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"Russian Killer" No. 2: A Challenging Geometric Theorem with Human and Machine Proofs[1)]

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Abstract. This article presents one human proof and three machine proofs of a challenging geometric theorem that gives a beautiful representation of the area of an arbitrary quadrilateral in terms of its four sides and four internal angles. These proofs demonstrate the power, capability and features of automated deduction methods and tools, which reduce qualitative difficulty to quantitative complexity versus traditional methods with individual ingenious ideas, for mathematical theorem proving. The present case study not only results in four probably new proofs of a hard theorem but also contributes to understanding the significance of developing effective algorithms and software tools for automated theorem proving in mathematics using advanced computing technology.

1. Introduction to the Problem

In February 1998 Sergey Markelov [6] from the Moscow Center for Continuous Mathematics Education sent a set of five geometric theorems to Dongming Wang for testing the capability of his GEOTHER package [7], with an interest to challenge whether computer provers can prove really *hard* theorems. These theorems have been used to prepare the Moscow team for all-Russia school mathematics olympiad, and are called *killers to analytic ways of geometric-problem-solving*. They can be proved in geometric ways, but no analytic proof could be found even by expert geometers. Let us call these five theorems the *Russian killers* for short.

After a quick look at the five killers, Wang was convinced that some of them can be proved by GEOTHER in principle. For experimental purpose, he took the second of the killers which is stated below. This killer is very easy to explain and to understand. It provides a beautiful representation of the area of an arbitrary quadrilateral in terms of its four sides and four internal angles.

Theorem. *Let $ABCD$ be an arbitrary quadrilateral with sides $|AB| = k$, $|BC| = l$, $|CD| = m$, $|DA| = n$ and internal angles $2a, 2b, 2c, 2d$ at vertices A, B, C, D respectively, and let S be the area of the quadrilateral. Then*

$$4\,S = \frac{(k+l+m+n)^2}{\cot a + \cot b + \cot c + \cot d} - \frac{(l+n-k-m)^2}{\tan a + \tan b + \tan c + \tan d}.$$

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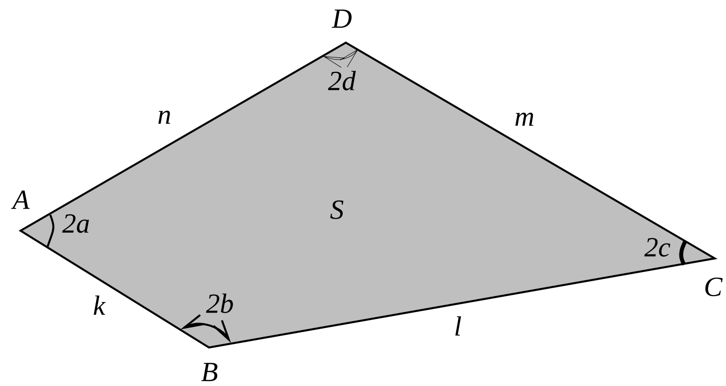

Fig. 1

The beauty of the expression lies partially at the separation of the sides and internal angles. The theorem generalizes the well-known *Brahmagupta formula* and the result for the case where the quadrilateral is inscribed a circle.

After a few trials, Wang announced a machine proof of the theorem using Wu's method [10] in GEOTHER in April 1998; this proof requires heavy polynomial computations. Meanwhile, he posted the theorem to several colleagues, soliciting other machine or human proofs. Soon after that, Hongbo Li announced another machine proof using Clifford algebra formalism, followed by the third machine proof given by Lu Yang, both in May 1998. The proofs of Li and Yang are short and took only a few seconds of computing time. Finally in later May 1998, Xiaorong Hou discovered an elegant and short geometric proof of the theorem. This proof reflects the common features of traditional geometric proofs, in which ingenious ideas are used individually to a great extent.

This article collects the four proofs. Its purpose is twofold: on the one hand different kinds of proofs of a difficult geometric theorem are presented that have clear interest for geometers. On the other hand, the proofs show the power and capability of automated deduction methods and tools for proving hard theorems (the Chinese provers against the Russian killers!), and the advantages and disadvantages of machine proofs versus human proofs. From this case study, one is brought to see the machine power and intelligence against human ingenuity in geometric problem solving.

2. A Traditional Geometric Proof

In this section is provided a geometric proof of the theorem worked out by the first author. The non-trivial ideas and special techniques used in the proof, besides their own value, may be a supplement to demonstrate the advantage, significance and power of machine proving, which is automatic, straightforward, and fast.

