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Geometric Constraint Solving with Conic[1)]

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Abstract. Most mechanical *CAD* systems use line and circle(ruler and compass) as basic drawing tools. In this paper we introduce a class of new drawing tools: the conic. We proved that the class of diagrams within the drawing scope of this new tool is larger than that can be drawn with line and circle. Actually, we proved that a diagram can be drawn with conic if and only if this diagram can be described with a sequence of triangularised equations of degree less than or equal to four. This allows us to maintain the elegance of geometric constraint solving with ruler and compass, because the solutions of cubic and quartic equations can be written explicitly.

1. Introduction

Contemporary *CAD*(Computer Aided Design) systems have been proven to be efficient for generating engineering drawings and modeling 3D objects. However, in the conception stage of the design process, most *CAD* systems still do not have all of the required flexibility. Designers are required to know beforehand the precise dimensions of the geometries, and changes are difficult. Nevertheless, in the product development cycle, several design changes are typically needed before the full requirements for functionality, manufacture and quality are met. Therefore, traditional *CAD* systems are not suitable for most conceptual design. These facts imply the need for a flexible tool for the conceptual design. A potential tool to meet these requirements is parametric design systems which provide the designers a way to create the initial design without knowing exact dimensions [?, ?, ?, ?, ?]. Moreover parametric design allows designers to make modifications to existing initial designs by changing parameter values. Parametric design has been incorporated into various *CAD*/*CAM* systems, such as Pro/Engineer and I-DEAS Master Series. These kinds of parametric modeling systems have promised to revolutionize mechanical *CAD*/*CAM* systems.

Geometry Constraint Solving (*GCS*) is the central topic in much of the current work of developing intelligent or parametric *CAD* system [?, ?, ?, ?, ?, ?]. There are three major approaches to *GCS*: numerical approach, symbolic approach and constructive approach. In the constructive approach, a pre-treatment is carried out to transform the constraint problem into a constructive form [?, ?, ?, ?] that is easy to draw. Once a constraint diagram is transformed into constructive form, all its solutions can be computed efficiently. Hence most parametric design systems adopt the constructive approach as a basic scheme for *GCS*. Other approaches are used if the constructive approach fails to give a solution.

A majority of the work based on constructive approach is to transform the constraint problem to an rc-constructible form which is a geometric configuration that can be constructed with ruler and compass . A direct extension of the rc-configuration is the diagrams

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of constructive type, that is, the points in the diagram can be constructed one by one but not necessarily with ruler and compass. This paper presents a new and powerful tool, conic, to broaden the scope of GCS . Our method follows the global propagation approach [?] and broadens the scope of this method . With this new method, we can not only solve lots of non-rc-costructible configurations that can not be drawn by the global propagation approach, but also make the solutions shorten. Moreover, this tool can be used in any constructive approach.

The rest of the paper is organized as follows. The second section describes the related work. This is followed by the geometric construction approach with conic. In the fourth section the scope of GCS with conic is given. Finally, we ended the paper by giving conclusive remarks and possible future work.

2. Related Work

2.1. Three Main Approaches

There are three major approaches for GCS: the numerical approach, the symbolic approach and the constructive approach.

In the numerical approach [?, ?, ?, ?], geometric constraints are converted into a system of simultaneous algebraic equations. Then, the algebraic equations are solved by various numerical techniques such as Newton-Raphson iterative numerical method. The main advantage of numerical approach is their generality and high speed. However, it also has some significant shortcomings. For example, the numerical approach requires good initial values, so that, changing of dimensions might make the GCS divergent. Besides this, the numerical approach can not distinguish between different roots in the solution space.

In the symbolic approach [?, ?], we also convert geometric constraints into a system of simultaneous algebraic equations. Instead of using numerical methods to solve the algebraic equations, we first use general symbolic methods such as Wu-Ritt's characteristic set method or the Gröbner basis method to change the algebraic equations to a new form which is easy to solve, and then, solve the new algebraic equations numerically. The symbolic approach provides a stable method to find all possible solution to equations, but it may require exponential running time.

