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Hyperbolic Conformal Geometry with Clifford Algebra[1)]

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Abstract.

In this paper we study hyperbolic conformal geometry following a Clifford algebraic approach. Similar to embedding an affine space into a one-dimensional higher vector space, we embed the hyperboloid model of the hyperbolic n-space in $\mathcal{R}^{n,1}$ into $\mathcal{R}^{n+1,1}$. The new model is convenient for the study of hyperbolic conformal properties. Besides investigating various properties of the model, we also study conformal transformations using their versor representations.

Key words: Clifford algebra, hyperbolic geometry, conformal transformations.

1. Introduction

Hyperbolic geometry is an important branch of mathematics and physics. Among various models of the hyperbolic n-space, the hyperboloid model, which identifies the space with a branch $\mathcal{H}^n$ of the set

$$\mathcal{D}^n = \{x \in \mathcal{R}^{n,1} | x \cdot x = -1\}, \tag{1..1}$$

has the following features:

- The model is isotropic in that at every point of the model the metric of the tangent space is the same.
- A hyperbolic line AB is the intersection of $\mathcal{H}^n$ with the plane decided by vectors A, B through the origin. When viewed from the origin, line AB can be identified with a projective line in $\mathcal{P}^n$. Similarly, a hyperbolic r-plane can be identified with a projective r-plane in $\mathcal{P}^n$. Here $0 \leq r \leq n-1$.

  The geometry of r-planes can therefore be studied within the framework of linear subspaces in $\mathcal{R}^{n,1}$.
- The tangent direction of a line l at a point A is a vector orthogonal to vector A in the plane decided by l and the origin. The angle of two intersecting lines is the Euclidean angle of their tangent directions at the intersection.

  This is the conformal property of the model.
- Let A, B be two points, and let $d(A, B)$ be their hyperbolic distance. Then $A \cdot B = -\cosh d(A, B)$.

  This helps to transform a geometric problem on distances to an algebraic problem on inner products.

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- A generalized circle is either a hyperbolic circle, or a horocycle, or a hypercycle (equidistant curve). A generalized circle is the intersection of $\mathcal{H}^n$ with an affine plane in $\mathcal{R}^{n,1}$. Similarly, a generalized r-sphere is the intersection of $\mathcal{H}^n$ with an affine $(r+1)$-plane.

  The geometry of generalized r-spheres can be studied within the framework of affine subspaces in $\mathcal{R}^{n,1}$.

- The hyperbolic isometries are orthogonal transformations of $\mathcal{R}^{n,1}$ keeping $\mathcal{H}^n$ invariant. In particular, they are all linear transformations.

- The model is similar to that of the spherical n-space in $\mathcal{R}^{n+1}$.

These features imply that we can use Clifford algebra, in particular the version formulated by Hestenes and Sobczyk (1984), to study the hyperboloid model. A study on two dimensional case is carried out in (Li, 1997).

Similar to the study of affine n-space within the framework of linear subspaces of an $(n+1)$-dimensional vector space, where the affine space is embedded as a hyperplane, to study hyperbolic conformal properties where the basic geometric objects are generalized spheres and planes, we can embed the hyperboloid model into the null cone of the space $\mathcal{R}^{n+1,1}$. The new model is called the homogeneous model (Li, Hestenes and Rockwood, 1999c), which will help to simplify the study of hyperbolic conformal geometry significantly.

Moreover, the homogeneous model provides a universal algebraic model for three geometries: Euclidean, spherical and hyperbolic ones. Different geometries correspond to different geometric interpretations of the same algebraic representations. The algebraic manipulation for the transfer from one geometry to another is just a rescaling of null vectors (Li, Hestenes and Rockwood, 1999c).

In this paper we first review the definition of the hyperboloid model, then study various properties of the homogeneous model. We also study conformal transformations using their versor representations. We use the terminology in hyperbolic geometry from (Iversen, 1992), (Ratcliffe, 1994) and (Li, 1997), and the terminology and notations in Clifford algebra from (Hestenes and Sobczyk, 1984).