Let

$$T = \frac{k+l+m+n}{2}, \quad t = \frac{l+n-k-m}{2}$$

and

$$\alpha = \tan a + \tan b + \tan c + \tan d, \quad \beta = \cot a + \cot b + \cot c + \cot d.$$

The formula to be proved becomes

$$S = \frac{T^2}{\beta} - \frac{t^2}{\alpha}.$$

The proof of the theorem consists of the following five steps.

STEP 1. We first note the following lemma, which is known and can be easily proved.

Lemma 1. If $ABCD$ is inscribed a circle, then

$$S = \frac{T^2}{\beta}, \quad |AB| = \frac{(\cot a + \cot b) \cdot T}{\beta}.$$

STEP 2. Let $ABCD$ be an arbitrary quadrilateral. Without loss of generality, we assume that $|BC| + |DA| > |AB| + |CD|$ and $|BC| \geq |DA|$. Then one can construct a diagram as in Fig. 2, which yields five points B', E, F, A', C', such that

$$FB' \parallel AD, \quad EB' \parallel CD, \quad B'C' \parallel BC, \quad A'B' \parallel AB;$$
$$|FB| + |EB'| = |BE| + |FB'| = T.$$

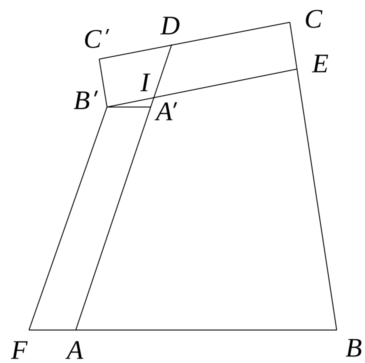

Fig. 2

So $FBEB'$ is inscribed a circle. The details of construction are given in step 5.

STEP 3. In what follows, let Δ_{ABCD} or Δ_{ABC} denote the area of an arbitrary quadrilateral $ABCD$ or triangle ABC. We have the following corollaries, of which the first two follow immediately from Lemma 1.

Corollary 1.

$$\Delta_{FBEB'} = \frac{T^2}{\beta}, \quad |FB| = \frac{(\cot a + \cot b) \cdot T}{\beta}.$$

Corollary 2.

$$\begin{aligned}|A'B'| + |C'D| &= |AF| + |B'E| - |CD| = |FB| - |AB| + |B'E| - |CD| \\ &= T - (|AB| + |CD|) = t,\\ |B'C'| + |DA'| &= |CE| + |DA| - |FB'| = |BC| - |BE| + |DA| - |FB'| \\ &= (|BC| + |DA|) - T = t,\end{aligned}$$

so $A'B'C'D$ is inscribed a circle. Moreover,

$$\Delta_{A'B'C'D} = \frac{t^2}{\alpha}, \quad |A'B'| = \frac{(\tan a + \tan b) \cdot t}{\alpha}.$$

Corollary 3. $\Delta_{FAA'B'} = \Delta_{ECC'B'}$.

Proof. Let

$$\gamma = \frac{1}{2} \sin d \cdot \cos b;$$

we have

$$\begin{aligned}
\Delta_{FAA'B'} &= |A'B'| \cdot |FB'| \cdot \sin 2a \\
&= \frac{(\tan a + \tan b) \cdot (\cot a + \cot d) \cdot t \cdot T \cdot \sin 2a}{\alpha \cdot \beta} \\
&= \frac{\sin(a+b) \cdot \sin(a+d) \cdot t \cdot T}{\alpha \cdot \beta \cdot \gamma}, \\
\Delta_{ECC'B'} &= |B'C'| \cdot |EB'| \cdot \sin 2c \\
&= \frac{(\tan c + \tan b) \cdot (\cot c + \cot d) \cdot t \cdot T \cdot \sin 2c}{\alpha \cdot \beta} \\
&= \frac{\sin(c+b) \cdot \sin(c+d) \cdot t \cdot T}{\alpha \cdot \beta \cdot \gamma}, \\
&= \frac{\sin(\pi - a - d) \cdot \sin(\pi - a - b) \cdot t \cdot T}{\alpha \cdot \beta \cdot \gamma}, \\
&= \Delta_{FAA'B'}.
\end{aligned}$$

Step 4. Note that I is the intersection point of $B'E$ and $A'D$ in Fig. 2. By Corollary 3, we have

$$\begin{aligned}
\Delta_{ABCD} &= \Delta_{ABEI} + \Delta_{IECD} \\
&= (\Delta_{FBEB'} - \Delta_{FAIB'}) + \Delta_{IECD} \\
&= \Delta_{FBEB'} - (\Delta_{FAA'B'} + \Delta_{B'A'I}) + \Delta_{IECD} \\
&= \Delta_{FBEB'} - (\Delta_{ECC'B'} - \Delta_{IECD}) - \Delta_{B'A'I} \\
&= \Delta_{FBEB'} - \Delta_{A'B'C'D}.
\end{aligned}$$

The following theorem is therefore established.