In the constructive approach, a pre-treatment is carried out to transform the constraint problem into a constructive form that is easy to draw. According to the method of transformation, the constructive approach has two classes: the graph analysis approach [?, ?] and rule-based approach [?, ?, ?]. In the graph analysis approach, geometric constraints are represented by constraint graphs. Then, with the help of graph theory, constraint graphs are converted into a constructive sequence through which we can draw the diagram. In the rule-based approach, automatic reasoning rules are used to find a constructive sequence for a GCS problem. The GCS problem is then determined by sequentially evaluating the rules in the constructive consequence. In this approach, it is easy to express complicated constraints such as tangent and, for users, to adjust the dimensions without repeated deducing. However, only rc-constructible configuration are considered in this approach. Some approaches can solve a large set of complex GCS problem, but they need to add auxiliary points that slowdown the search speed.

2.2. Global Propagation

Since the work reported in this paper is closely related to the global propagation, we will give a brief introduction to this method. The global propagation approach[?] takes the declarative descriptions of a geometric diagram as input and tries to determine the position of a geometric object from not only the constraint involving the geometric object but also implicit information derived from other constraints. The global propagation method determine the position of a point by the locus intersection method. To determine the locus of a point, global information about this point is needed. The global information needed in the propagation comes from a Geometric Information Base(GIB). The GIB for a configuration is a database containing all the properties of the configuration that can be deduced using a fixed set of geometric axioms. In the global propagation approach, besides the dimensional constraints specifying a distance or an angle, we use many implicit constraints such as the congruence of line segments or angles and ratio of line segments to specify more constraint problems than using dimensional constraints alone. The details of the global propagation approach can be found in [?].

By analyzing the global propagation approach, we can easily find that the key of the global propagation approach is to find the locus of the point. After knowing two loci that a point is on, we can easily know the position of the point by computing the intersection point of the two loci. However, the loci used in the global propagation approach is limited to the line and the circle. Using lines and circles , we can only determine the positions of the points whose dimensions are obtained from a linear or quadratic equation. Nevertheless, many problems from practical CAD/CAM need to solve equations with degree higher than two. For example, for two given circles O_1 and O_2, draw a square $ABCD$ with points A and B on the circle O_1 and points C and D on the circle O_2 (Figure 1). For convenience, let us assume that the radii of circles O_1 and O_2 are 3 and 5 respectively and $|O_1O_2| = 10$, we have $X^4 - 30X^3 + 316X^2 - 1320X + 13456 = 0$, where X is the length of one side of the square $ABCD$. Using the decision method for rc-constructible configuration[?], we can prove that the square can not be drawn with ruler and compass only. In order to remedy this defect, we introduce a new class of locus—conic. It can be proved later that any point whose dimension is computed from a cubic or quartic equation will be the intersection point of two conics.

3. Geometric Construction with Conic

3.1. Constructive Sequence with Conic

A diagram can be drawn with ruler, compass and conic if the points in the diagram can be listed in an order($P_1, P_2, ..., P_m$) such that each point P_i can be drawn using the following three basic constructions:

1. construction POINT(P): takes a free point P in the plane.
2. construction ON(P, O): takes a semi-free point P on a geometric object O.
3. construction INTERSECTION(P, O_1, O_2): takes the intersection P of two geometric objects O_1 and O_2.

Fig. 1. A simple example

In the above constructions, O, O_1 and O_2 are one of three geometric objects: line, circle and conic.

Conic are represented as functions of their focal points and directrixes. Lines and circles are represented as functions of their characteristic points such as the center of circle. Hence we can obtain other constructions on the basis of the three basic constructions. Let LINE(P,Q) be the line passing through points P and Q; PLINE(R, P, Q) the line passing through point R and parallel to LINE(P, Q); TLINE(R, P, Q) the line passing through point R and perpendicular to LINE(P, Q); CIRCLE(O, r) the circle with center O and radius r; PCIRCLE(O, P) the circle with center O and passing through point P; ACIRCLE(P, Q, α) the circle(or an arc)consisting of the points R satisfying $\angle PRQ = \alpha$; CCIRCLE(P, Q, R) the circle passing through three points P, Q, R; DCIRCLE(P, Q) the circle with LINE(P, Q) as a diameter [**?**]. For conic, let KCONICS(P, L, k) be the hyperbola, parabola or ellipse with focal point P and directrix L according to the cases of $k > 1, k = 1$, or $k < 1$; HYPERBOLA(P, Q, d) the hyperbola consisting of the points R satisfying $||RP|-|RQ|| = d$; RHYPERBOLA(P, Q, d) the branch close to point Q of the hyperbola consisting of the points R satisfying $|RP| - |RQ| = d, (d > 0)$; ELLIPSE(P, Q, d) the ellipse consisting of the points R satisfying $|RP| + |RQ| = d$.