2. The hyperboloid model

For hyperbolic conformal geometry, the space $\mathcal{H}^n$ is not big enough, and we need its double covering space $\mathcal{D}^n$ defined in (**??**), called the double-hyperbolic n-space, or the hyperboloid model of the double-hyperbolic n-space. It has two branches, denoted by $\mathcal{H}^n$ and $-\mathcal{H}^n$ respectively.

2.1. Oriented generalized points, planes and spheres at infinity

Definition 2.1. An oriented generalized point of $\mathcal{D}^n$ is either a point, or an end, or a direction. A point is an element in $\mathcal{D}^n$. An end (or oriented point at infinity) is a null half 1-space of $\mathcal{R}^{n,1}$. A direction (or oriented imaginary point) is a Euclidean half 1-space of $\mathcal{R}^{n,1}$. A point at infinity is a null 1-space of $\mathcal{R}^{n,1}$.

Algebraically, any oriented generalized point can be represented by a vector in $\mathcal{R}^{n,1}$; two vectors represent the same oriented generalized point if and only if they differ by a positive scale. A point at infinity is represented by a null vector, and two null vectors represent the same point at infinity if and only if they differ by a nonzero scale.

Lemma 2.1. Let c be a point at infinity, p, q be points. Then p, q are on the same branch of $\mathcal{D}^n$ if and only if $c \cdot p\, c \cdot q > 0$.

Definition 2.2. Let c be an end, $p \in \mathcal{H}^n$. If $c \cdot p < 0$, c is called an end of the branch $\mathcal{H}^n$; otherwise it is called an end of the branch $-\mathcal{H}^n$.

Definition 2.3. An r-plane of $\mathcal{D}^n$ is the intersection of $\mathcal{D}^n$ with an $(r+1)$-space of $\mathcal{R}^{n,1}$.

In $\mathcal{G}_{n,1}$, an r-plane is represented by an $(r+1)$-blade corresponding to the $(r+1)$-space. When $r = 0$, an 0-plane is a pair of antipodal points; when $r = n-1$, an $(n-1)$-plane is called a hyperplane.

Definition 2.4. The sphere at infinity of $\mathcal{D}^n$ is the whole set of points at infinity. An r-sphere at infinity of $\mathcal{D}^n$ is the intersection of the sphere at infinity with an $(r+1)$-plane of $\mathcal{D}^n$, also called the sphere at infinity of the $(r+1)$-plane.

When $r = 0$, a 0-sphere at infinity is a pair of points at infinity.

2.2. Generalized spheres and total spheres

Definition 2.5. A generalized sphere is either a sphere, or a horosphere, or a hypersphere. It is decided by a pair (c, ρ), where c is a vector in $\mathcal{R}^{n,1}$ representing an oriented generalized point, called the center of the generalized sphere, and $\rho > 0$ is the generalized radius.

1. When c is a point, the set $\{p \in \mathcal{D}^n | p \cdot c = -(1+\rho)\}$ is called a sphere. It is on the same branch with point c.
2. When c is an end, the set $\{p \in \mathcal{D}^n | p \cdot c = -\rho\}$ is called a horosphere. It is on the same branch with the end c.
3. When c is a direction, the set $\{p \in \mathcal{D}^n | p \cdot c = -\rho\}$ is called a hypersphere. The hyperplane of $\mathcal{D}^n$ orthogonal to c is called the axis of the hypersphere.

Definition 2.6. A generalized r-sphere is a generalized sphere in an $(r+1)$-plane taken as a double-hyperbolic $(r+1)$-space.

When $r = 0$, a 0-sphere is a pair of points on the same branch of $\mathcal{D}^n$; a 0-horosphere is a point and a point at infinity; a 0-hypersphere is a pair of non-antipodal points on different branches of $\mathcal{D}^n$.

Definition 2.7. A total sphere of $\mathcal{D}^n$ refers to a generalized sphere, or a hyperplane, or the sphere at infinity. A total r-sphere is an r-dimensional generalized sphere, plane, or sphere at infinity.