Theorem 1. $\Delta_{ABCD} = \Delta_{FBEB'} - \Delta_{A'B'C'D}$.

Step 5. Construct the diagram in Fig. 3 according to the steps detailed below. It is a simple exercise to verify that the constructed diagram satisfies the requirements given in step 2.

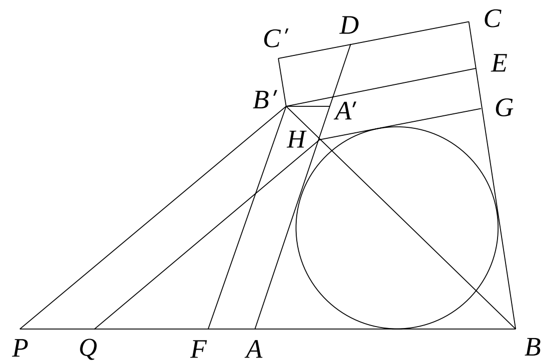

Fig. 3

- Draw the inscribed circle of the three sides AB, BC, DA.
- Draw a tangent line GH of the circle that is parallel to line CD. This produces a point G on the segment BC and a point H on the segment DA.
- Mark off a segment BP of length T and a segment BQ of length $(|AB|+|BG|+|GH|+|HA|)/2$ on line BA from point B.
- Draw the parallel to line QH through point P that intersects line BH at point B'.
- Draw the parallel to line DC through point B' that intersects line BC at point E.
- Draw the parallel to line DA through point B' that intersects line BA at point F.
- Draw the parallel to line AB through point B' that intersects line DA at point A'.
- Draw the parallel to line BC through point B' that intersects line CD at point C'.

This completes the proof of the theorem.

The geometric constructions used to reduce the problem in the above proof are clearly crucial and well thought out. It is not trivial to figure out such constructions and proofs even for geometry experts. The reader is urged to work out his own geometric proofs.

3. A Machine Proof Using Wu's Method

In contrast with the geometric proof presented in the preceding section, the machine proof explained below is straightforward. Instead of ingenious ideas, we simply write a short natural specification of the theorem, apply a geometry theorem prover — `Wprover` in GEOTHER [7] — developed on the basis of Wu's method [10], and let the machine do the computation and proving.

According to the geometric hypotheses, we have the following relations

$$\begin{aligned}
h_1 &= 2S - kn\sin 2a - lm\sin 2c = 0,\\
h_2 &= 2S - kl\sin 2b - mn\sin 2d = 0,\\
h_3 &= k^2 + n^2 - 2kn\cos 2a - l^2 - m^2 + 2lm\cos 2c = 0,\\
h_4 &= k^2 + l^2 - 2kl\cos 2b - m^2 - n^2 + 2mn\cos 2d = 0,\\
h_5 &= \sin(a+b+c+d) = 0.
\end{aligned}$$

Let

$$\sin a = x_1, \quad \sin b = x_2, \quad \sin c = x_3, \quad \sin d = x_4,$$
$$\cos a = y_1, \quad \cos b = y_2, \quad \cos c = y_3, \quad \cos d = y_4.$$

By expanding the sine and cosine of double angles and sum of angles with simple substitution, the above $h_1, \ldots, h_5$ will become expressions in $x_1, \ldots, x_4, y_1, \ldots, y_4$ and S, k, l, m, n. Clearly,

$$\begin{aligned} h_6 &= x_1^2 + y_1^2 - 1 = 0, \\ h_7 &= x_2^2 + y_2^2 - 1 = 0, \\ h_8 &= x_3^2 + y_3^2 - 1 = 0, \\ h_9 &= x_4^2 + y_4^2 - 1 = 0. \end{aligned}$$

Then

$$h_1 = 0, \ldots, h_9 = 0$$

constitute the hypothesis of the theorem, and the conclusion to be proved is

$$g = S - \frac{T^2}{\dfrac{y_1}{x_1} + \dfrac{y_2}{x_2} + \dfrac{y_3}{x_3} + \dfrac{y_4}{x_4}} + \frac{t^2}{\dfrac{x_1}{y_1} + \dfrac{x_2}{y_2} + \dfrac{x_3}{y_3} + \dfrac{x_4}{x_4}} = 0.$$

The statement of the theorem implies that the denominators do not vanish (e.g., $x_i \neq 0$ and $y_i \neq 0$ for every i). So we only need to prove that $h_1 = 0, \ldots, h_9 = 0$ imply that the numerator g^* of g is 0. For this purpose, let us simply apply Wu's method of automated geometry theorem proving [10].