3.2. New Constraints

With these new construction tools , we may use three new constraints:

- **SUM(A,B,C,s):** the sum of $|AB|$ and $|AC|$ is a fixed value s, where A,B,C are points.
- **DIF(A,B,C,d):** the difference of $|AB|$ and $|AC|$ is a fixed value d, where A,B,C are points.
- **EQU(A,P,L):** the distance from A to P is equal to the distance from A to line L, where A,P are points and L is a line.

Let us assume that B and C are fixed points. The locus of the point A that satisfies the constraint **SUM(A,B,C,s)** is an ellipse whose focal points are B and C. The locus of the

Fig. 2. Conic

point A that satisfies the constraint **DIF(A,B,C,s)** is a hyperbola whose focal points are B and C. The locus of the point A that satisfies the constraint **EQU(A,P,L)** is a parabola with focal point P and directrix L (Figure 2). With these new predicates, we may use any constructive approach of GCS to find construction sequence for a constraint problem.

3.3. Algorithm

The proposed method takes a constraint problem as input, and outputs a construction sequence for it. In the following, we will list the main steps of the method.

1. For a constraint problem $\{[Q_1, ..., Q_m], [P_1, ..., P_n], [C_1, ..., C_m]\}$, let $CT = [C_1, ..., C_m]$ be the constraint set, $CS = \phi$ the construction sequence, $QS = [Q_1, ..., Q_m]$ the points with given construction order, and $PS = [P_1, ..., P_n]$ the rest points to be constructed. We assume that the problem is not over-constraint, i.e., we have $|CT| \leq 2 \times |PS| - 3$.

2. Repeat the following steps first for QS and then for PS until both QS and PS become empty.

3. Take a point P from QS or PS. For each constraint $T \in CT$ involving P, decide the locus Lc of P satisfying T, assuming all points constructed in CS are known. We then obtain a set of triples: $\{(P, T_1, Lc_1), ..., (P, T_s, Lc_s)\}$. We consider three cases.
   - $CS = 0$. Point P is an arbitrarily chosen(free) point. We add a new construction CS=POINT(P) to CS.
   - There exist $i \neq j$ such that $T_i \neq T_j$ and Lc_i and Lc_j are not parallel lines or concentric circles or conic having no intersection points. We add a new construction CS=INTERSECTION(P, Lc_i, Lc_j) to CS and remove T_i and T_j from CT.
   - Point P is a semi-free point that can move freely on a line, a circle or a conic. We add a new construction CS=ON(P, Lc_i) to CS and remove T_i from CT.

4. Now we check whether the remaining problem is over-constraint, that is, whether $|CT| > 2 \times |PS|$.
   - If it is true, the construction sequence is invalid. If P is from QS, the order given by the user can not be constructed and the method terminates. Otherwise, restore the removed constraints and repeat the preceding step for a new point from PS.
   - If it is not true, point P is constructed. We need to repeat the preceding step for a new point.

The crucial step of the algorithm is how to determine the locus Lc from a constraint T in step 3.

3.4. Solving *Apollonius's drawing problem* with Conic

Let us consider the well-known *Apollonius's drawing problem* which has been considered in[**?**, **?**]: determining a circle satisfying condition $(C_i, C_j, C_k), (1 \leq i, j, k \leq 3)$, where,

C_1 : the circle passes through a known point.

C_2 : the circle is tangent to a known line.

C_3 : the circle is tangent to a known circle.