3. The homogeneous model

Let a_0 be a fixed vector in $\mathcal{R}^{n+1,1}$, $a_0^2 = 1$. The space represented by $a_0^{\sim}$ is Minkowskii, denoted by $\mathcal{R}^{n,1}$. The mapping

$$x \mapsto x - a_0, \text{ for } x \in \mathcal{D}^n, \tag{3..2}$$

maps $\mathcal{D}^n$ in a one-to-one manner onto the set

$$\mathcal{N}^n_{a_0} = \{x \in \mathcal{R}^{n+1,1} | x \cdot x = 0, x \cdot a_0 = -1\}. \tag{3..3}$$

Its inverse is $P_{a_0^\sim} = P^\perp_{a_0}$. The set $\mathcal{N}^n_{a_0}$, together with the mapping $P_{a_0^\sim}$, is called the homogeneous model of the double-hyperbolic n-space.

In this model, an end or point at infinity is represented by a null vector orthogonal to a_0; a direction is represented by a vector of unit square orthogonal to a_0.

3.1. Representations of total spheres

Lemma 3.1. Let p, q be null vectors representing two points on the same branch of $\mathcal{D}^n$. Let $d(p, q)$ be the hyperbolic distance between the two points. Then $p \cdot q = 1 - \cosh d(p, q)$.

Proposition 3.2. A point p is on the sphere (c, ρ), when p, c are understood to be null vectors representing the points, if and only if $p \cdot c = -\rho$.

Proposition 3.3. A point p is on the horosphere (or hypersphere) (c, ρ), where p is understood to be the null vector representing the point, if and only if $p \cdot c = -\rho$.

The following is the first main theorem on the homogeneous model.

Theorem 3.4. Let $B_{r-1,1}$ be a Minkowskii r-blade in $\mathcal{G}_{n+1,1}$, $2 \le r \le n+1$. Then $B_{r-1,1}$ represents a total $(r-2)$-sphere: a point represented by a null vector p is on the total $(r-2)$-sphere if and only if $p \wedge B_{r-1,1} = 0$. Furthermore, if

1. $a_0 \cdot B_{r-1,1} = 0$, then $B_{r-1,1}$ represents an $(r-2)$-sphere at infinity.
2. $a_0 \cdot B_{r-1,1}$ is Euclidean, then $B_{r-1,1}$ represents an $(r-2)$-sphere.
3. $a_0 \cdot B_{r-1,1}$ is degenerate, then $B_{r-1,1}$ represents an $(r-2)$-horosphere.
4. $a_0 \cdot B_{r-1,1}$ is Minkowskii, but $a_0 \wedge B_{r-1,1} \neq 0$, then $B_{r-1,1}$ represents an $(r-2)$-hypersphere.
5. $a_0 \wedge B_{r-1,1} = 0$, then $B_{r-1,1}$ represents an $(r-2)$-plane.

When $r = n+1$, the dual form of the above theorem is

Theorem 3.5. Let s be a vector of positive signature in $\mathcal{R}^{n+1,1}$, then $s^\sim$ represents a total sphere. If

1. $a_0 \wedge s = 0$, then $s^\sim$ represents the sphere at infinity. The sphere at infinity is represented by $a_0^\sim$.
2. $a_0 \wedge s$ is Minkowskii, then $s^\sim$ represents a sphere. The sphere (c, ρ) is represented by $(c - \rho a_0)^\sim$, where c is a null vector representing the point.
3. $a_0 \wedge s$ is degenerate, then $s^\sim$ represents a horosphere. The horosphere (c, ρ) is represented by $(c - \rho a_0)^\sim$.
4. $a_0 \wedge s$ is Euclidean, but $a_0 \cdot s \neq 0$, then $s^\sim$ represents a hypersphere. The hypersphere (c, ρ) is represented by $(c - \rho a_0)^\sim$.

5. $a_0 \cdot s = 0$, then $s^\sim$ represents a hyperplane. A hyperplane with normal direction c is represented by $c^\sim$.

Proof. Without loss of generality we prove Theorem **??** only. If

1. $s \wedge a_0 = 0$, then any null vector in the space $s^\sim$ represents a point at infinity, and vice versa.