With loss of generality, take $n = 1$. Then g^* becomes a polynomial consisting of 91 terms. With respect to the variable ordering

$$y_2 \prec x_2 \prec y_3 \prec x_3 \prec y_4 \prec x_4 \prec y_1 \prec x_1 \prec l \prec k \prec m \prec S,$$

the set of hypothesis-polynomials $h_1, \ldots, h_9$ may be easily *triangularized* by variable elimination into an "almost equivalent" set (called a *quasi-characteristic set*) of 8 polynomials:

$$\begin{aligned} c_1 &= h_7, \quad c_2 = h_8, \quad c_3 = h_9, \\ c_4 &= y_1^2 + 2(2y_2^2y_3y_4x_3 + 2y_2y_3^2y_4x_2 - y_2y_4x_2 - y_3y_4x_3)x_4 - 4y_2^2y_3^2y_4^2 \\ &\quad - 2y_2y_3x_2x_3 + 4y_2y_3y_4^2x_2x_3 + 2y_2^2y_3^2 + 2y_2^2y_4^2 + 2y_3^2y_4^2 - y_2^2 - y_3^2 - y_4^2, \\ c_5 &= I_5x_1 + y_1y_3y_4x_2 - y_1x_2x_3x_4 + y_1y_2y_4x_3 + y_1y_2y_3x_4, \\ c_6 &= I_6k + (y_2y_3^2y_4x_3 - y_2y_3x_4 + y_2y_3^3x_4 - y_3^2x_2x_3x_4 + y_3^3y_4x_2 - y_3y_4x_2)l \\ &\quad - y_2y_4^3x_3 + y_3y_4x_2 + y_2y_4x_3 - y_3y_4^3x_2 - y_2y_3y_4^2x_4 + y_4^2x_2x_3x_4, \\ c_7 &= I_7m + (2y_2^2l - l - 2y_1^2 + 1)k - l^2 + 1, \\ c_8 &= S - y_2x_2lk - y_4x_4m, \end{aligned}$$

where

$$\begin{aligned} I_5 &= y_2y_3y_4 - y_2x_3x_4 - y_4x_2x_3 - y_3x_2x_4, \\ I_6 &= 3y_2^2y_3y_4x_2 - y_2^2x_2x_3x_4 + y_2^3y_4x_3 + 3y_2^3y_3x_4 - y_2y_4x_3 - 2y_2y_3x_4 \\ &\quad - y_3y_4x_2 + 4y_2^2y_3^2x_2x_3x_4 - 4y_2^2y_3^3y_4x_2 - 4y_2^3y_3^2y_4x_3 - 4y_2^3y_3^3x_4 \\ &\quad + y_3^3y_4x_2 - y_3^2x_2x_3x_4 + 3y_2y_3^2y_4x_3 + 3y_2y_3^3x_4, \\ I_7 &= 2ly_3^2 - l - 2y_4^2 + 1. \end{aligned}$$

During the computation, a polynomial factor $l^2 - 1$ is removed. The "almost equivalence" means that $h_1 = 0, \ldots, h_9 = 0$ are equivalent to $c_1 = 0, \ldots, c_8 = 0$ under the subsidiary condition that

$$(l^2 - 1) I_5 I_6 I_7 \neq 0. \qquad (*)$$

With this condition assumed, proving the theorem is reduced to verifying whether the *pseudo-remainder* R of g^* with respect to $\{c_1, \ldots, c_8\}$ is identically equal to 0. The remainder R is indeed 0: the verification is not easy and takes about 320 seconds of CPU time in Maple V.3 on an Alpha station. Some of the occurring polynomials are very large. R is obtained successively as follows: Compute first the pseudo-remainder R_8 of g^* with respect to c_8 in S, then the pseudo-remainder R_7 of R_8 with respect to c_7 in m, and so forth. R is the last pseudo-remainder R_1 of R_2 with respect to c_1 in x_2. Here, $S, m, \ldots, x_2$ are the leading variables of $c_8, c_7, \ldots, c_1$ respectively.

Let us use an index triple $[t\ x\ d]$ to characterize an arbitrary polynomial P, where t is the number of terms in P, x the leading variable of P and d the degree of P in x. The reduction of g^* to 0 using $c_8, \ldots, c_1$ may be sketched as follows:

$$g^* = [93\ S\ 1] \longrightarrow [106\ m\ 2] \longrightarrow [680\ k\ 2]$$

$$\longrightarrow \left\{ \begin{array}{l} [4529\ x_1\ 2] \longrightarrow [6541\ y_1\ 6] \longrightarrow [19013\ x_4\ 9], \\ [3432\ x_1\ 2] \longrightarrow [5221\ y_1\ 6] \longrightarrow [17586\ x_4\ 9], \\ [2450\ x_1\ 2] \longrightarrow [3690\ y_1\ 4] \longrightarrow [11066\ x_4\ 8], \\ [1015\ x_1\ 2] \longrightarrow [1543\ y_1\ 2] \longrightarrow [6276\ x_4\ 7], \\ [4034\ x_1\ 2] \longrightarrow [6067\ y_1\ 6] \longrightarrow [17813\ x_4\ 9] \end{array} \right\}$$