There are altogether ten drawing problems. Although these drawing problems can be solved by the global propagation approach with ruler and compass[**?**], the solution thus given involves many auxiliary points. Hence the complexity of computing has been increased. In [**?**], the Gröbner basis is used to solve these problems. We need only compute one intersection point by introducing conic. First, let us simply analyze *Apollonius's drawing problem* below. Determining a circle need two elements: the center and radius of the circle. If we know the center of the circle, the radius of the circle can be easily obtained by computing the distance from the center of the circle to the known point or the known line . The key step is determining the center of the circle. Note that the circle satisfies three constraints. If we ignore a constraint, the remaining two constraints would determine a locus for the center of the circle . Hence, we will have three loci for three given constraints and the center of the circle will be determined by computing the intersection point of any two loci. Subsequently, as long as we know the locus of the center determined by any two constraints, we will know the position of the center of the circle and solve the *Apollonius's drawing problem*.

According to the three given constraints, we need consider six cases (C_1, C_1), (C_1, C_2), (C_1, C_3), (C_2, C_2), (C_2, C_3), (C_3, C_3). In the figures below(from Figure 3 to Figure 6), the bold circle and bold line represent the known circle and known line; the dotted circle and dotted line represent the unknown circle and auxiliary line respectively. Others are the loci of the center of the circle.

Fig. 3. Case (C_1, C_3): point inside of or on circle

Fig. 4. Case (C_1, C_3): point outside of circle

$(\mathbf{C_1}, \mathbf{C_1})$: The circle passes through two known points. Clearly, the locus of the center of the circle is the perpendicular bisector of the segment connecting two known points.

$(\mathbf{C_1}, \mathbf{C_2})$: The circle passes through a known point and is tangent to a known line. In this case, the distance from the center of the circle to the known point equals the distance from the center of the circle to the known line. The locus of the center of the circle is a parabola whose focal point is the known point and directrix is the known line.

$(\mathbf{C_2}, \mathbf{C_2})$: The circle is tangent to two known lines simultaneously. If two known lines are parallel, the locus of the center of the circle is a parallel line equidistant with two known lines. Otherwise, it is the two bisectors of the angle formed by two known lines.

Fig. 5. Case (C_2, C_3): circle and line are intersecting or tangent

$(\mathbf{C_1}, \mathbf{C_3})$: The circle passes through a known point and is tangent to a known circle. Different positions of the known point and circle will result in different loci for the center . When the known point is inside the known circle(Figure 3), the absolute value of the sum of the distance from the center of the circle to the known point and from the center of the circle to the center of the known circle is a fixed value. Hence according to the constraint **SUM(A,B,C,d)**, the locus is an ellipse whose focal points are the known point and the center of the known circle. When the known point is on the known circle(Figure 3), the locus is the line passing through the known point and the center of the known circle. When the known point is outside of the known circle(Figure 4), the absolute value of the differences between the distances from the center of the circle to the known point and from the center of the circle to the center of the known circle is a fixed value. Hence according to the constraint **DIF(A,B,C,d)**, the locus of the center of the circle is a hyperbola whose focal point is the known point and the center of the known circle.

In what below, we only give the conclusion without analyzing.

$(\mathbf{C_2}, \mathbf{C_3})$: The circle is tangent to a known line and a circle simultaneously. When the known circle and the known line are intersecting(Figure 5, left), the locus is a parabola whose focal point is the center of the known circle and the directrix is a line having fixed distance to the known line. When the known line is tangent to the known circle(Figure 5, right), the locus is the line perpendicular to the known line and passing through the center of the known circle or the parabola whose focal point is the center of the known circle and the directrix is a line having fixed distance to the known line. When the known circle and the known line are separate(Figure 6), the locus is the parabola whose focal point is the center of the known circle and the directrix is a line having fixed distance to the known line.

$(\mathbf{C_3}, \mathbf{C_3})$: The circle is tangent to two known circles simultaneously. When the two known

Fig. 6. Case (C_2, C_3):circle and line are separate

circles and the circle are tangent at the same point, the locus is a line passing through the center of two known circles. When the two known circles are interiorly tangent , the locus is an ellipse whose focal points are the centers of the two known circles. When the two known circles are intersecting, at the same time, the circle is an inscribed circle of one known circle and a circumscribed circle of another known circle, the locus is an ellipse whose focal points are the centers of the two known circles. Except the cases above, the locus is a hyperbola whose focal points are the centers of the two known circles.