2. $s \wedge a_0$ is Minkowskii, then $s \cdot a_0 \neq 0$. Let ϵ be the sign of $s \cdot a_0$. Let

$$\mathbf{c} = -\epsilon \frac{P_{a_0}^{\perp}(s)}{|P_{a_0}^{\perp}(s)|}, \quad \rho = \frac{|a_0 \cdot s|}{|a_0 \wedge s|} - 1. \tag{3..4}$$

Then $\mathbf{c}$ is a point, $\rho > 0$, as $|a_0 \wedge s|^2 = (a_0 \cdot s)^2 - s^2 < (a_0 \cdot s)^2$.

Let $s' = -\epsilon s / |a_0 \wedge s|$. Then

$$s' = \mathbf{c} - (1+\rho) a_0 = c - \rho a_0, \tag{3..5}$$

where $c = \mathbf{c} - a_0$. A point represented by a null vector p is on the sphere $(\mathbf{c}, \rho)$ if and only if $p \cdot s' = 0$, which is equivalent to $p \wedge s^\sim = 0$.

3. $s \wedge a_0$ is degenerate, then $|s \cdot a_0| = |s| \neq 0$. Let ϵ be the sign of $s \cdot a_0$. Let

$$c = -\epsilon P_{a_0}^{\perp}(s), \quad \rho = |a_0 \cdot s| = |s|. \tag{3..6}$$

then c is an end, $\rho > 0$.

Let $s' = -\epsilon s$. Then

$$s' = c - \rho a_0. \tag{3..7}$$

A point represented by a null vector p is on the horosphere (c, ρ) if and only if $p \wedge s^\sim = 0$.

4. $s \wedge a_0$ is Euclidean, but $s \cdot a_0 \neq 0$, let ϵ be the sign of $s \cdot a_0$. Let

$$c = -\epsilon \frac{P_{a_0}^{\perp}(s)}{|P_{a_0}^{\perp}(s)|}, \quad \rho = \frac{|a_0 \cdot s|}{|a_0 \wedge s|}. \tag{3..8}$$

Then c is a direction, $\rho > 0$.

Let $s' = -\epsilon s / |a_0 \wedge s|$. Then

$$s' = c - \rho a_0. \tag{3..9}$$

A point represented by a null vector p is on the hypersphere (c, ρ) if and only if $p \wedge s^\sim = 0$.

5. $s \cdot a_0 = 0$, a point represented by a null vector p is on the hyperplane normal to s if and only if $p \wedge s^\sim = 0$.

□

3.2. Relations between two total spheres

Definition 3.1. Two hyperplanes are said to be parallel if their spheres at infinity have one and only one common point at infinity. They are said to be ultra-parallel if they have a unique common perpendicular line.

Definition 3.2. Two spheres (or horospheres, or hyperspheres) are said to be concentric if their centers are collinear. Two hyperspheres are said to be tangent at infinity if they do not intersect and their axes are parallel.

Theorem 3.6. Let $s_1^\sim, s_2^\sim$ represent two distinct total spheres other than the sphere at infinity of $\mathcal{D}^n$. Then if

1. $a_0 \wedge s_1 \wedge s_2 = 0$, they are concentric spheres, horospheres, or hyperspheres if and only if $a_0 \cdot (s_1 \wedge s_2)$ is negative-signatured, null, or positive-signatured respectively.
2. $s_1 \wedge s_2$ is Euclidean, $a_0 \wedge s_1 \wedge s_2 \neq 0$ and not both total spheres are hyperplanes, they intersect and the intersection is the generalized $(n-2)$-sphere $(s_1 \wedge s_2)^\sim$.
3. $s_1 \wedge s_2$ is degenerate, $a_0 \wedge s_1 \wedge s_2 \neq 0$ and not both total spheres are hyperplanes, they are tangent to each other at the point or point at infinity corresponding to null vector $P_{s_1}(s_2)$.
4. $s_1 \wedge s_2$ is Minkowskii, $a_0 \wedge s_1 \wedge s_2 \neq 0$ and not both total spheres are hyperplanes, they do not intersect. There is a unique pair of points or a point and a point at infinity that are inversive with respect to both total spheres.
5. both are hyperplanes, they intersect, are parallel or ultra-parallel if and only if $s_1 \wedge s_2$ is Euclidean, degenerate or Minkowskii respectively.

