end{aligned}$$

Because one has to determine non-multiplicative variables for every element of the set in each step, the more steps, the more complex. Our algorithm is an effective improvement for their algorithm.

7. Conclusion

The vector representation of involutive division is useful to study the structure of involutive divisions, and to find new divisions. So far we have known Thomas type, Janet type, Pommaret, induced divisions [6] and division II. There seems to be Pommaret type and II type divisions. All divisions listed above are 'good' in the sense of they are Artinian and constructive. The following problems are not solved yet.

Problem what is about Pommaret type (II type) divisions ?

Conjecture the number of good divisions is finite.

AcknowledgementΦ Φle

The authors would like to thank Vladimir Gerdt and Yuri Blinkov for sending their up today work. This work was supported in part by An outstanding Youth Grant from Chinese NSF, a National Key Projection on Basic research, and by the Educational Committee of Liaoning Province, P.R. China

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