Now let us consider the problem (C_1, C_3, C_3) (Figure 7). Based on the above analysis, the center of the circle is the intersection of two hyperbolas. Hence this problem has four solutions. From this example, we may give simpler construction steps for rc-constructible problem. We essentially need one construction INTERSECTION(O, RHYPERBOLA(O_1, A, R_1), RHYPERBOLA$(O_1, O_2, R_1 - R_2)$) to find the center of circle O. In the global propagation approach, we need 5 constructions(Figure 8) [**?**].

Fig. 7. Solving *Apollonius's drawing problems* with conic

3.5. Other Examples

Example 1 Draw an equilateral triangle ABC, if B, C are on a known circle O and A is a known point that is outside of the known circle (Figure 9, left).

We remove the constraint that B and C are on a known circle O and assume that B and C are symmetric with line OA. Note that B and C are symmetrical according to the line OA, hence we only need construct point B. Since $\triangle ABC$ is an equilateral triangle, it is always true that $|AB|$ equals to two times the distance from point B to line OA. According to the definition of hyperbola, B is the intersection of a hyperbola and a circle, at this time, the hyperbola is degenerated to two lines. Therefore, this problem has two solutions. The following is one of the construction sequence for this problem:

1. POINT(A);
2. POINT(O);
3. CIRCLE(O, R);
4. LINE(O, A);
5. INTERSECTION(B, KCONICS(A, LINE(O, A), 2), CIRCLE(O, R));
6. INTERSECTION(C, PCIRCLE(B, A),PCIRCLE(A, B));
7. LINE(A, B);
8. LINE(A, C);
9. LINE(B, C).

Fig. 8. Solving *Apollonius's drawing problems* with the global propagation approach

Example 2 Draw a $\triangle ABC$, if $|AB|$, $\angle ACB$ and $|BC|+|AC|$ are known (Figure 9, right).

Suppose A and B are known. Since $\angle ACB$ is known, C is on a circle. Since $|BC|+|AC|$ is known, from the constraint **SUM(C,A,B,d)**, the locus of C is a hyperbola. Hence point C is the intersection of a circle and an ellipse, since $|AB|$ is already given. This problem has four solutions. One of the construction sequence is as follows:

1. POINT(A);
2. POINT(B);
3. LINE(A, B);
4. INTERSECTION(C, ACIRCLE(A, B, α), ELLIPSE($A, B, |AC| + |BC|$));
5. LINE(A, C);
6. LINE(B, C).

To this example, although point C is the intersection of a circle and an ellipse, point C is still rc-constructible. This is a consequence of a theorem proved by us that if one of the symmetric axes of one conic is the same as one of the symmetric axes of another conic, the intersection of the two conics is rc-constructible. The above two examples are all rc-constructible, in the following, we will enumerate two non-rc-constructible problems.

Example 3 For three given circles O_1, O_2, O_3 and a fixed point C that is on circle O_1, draw a circle O tangent to two circles O_2, O_3 and its center satisfies the constraint $|CO| \times |OD| = K^2$, where K is a fixed value and D is the intersection of the circle O_3 and line CO (Figure 10).

First, we only consider that circle O is tangent to two circles O_2, O_3. Since $||OO_2| - |OO_3|| = |R_2 - R_3|$, where R_1, R_2, R_3 are the radius of circle O_1, O_2, O_3 respectively, point O is on a hyperbola whose focal points are O_2 and O_3. Second, we only consider the constraint

Fig. 9. Rc-constructible configuration examples

$|CO| \times |OD| = K^2$. Based on the tangency property, the length of tangent line from O to circle O_1 is a fixed value, hence O is on a concentric circle of circle O_1. This problem has two solutions, one of the construction sequence is as follows:

1. POINT(O_1);
2. POINT(O_2);
3. POINT(O_3);
4. CIRCLE(O_1, R_1);
5. CIRCLE(O_2, R_2);
6. CIRCLE(O_3, R_3);
7. INTERSECTION(O, RHYPERBOLA($O_1, O_2, R_1 - R_2$), CIRCLE($O_3, \sqrt{K^2 - R_3^2}$));
8. CIRCLE($O, |OO_1| - R_1$).