Proof. The intersection of the two spaces $s_1^\sim, s_2^\sim$ corresponds to the blade $s_1^\sim \vee s_2^\sim = (s_1 \wedge s_2)^\sim$, which is Minkowskii, degenerate or Euclidean, if and only if its null 1-spaces correspond to a total $(n-2)$-sphere, a point or point at infinity, or no point and point at infinity, respectively, i. e., if and only if the total spheres intersect, are tangent or do not intersect. In the last case $s_1 \wedge s_2$ represents a generalized 0-sphere and is invariant by the reflection with respect to either s_1 or s_2. Geometrically, the generalized 0-sphere is a pair of points or a point and a point at infinity inversive with respect to both total spheres. □

More specific conclusions can be established for the intersections and tangencies of various pairs of total spheres. Below we present a theorem on two hyperspheres.

Lemma 3.7. Let $c_1^\sim, c_2^\sim$ be two non-intersecting hyperplanes. Let A_1, A_2 be points on the same branch of $\mathcal{D}^n$ such that A_1 is on $c_1^\sim$ and A_2 is on $c_2^\sim$. Then $C = c_1 \cdot c_2\, A_1 \cdot c_2\, A_2 \cdot c_1 < 0$.

Proof. Since the hyperplanes do not intersect, $c_1 \cdot c_2 \neq 0$, and $A_1 \cdot c_2 \neq 0$ for any point A_1 on hyperplane $c_1^\sim$. Similarly, $A_2 \cdot c_1 \neq 0$ for any point A_2 on hyperplane $c_2^\sim$. So $C \neq 0$ and its sign depends on c_1, c_2 only.

When the hyperplanes are ultra-parallel, let A_1A_2 be the common perpendicular line, A_1, A_2 be the corresponding feet on the two hyperplanes on the same branch of $\mathcal{D}^n$. Then

$$A_1 = \pm\frac{c_1(c_1 \wedge c_2)}{|c_1 \wedge c_2|}, \quad A_2 = \pm\frac{c_1 \cdot c_2}{|c_1 \cdot c_2|}\frac{c_2(c_1 \wedge c_2)}{|c_1 \wedge c_2|}. \tag{3..10}$$

So

$$C = -|c_1 \cdot c_2||c_1 \wedge c_2| < 0. \tag{3..11}$$

When they are parallel, since C is a continuous function of its variables, $C \leq 0$. As $C \neq 0$, we get $C < 0$.

□

Definition 3.3. Two hyperspheres are said to be same-sided if their axes are either identical or do not intersect, and for any line intersecting both hyperspheres and both axes at points S_1, S_2, A_1, A_2 on the same branch of $\mathcal{D}^n$, the order of S_1, A_1 is the same with the order of S_2, A_2 on the line.

Proposition 3.8. Two hyperspheres (c_1, ρ_1) and (c_2, ρ_2) are same-sided if and only if $c_1 \cdot c_2 \geq 1$.

Proof. Let there be a line intersecting both hyperspheres and both axes at points S_1, S_2, A_1, A_2 on a branch of $\mathcal{D}^n$. Then

$$S_i \cdot c_i = -\rho_i,\ A_i \cdot c_i = 0,\ S_i \wedge A_1 \wedge A_2 = 0,\ \text{for } i = 1, 2. \tag{3..12}$$

If $A_1 \neq A_2$, then $S_1 \wedge A_1 = \lambda A_1 \wedge A_2$. Making inner product on both sides of the equality with C_1, we get $\lambda = -\rho_1/A_2 \cdot c_1$. Similarly, we have $S_2 \wedge A_2 = \mu A_1 \wedge A_2$ where $\mu = -\rho_2/A_1 \cdot c_2$. The pair S_1, A_1 have the same order with the pair S_2, A_2 on the line if and only if $S_1 \wedge A_1$ and $S_2 \wedge A_2$ have the same orientation, which is equivalent to

$$\lambda\mu = -\frac{\rho_1\rho_2}{A_2 \cdot c_1\, A_1 \cdot c_2} > 0. \tag{3..13}$$