$$\longrightarrow \left\{ \begin{array}{l} [687\ x_3\ 9],\ [210\ x_3\ 8],\ [549\ x_3\ 9],\ [656\ x_3\ 8],\ [505\ x_3\ 7], \\ [327\ x_3\ 9],\ [803\ x_3\ 9],\ [697\ x_3\ 9],\ [647\ x_3\ 9],\ [582\ x_3\ 9], \\ [524\ x_3\ 9],\ [688\ x_3\ 9],\ [667\ x_3\ 9],\ [420\ x_3\ 9],\ [693\ x_3\ 9], \\ [697\ x_3\ 9],\ [688\ x_3\ 9],\ [283\ x_3\ 9],\ [684\ x_3\ 9],\ [519\ x_3\ 9], \\ [432\ x_3\ 9],\ [622\ x_3\ 8],\ [523\ x_3\ 8],\ [554\ x_3\ 9],\ [538\ x_3\ 9], \\ [549\ x_3\ 9],\ [549\ x_3\ 9],\ [544\ x_3\ 9],\ [376\ x_3\ 9],\ [730\ x_3\ 8], \\ [732\ x_3\ 8],\ [699\ x_3\ 8],\ [696\ x_3\ 8],\ [711\ x_3\ 8],\ [646\ x_3\ 7], \\ [602\ x_3\ 7],\ [799\ x_3\ 9],\ [810\ x_3\ 9],\ [790\ x_3\ 9],\ [218\ x_3\ 8], \\ [558\ x_3\ 9],\ [556\ x_3\ 9],\ [549\ x_3\ 9],\ [494\ x_3\ 9],\ [170\ x_3\ 7], \\ [437\ x_3\ 8],\ [313\ x_3\ 8],\ [165\ x_3\ 6],\ [425\ x_3\ 7],\ [293\ x_3\ 7], \\ [641\ x_3\ 7],\ [649\ x_3\ 7],\ [621\ x_3\ 7],\ [157\ x_3\ 8],\ [444\ x_3\ 9], \\ [308\ x_3\ 9],\ [665\ x_3\ 9],\ [545\ x_3\ 9],\ [796\ x_3\ 9],\ [718\ x_3\ 9], \\ [780\ x_3\ 9],\ [804\ x_3\ 9] \end{array} \right\}$$

$$\longrightarrow \{0\}.$$

Therefore, the theorem is proved to be true under the subsidiary condition $(*)$. The geometric meaning of the algebraic condition can be interpreted (automatically) in most cases, and we do not enter into the details of interpretation. In general, the subsidiary condition corresponds to some degenerate cases in which the geometric theorem may be false or meaningless. Moreover, whether the theorem is true in a special or degenerate case can be checked by using the same method.

In the above proof of the theorem, we had no attempt to use special techniques to simplify the algebraic formulation, so the involved algebraic computations are very heavy. However, this is already within the reach of a PC Pentium nowadays. The splitting technique in the reduction is due to Wu [9]. Direct computation of the pseudo-remainder on our machines without this technique is still not possible. With some thought and reasoning, the machine proof may be considerably simplified by using different formulations. This can be seen from the formulation and proof in Section 5.

4. A Machine Proof Using Clifford Algebra

In the previous machine proof geometric relations are expressed as polynomial equations with distances of segments, sines and cosines of angles as variables, and computations are performed with polynomials. It is known that geometric problems may also be formalized in other algebras. In the next proof, Clifford algebra is used to represent geometric relations, where each algebraic expression has a clearer geometric meaning. In this case, computations have to be carried out according to the rules in Clifford algebra. The reader is referred to Chapter 1 of [2] for a geometric introduction to Clifford algebra and to [1, 4, 5, 8] for some recent developments on Clifford algebra approaches for automated geometric theorem proving.

Let the points A, B, C, D be considered as vectors from the origin to the points. Then $\mathbf{k} = B - A$, $\mathbf{l} = C - B$, $\mathbf{m} = D - C$ and $\mathbf{n} = A - D$ are also vectors. Their corresponding unit vectors are denoted by $\bar{\mathbf{k}}, \bar{\mathbf{l}}, \bar{\mathbf{m}}$ and $\bar{\mathbf{n}}$.