4. Scope of *GCS* with Conic

Corresponding to the construction order of points in diagram, there is a construction sequence($CS_1, CS_2, ..., CS_m$), where each CS_i is constructions. Algebraically, each CS_i corresponds to a system of algebraic equations. Hence, a construction sequence corresponds to

Fig. 10. Non rc-constructible configuration example

a system of algebraic equations. Since lines, circles and conic are functions of points, we need only consider points.

First, let us consider the three basic constructions. If P is introduced by POINT(P), P is a free point. No equation is introduced. If P is introduced by ON(P, O), we need one algebraic equation, $f(u,x)=0$, to describe P, where u is a parameter and x is a dependent variable. For a given u, we can compute x from $f(u,x)=0$. Hence $f(u,x)=0$ will cause infinite solutions. The construction INTERSECTION(P, O_1, O_2) will produce a system of algebraic equations:

$$\begin{cases} f_1(x_1,x_2)=0 \\ f_2(x_1,x_2)=0 \end{cases} \tag{1}$$

where $f_1(x_1,x_2)=0$ and $f_2(x_1,x_2)=0$ are the equations of the two geometric objects respectively.

Let us consider the system of algebraic equations produced by the construction sequence $(CS_1, CS_2, ..., CS_n)$. It is easy to see that by properly choosing coordinates the equations are of the following form:

$$\begin{cases} f_1(u_1,x_1)=0 \\ f_2(u_2,x_2)=0 \\ f_3(u_1,x_1,x_3,x_4)=0 \\ f_4(u_2,x_2,x_3,x_4)=0 \\ \qquad \vdots \\ \qquad \vdots \\ f_l(u_1,...,u_q,x_1,...,x_p)=0 \end{cases} \tag{2}$$

where, $degree(f_i) \leq 2 (1 \leq i \leq l)$.

Let FS be the system of algebraic equations (2). Based on the *Wu-Ritt's zero decomposition algorithm*[**?**], FS can be transformed into a system of algebraic equations TS in triangular form and TS can be written as:

$$\begin{cases} F_1(u_1,...,u_q,x_1)=0 \\ F_2(u_1,...,u_q,x_1,x_2)=0 \\ \quad\vdots \\ \quad\vdots \\ F_p(u_1,...,u_q,x_1,...,x_p)=0. \end{cases} \tag{3}$$

Since the variables are introduced one by one or two by two, by the *Bezout Theorem*, $degree(F_i) \leq 4 (1 \leq i \leq l)$. The solution of a system of algebraic equations in triangular form is easy to compute. For a set of numerical values for the $u_i (1 \leq i \leq q)$, $F_1(u_1,...,u_q,x_1)=0$ becomes a univariate equation. We can find the solutions for x_1 from this univariate equation. Next, by substituting the numerical solution for x_1 into equation $F_2(u_1,...,u_q,x_1,x_2)$, $F_2(u_1,...,u_q,x_1,x_2)=0$ becomes a univariate equation. We now solve x_2 from this univariate equation, and so on. Further, since $degree(F_i) \leq 4 (1 \leq i \leq l)$, their solutions can be found by explicit radicals formulas [**?**].

We have proved that a construction sequence of lines, circles and conic will lead to a triangular set of polynomials with degree less than five. Now, we will prove that each triangularised polynomial equations with degree less than five can be generalized by a construction sequence of lines, circles and conic. Note that we need only to solve a univariate equation with degree less than five in the process of solving TS. Hence we only need to prove that the root of any univariate equation with degree less than five can be constructed with lines, circles and conic. The cases of the univariate equations with degree less than three are trivial and obvious. We only consider the cases of the univariate equation with degree three and four.

Lemma1 *The root of any cubic univariate equation can be constructed with two conics.*
Proof. Let the cubic univariate equation be

$$p(x) = x^3 + ax^2 + bx + c \tag{4}$$

where a,b,c $\in R$. We can construct two conics $f_1(x,y)=0$ and $f_2(x,y)=0$, where,

$$f_1(x,y) = y - x^2 \tag{5}$$

$$f_2(x,y) = (x+c_1)(y+c_2) + f. \tag{6}$$

By (5) we have

$$y = x^2 \tag{7}$$

substituting y from (7) into (6), we obtain

$$f_2(x,y) = (x+c_1)(x^2+c_2) + f \tag{8}$$

$$= x^3 + c_1x^2 + c_2x + f + c_1c_2. \tag{9}$$

Comparing the coefficients of this cubic equation with that of $p(x)$, we have

$$\begin{cases} a &= c_1 \\ b &= c_2 \\ c &= f + c_1 c_2. \end{cases} \tag{10}$$

Clearly, the roots of $x^3 + ax^2 + bx + c = 0$ can be constructed by two conics, $y = x^2$ and $(x + a)(y + b) + c - ab = 0$.