If the axes do not intersect, then $(c_1 \wedge c_2)^2 = (c_1 \cdot c_2)^2 - 1 \geq 0$. $A_1 \neq A_2$ is always true, and the conclusion follows from Lemma **??**. If the axes intersect, then $|c_1 \cdot c_2| < 1$. If the axes are identical, then $A_1 = A_2 = A$, $c_1 = \epsilon c_2$ where $\epsilon = \pm 1$. Let a be a tangent direction of the line at point A, then $a \cdot A = 0$. We have

$$S_1 \wedge A = -\frac{\rho_1}{\epsilon a \cdot c_2}A_1 \wedge A_2, \quad S_2 \wedge A = -\frac{\rho_2}{a \cdot c_2}A_1 \wedge A_2. \tag{3..14}$$

$S_1 \wedge A_1$ and $S_2 \wedge A_2$ have the same orientation if and only if $\epsilon = 1$.

□

Theorem 3.9. Let $s_1^\sim, s_2^\sim$ represent two distinct hyperspheres. Then if they

1. intersect, the intersection is an $(n-2)$-dimensional
   (a) sphere, if their axes are ultra-parallel;
   (b) horosphere, if their axes are parallel;

(c) hypersphere, if their axes intersect.

The center and radius of the intersection are the same with those of the generalized sphere $(P_{s_1\wedge s_2}(a_0))^{\sim}$.

2. are tangent, their axes are parallel or ultra-parallel. The tangency occurs at infinity if and only if the hyperspheres have parallel axes and equal radii, and are same-sided.

3. neither intersect nor are tangent, their axes are ultra-parallel.

Proof. Let $s_i = c_i - \rho_i a_0$ for $i = 1, 2$, where $c_i \cdot a_0 = 0$, $c_i^2 = 1$ and $\rho_i > 0$. When the hyperplanes

1. intersect, then $(s_1 \wedge s_2)^2 < 0$. The Minkowskii blade $P_{s_1\wedge s_2}(a_0)^{\sim}$ represents a generalized sphere, since neither $a_0 \cdot P_{s_1\wedge s_2}(a_0)^{\sim}$ nor $a_0 \wedge P_{s_1\wedge s_2}(a_0)^{\sim}$ equals zero. Using the formula
$$(s_1 \wedge s_2)^{\sim} = \frac{(s_1 \wedge s_2)^2}{(a_0 \cdot (s_1 \wedge s_2))^2}(a_0 \cdot (s_1 \wedge s_2))P_{s_1\wedge s_2}(a_0)^{\sim}, \tag{3..15}$$
we get that the total $(n-2)$-sphere $(s_1 \wedge s_2)^{\sim}$ is the intersection of the hyperplane $(a_0 \cdot (s_1 \wedge s_2))^{\sim}$ with the generalized sphere $P_{s_1\wedge s_2}(a_0)^{\sim}$. Since
$$P_{a_0}^{\perp}(P_{s_1\wedge s_2}(a_0)) \cdot (a_0 \cdot (s_1 \wedge s_2)) = P_{a_0}(P_{s_1\wedge s_2}(a_0)) \cdot (a_0 \cdot (s_1 \wedge s_2)) = 0, \tag{3..16}$$
the center of the generalized sphere is in the hyperplane. Therefore, $(s_1 \wedge s_2)^{\sim}$ is a generalized $(n-2)$-sphere whose center and radius are the same with those of $P_{s_1\wedge s_2}(a_0)^{\sim}$.

   The intersection is an $(n-2)$-dimensional sphere, horosphere or hypersphere if the blade $a_0 \cdot P_{s_1\wedge s_2}(a_0)^{\sim} = (a_0 \wedge P_{s_1\wedge s_2}(a_0))^{\sim}$ is Euclidean, degenerate or Minkowskii respectively. Using
$$(s_1 \wedge s_2)^4(a_0 \wedge P_{s_1\wedge s_2}(a_0))^2 = (c_1 \wedge c_2)^2(\rho_2 c_1 - \rho_1 c_2)^2 \tag{3..17}$$
and
$$(s_1 \wedge s_2)^2 = (c_1 \wedge c_2)^2 - (\rho_2 c_1 - \rho_1 c_2)^2, \tag{3..18}$$
we get that $(a_0 \wedge P_{s_1\wedge s_2}(a_0))^2$ has the same sign with $(c_1 \wedge c_2)^2$. From this we get all the conclusions on the intersection using Theorem **??**.