Let $I\!I$ be a unit bivector of the plane, which represents geometrically the oriented parallelogram formed by two unit vectors. Let

$$x = \exp(2aI\!I), \quad y = \exp(2bI\!I), \quad z = \exp(2cI\!I), \quad w = \exp(2dI\!I).$$

Then the hypotheses of the theorem may be expressed as follows.

- Quadrilateral constraint: $\mathbf{k} + \mathbf{l} + \mathbf{m} + \mathbf{n} = 0$.
- Length constraints: $\mathbf{k} = k\,\bar{\mathbf{k}}, \quad \mathbf{l} = l\,\bar{\mathbf{l}}, \quad \mathbf{m} = m\,\bar{\mathbf{m}}, \quad \mathbf{n} = n\,\bar{\mathbf{n}}$.
- Angle constraints: $\bar{\mathbf{n}}\,\bar{\mathbf{k}} = -x, \quad \bar{\mathbf{k}}\,\bar{\mathbf{l}} = -y, \quad \bar{\mathbf{l}}\,\bar{\mathbf{m}} = -z, \quad \bar{\mathbf{m}}\,\bar{\mathbf{n}} = -w$.
- Unit magnitude constraints: $\bar{\mathbf{k}}\,\bar{\mathbf{k}} = 1, \quad \bar{\mathbf{l}}\,\bar{\mathbf{l}} = 1, \quad \bar{\mathbf{m}}\,\bar{\mathbf{m}} = 1, \quad \bar{\mathbf{n}}\,\bar{\mathbf{n}} = 1$.
- Inequality constraints: $\bar{\mathbf{k}}, \bar{\mathbf{l}}, \bar{\mathbf{m}}, \bar{\mathbf{n}} \neq 0, \quad x, y, z, w \neq 0$.
- Area constraint: As $S = \Delta_{ADC} + \Delta_{ACB}$, we have

$$SI\!I = -\frac{1}{2}(\mathbf{m} \wedge \mathbf{n} + \mathbf{k} \wedge \mathbf{l}) = \frac{1}{4}(\mathbf{n}\,\mathbf{m} - \mathbf{m}\,\mathbf{n} + \mathbf{l}\,\mathbf{k} - \mathbf{k}\,\mathbf{l}).$$

Let $s = 4SI\!I$; then $s = \mathbf{n}\,\mathbf{m} - \mathbf{m}\,\mathbf{n} + \mathbf{l}\,\mathbf{k} - \mathbf{k}\,\mathbf{l}$.

- Trigonometric transformations: Let $\tan(x\mathbb{I}) = \mathbb{I}\,\tan(x)$ for any scalar x; then

$$\tan(x\mathbb{I}) = \frac{\exp(2x\mathbb{I}) - 1}{\exp(2x\mathbb{I}) + 1}.$$

It follows that

$$\tan(a\mathbb{I}) = \frac{x-1}{x+1}, \quad \tan(b\mathbb{I}) = \frac{y-1}{y+1}, \quad \tan(c\mathbb{I}) = \frac{z-1}{z+1}, \quad \tan(d\mathbb{I}) = \frac{w-1}{w+1}.$$

Let $R = 4(T^2/\beta - t^2/\alpha)$ and $r = R\mathbb{I}$. Then

$$\begin{aligned} r \;=\; & \frac{4\,T^2}{\dfrac{1}{\tan(a\mathbb{I})} + \dfrac{1}{\tan(b\mathbb{I})} + \dfrac{1}{\tan(c\mathbb{I})} + \dfrac{1}{\tan(d\mathbb{I})}} \\ & + \frac{4\,t^2}{\tan(a\mathbb{I}) + \tan(b\mathbb{I}) + \tan(c\mathbb{I}) + \tan(d\mathbb{I})}. \end{aligned}$$

So the conclusion to be proved has the form

$$g^\star = s - r = 0.$$

The proof of the theorem now proceeds in a way similar to that by Wu's method. The hypothesis-expressions are first triangularized and then used to reduce the conclusion-expression to 0. The triangulation and reduction process, which is described in [4, 5] and is called a *vectorial equations solving method*, is however different.