Lemma2 *The root of any quartic univariate equation can be constructed with two conics.* The proof of lemma2 can be obtained similarly, where, the conic for

$$p(x) = x^4 + ax^3 + bx^2 + cx + d \tag{11}$$

are of the form

$$f_1(x, y) = x^2 + dy^2 + ax + cy + b \tag{12}$$

$$f_2(x, y) = xy - 1. \tag{13}$$

By **Lemma1** and **Lemma2**, we have proved the following result.

Theorem. *A diagram can be drawn with lines, circles and conics if and only if the coordinates of the points in the diagram can be expressed as roots of a sequence of algebraic equations with degree less than or equal to four.*
From the above theorem, we have two consequences:

1. By introducing conic, the drawing scope has been essentially enlarged than using ruler and compass alone.

2. Since roots of equations of degree less than four can be written as formulas of radicals [**?**], as the quadratic equations, the computation process is still as simple as the case of using ruler and compass.

Theoretically, the well-known non-rc-constructible drawing problem: trisection of angle(see Example 4) and draw a cube whose volume is double another known cube can be draw by conic.

Example 4 (*Trisection of angle*) Try to trisect an given angle θ.
This diagram can not be drawn with ruler and compass and the problem can be reduced to solve the following equation

$$x^3 - 3x - m = 0 \quad (m = 2\cos\theta)$$

the roots of which are the intersection of two conics:

$$f_1(x, y) = x^2 - y - 3$$

$$f_2(x, y) = xy - m.$$

Therefore, the problem of trisection of angle can be solved with conic.

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Example 5 (*Pappus's Problem*)[**?**, **?**] Put a rectangle into a rectangle as shown in Figure 11.

In order to find the correct position of the rectangle in the rectangular trough, we only need to determine the length of EB. Suppose $AC = a, AB = b, EF = h$ and $EB = x$, we have

$$x^4 - 2bx^3 + (b^2 + a^2)x^2 - a^2h^2 = 0. \tag{14}$$

Generally, the roots of this equation can not be drawn with ruler and compass except that a, b and h equal particular values respectively. Based on the method proposed in Section 4, we can construct the roots of this equation with the two conics below:

$$f_1(x, y) = x^2 - a^2h^2y^2 - 2bx + b^2 + a^2 \tag{15}$$

$$f_2(x, y) = xy - 1. \tag{16}$$

That is, we can draw this diagram with the proposed method.

Now, let us consider the constraint problem in Figure 1. With the given conditions, we know that if one of A, B, C and D is known, the square $ABCD$ can be drawn. We will determine C. Since points B and C are on circle $O_1, BC = BA$, and $BC \perp BA$, by properly choosing coordinates, we have the locus of point C, $X^2 - 4XY + 4Y^2 = R_1^2$, where R_1 is the radius of circle O_1. This is a conic. Since point C lies on the circle O_2, we can construct point C with circle O_2 and the above conic, that is, the square $ABCD$ can be drawn with conic.

Fig. 11. *Pappus's Problem*

5. Conclusion

1. In the geometric construction approach of GCS, we use line, circle and conic as drawing tools.
2. The drawing scope by introducing conic is essentially larger than using line and circle only.
3. For some diagrams that can be drawn with ruler and compass, we may give much simpler construction procedures by using conic.
4. To draw a diagram with the three tools, we need to solve a sequence of equations with degree ≤ 4.
   Conversely, any sequence of equations with degree ≤ 4 can be obtained by a construction sequence of line, circle and conic.
5. Since equations with degree ≤ 4 can be solved explicitly by radicals, the computation procedure is still complete, stable, and efficiently as the case of ruler and compass construction.
6. We can explicitly obtain a construction sequence having geometric meaning for all GCS whose dimensions are computed from a cubic or quartic equation like a quadratic equation.

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