To make triangulation simple, we choose the following basic variables together with an ordering:

$$\begin{aligned} & k \prec n \prec x \prec \tan(a\mathbb{I}) \prec l \prec y \prec \tan(b\mathbb{I}) \prec m \prec z \prec \tan(c\mathbb{I}) \prec w \prec \tan(d\mathbb{I}) \\ & \quad \prec \bar{\mathbf{k}} \prec \bar{\mathbf{l}} \prec \bar{\mathbf{m}} \prec \bar{\mathbf{n}} \prec \mathbf{k} \prec \mathbf{l} \prec \mathbf{m} \prec \mathbf{n} \prec s. \end{aligned}$$

The triangulation process for the hypothesis-expressions then consists in solving vectorial equations. The computations are quite simple and can be done by hand. We omit the details of triangulation and produce the obtained result as follows:

$$\tan(aI\!I) = \frac{x-1}{x+1},$$
$$\tan(bI\!I) = \frac{y-1}{y+1},$$
$$m^2 = k^2 + l^2 + n^2 - kl(y + \frac{1}{y}) - kn(x + \frac{1}{x}) + ln(xy + \frac{1}{xy}),$$
$$z = (l - \frac{k}{y} + \frac{n}{xy})/m,$$
$$\tan(cI\!I) = (l - \frac{k}{y} + \frac{n}{xy} - m)/(l - \frac{k}{y} + \frac{n}{xy} + m),$$
$$w = (n - \frac{k}{x} + \frac{l}{xy})/m,$$
$$\tan(dI\!I) = (n - \frac{k}{x} + \frac{l}{xy} - m)/(n - \frac{k}{x} + \frac{l}{xy} + m),$$
$$\bar{\mathbf{l}} = -\frac{1}{y}\bar{\mathbf{k}},$$

$$\bar{\mathbf{m}} = (-k + \frac{l}{y} + n\,x)\bar{\mathbf{k}}/m,$$
$$\bar{\mathbf{n}} = -x\bar{\mathbf{k}},$$
$$\mathbf{k} = k\,\bar{\mathbf{k}},$$
$$\mathbf{l} = -\frac{l}{y}\bar{\mathbf{k}},$$
$$\mathbf{m} = (-k + \frac{l}{y} + n\,x)\bar{\mathbf{k}},$$
$$\mathbf{n} = -n\,x\bar{\mathbf{k}},$$
$$s = kn(x - \frac{1}{x}) + kl(y - \frac{1}{y}) - ln(xy - \frac{1}{xy}).$$

Finally we need to verify whether the conclusion-expression $g^\star$ can be reduced to 0 by the above triangularized sequence of expressions. For this verification, a computer algebra system is required to carry out the computation. The following Maple session shows how $g^\star$ is reduced to 0 by simple substitution and pseudo-division.

```
> r:=(k+l+m+n)^2/(1/tan(aI)+1/tan(bI)+1/tan(cI) +1/tan(dI))
> +(k-l+m-n)^2/(tan(aI)+tan(bI)+tan(cI)+tan(dI)):
> tan(aI):=(x-1)/(x+1):
> tan(bI):=(y-1)/(y+1):
> eqn(m):=m^2-(k^2+l^2+n^2-k*l*(y+1/y)- k*n*(x+1/x)
> +l*n*(x*y+1/(x*y))):
> tan(cI):=(l-k/y+n/(x*y)-m)/(l-k/y+n/(x*y)+m):
> tan(dI):=(n-k/x+l/(x*y)-m)/(n-k/x+l/(x*y)+m):
```

```
> s:=k*n*(x-1/x)+k*l*(y-1/y)-l*n*(x*y-1/(x*y)):
> g:=s-r:
> st:=time():g:=numer(g):time()-st;
                          .782
> nops(g);
                          855
> st:=time():prem(g,eqn(m),m);time()-st;
                          0
                          .094
```

Thus the theorem is proved to be true under some non-degeneracy conditions. The above computation was done with Maple V.5 on a Pentium Pro 200MHz with 64M memory.

5. A Machine Proof Using Complex Numbers

Assume that $ABCD$ is not a parallelogram; otherwise, the statement is trivial. So, we may let lines AD and BC intersect at E. The area S may be expressed in three forms:

$$S = \Delta_{DAB} + \Delta_{BCD}, \quad S = \Delta_{ABC} + \Delta_{CDA}, \quad S = \Delta_{ABE} - \Delta_{DCE},$$

which are equivalent to the following equations:

$$\begin{aligned} f_1 = h_1 &= 2S - k\,n\sin 2a - l\,m\sin 2c = 0, \\ f_2 = h_2 &= 2S - k\,l\sin 2b - m\,n\sin 2d = 0, \\ f_3 &= 2S - \frac{k^2}{\cot\,2a + \cot\,2b} - \frac{m^2}{\cot\,2c + \cot\,2d} = 0. \end{aligned}$$

In order to reduce the number of variables and the computational complexity, we denote $e^{2a\,i}$, $e^{2b\,i}$, $e^{2c\,i}$, $e^{2d\,i}$ by u_1, u_2, u_3, u_4 respectively. We thus have:

$$\begin{aligned} \cos 2a &= \frac{1}{2}(u_1 + \frac{1}{u_1}), & \sin 2a &= -\frac{i}{2}(u_1 - \frac{1}{u_1}), \\ \cos 2b &= \frac{1}{2}(u_2 + \frac{1}{u_2}), & \sin 2b &= -\frac{i}{2}(u_2 - \frac{1}{u_2}), \\ \cos 2c &= \frac{1}{2}(u_3 + \frac{1}{u_3}), & \sin 2c &= -\frac{i}{2}(u_3 - \frac{1}{u_3}), \\ \cos 2d &= \frac{1}{2}(u_4 + \frac{1}{u_4}), & \sin 2d &= -\frac{i}{2}(u_4 - \frac{1}{u_4}), \\ \cot a &= i\,\frac{u_1 + 1}{u_1 - 1}, & \tan a &= -i\,\frac{u_1 - 1}{u_1 + 1}, \\ \cot b &= i\,\frac{u_2 + 1}{u_2 - 1}, & \tan b &= -i\,\frac{u_2 - 1}{u_2 + 1}, \\ \cot c &= i\,\frac{u_3 + 1}{u_3 - 1}, & \tan c &= -i\,\frac{u_3 - 1}{u_3 + 1}, \\ \cot d &= i\,\frac{u_4 + 1}{u_4 - 1}, & \tan d &= -i\,\frac{u_4 - 1}{u_4 + 1}. \end{aligned}$$

Note that u_1, u_2, u_3, u_4 are not all independent; in fact, $u_1u_2u_3u_4 = 1$ because $2a+2b+2c+2d = 2\pi$. So,

$$u_4 = \frac{1}{u_1u_2u_3}.$$

For brevity and without loss of generality, we may let

$$k = 1, \quad S = i\,F.$$

Substituting all the above equalities into the polynomials f_3, f_2, f_1 and removing all the extra/trivial factors, respectively, we obtain

$$\begin{aligned}
\bar{h}_1 &= m^2(u_1^2u_2^2u_3^4 - u_1^2u_2^2u_3^2 - u_3^2 + 1) + (u_1^2u_2^2 - u_1^2 - u_2^2 + 1)u_3^2 \\
&\quad + 4F(u_1^2u_2^2 - 1)u_3^2, \\
\bar{h}_2 &= -lu_1(u_2^2 - 1)u_3 + mn(u_1^2u_2^2u_3^2 - 1) - 4Fu_1u_2u_3, \\
\bar{h}_3 &= lmu_1(u_3^2 - 1) + n(u_1^2 - 1)u_3 + 4Fu_1u_3.
\end{aligned}$$

Doing the same substitution and simplification for the conclusion-polynomial

$$S - \frac{T^2}{\beta} + \frac{t^2}{\alpha},$$

we have

$$\begin{aligned}
\bar{g} &= 1 - 2\,(1+m)(l+n)p_1 + p_2 + p_3 - 2\,(1+m)(l+n)(2+p_2)u_1u_2u_3 \\
&\quad + (l^2 + m^2 + n^2 + 2\,ln + 2\,m)(1 + p_2 + p_3) - 4F(1 - p_2 + p_3 - 2\,u_1^2u_2^2u_3^2),
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= u_3 + u_2 + u_1, \\
p_2 &= u_2u_3 + u_1u_3 + u_1u_2, \\
p_3 &= u_1^2u_2^2u_3^2 + u_1u_2u_3^2 + u_1u_2^2u_3 + u_1^2u_2u_3.
\end{aligned}$$

When expanded, $\bar{g}$ is a polynomial of 84 terms in $l, m, n, u_1, u_2, u_3, F$.

Now, what we need is only to verify whether the following holds

$$\bar{h}_1 = 0, \bar{h}_2 = 0, \bar{h}_3 = 0 \implies \bar{g} = 0,$$

where u_1, u_2, u_3, F are considered as independent parameters and l, m, n as dependent variables. This may be done by computing first the pseudo-remainder $\tilde{h}_2$ of $\bar{h}_2$ with respect to $\bar{h}_3$ in l and then the pseudo-remainders of $\bar{g}$ with respect to $\bar{h}_3$ in l, $\tilde{h}_2$ in n and $\bar{h}_1$ in m successively. The final pseudo-remainder was found to be 0, so the theorem is proved. With the pseudo-division function in Maple V.3, the computation of the pseudo-remainders took about 3 seconds on a Pentium 200MHz, while the whole program, including substitution, simplification and pseudo-division, may be run in less than 6 seconds of CPU time.

A method for proving geometric theorems involving trigonometric identities similar to the one used in the above proof may be found in [3].

Remark. Finding machine proofs for the other four Russian killers is also of fun and interest. In fact, D. Wang has found a simple proof (in less than one second of computing time) for the fifth killer, and L. Yang has given a machine proof for the third killer. The details will be reported elsewhere.

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