2. are tangent, then $(s_1 \wedge s_2)^2 = 0$. If the axes intersect, then $c_1 \wedge c_2$ is Euclidean, so $\rho_2 c_1 - \rho_1 c_2$ has positive signature. As a result, $(s_1 \wedge s_2)^2 < 0$ by (**??**), which is a contradiction.

   The tangency occurs at infinity if $(s_1 \wedge s_2)^2 = (c_1 \wedge c_2)^2 = 0$, which is equivalent to $c_1 \cdot c_2 = 1$ and $\rho_1 = \rho_2$.

3. neither intersect nor are tangent, then $(s_1 \wedge s_2)^2 > 0$. By (**??**), $(\rho_2 c_1 - \rho_1 c_2)^2 < (c_1 \wedge c_2)^2$. If $c_1 \wedge c_2$ is not Minkowskii, then $(c_1 \wedge c_2)^2 \leq 0$; on the other hand, $\rho_2 c_1 - \rho_1 c_2$ has non-negative square because it is a vector in $c_1 \wedge c_2$. As a result, $(\rho_2 c_1 - \rho_1 c_2)^2 \geq (c_1 \wedge c_2)^2$, which is a contradiction.

□

3.3. Bunches of total spheres

The main content in the previous subsection is a special case of the content in this subsection.

Definition 3.4. A bunch of total spheres decided by r total spheres represented by r Minkowskii $(n+1)$-blades $B_1, \ldots, B_r$, is the set of total spheres represented by $\lambda_1 B_1 + \ldots + \lambda_r B_r$, where the λ's are scalars.

When $B_1 \vee \cdots \vee B_r \neq 0$, the integer $r-1$ is called the dimension of the bunch. A one-dimensional bunch is also called a pencil. The dimension of a bunch is allowed to be between 1 and $n-1$. The blade $A_{n-r+2} = B_1 \vee \cdots \vee B_r$ can be used to represent the bunch. There are five classes:

1. When $a_0 \cdot A_{n-r+2} = 0$, the bunch is called a concentric bunch. It is composed of the sphere at infinity and the generalized spheres whose centers lie in the subspace $(a_0 \wedge A_{n-r+2})^{\sim}$ of $\mathcal{R}^{n,1}$.

2. When A_{n-r+2} is Minkowskii and $a_0 \cdot A_{n-r+2}, a_0 \wedge A_{n-r+2} \neq 0$, the bunch is called a concurrent bunch. It is composed of total spheres containing the generalized $(n-r)$-sphere A_{n-r+2}.

   In particular, when A_{n-r+2} represents an $(n-r)$-hypersphere, the bunch is composed of hyperspheres only. $a_0 \wedge (a_0 \cdot A_{n-r+2})$ represents the axis of the $(n-r)$-hypersphere, and is the intersection of all axes of the hyperspheres in the bunch.

3. When A_{n-r+2} is degenerate and $a_0 \cdot A_{n-r+2}, a_0 \wedge A_{n-r+2} \neq 0$, the bunch is called a tangent bunch. Any two non-intersecting total spheres in the bunch are tangent to each other. The tangency occurs at the point or point at infinity corresponding to the unique null 1-space in the space A_{n-r+2}.

4. When A_{n-r+2} is Euclidean and $a_0 \cdot A_{n-r+2}, a_0 \wedge A_{n-r+2} \neq 0$, the bunch is called a Poncelet bunch. $A^{\sim}_{n-r+2}$ represents a generalized $(r-2)$-sphere, called Poncelet sphere, which is self-inversive with respect to every total sphere in the bunch.

5. When $a_0 \wedge A_{n-r+2} = 0$, the bunch is called a hyperplane bunch. It is composed of hyperplanes (1) perpendicular to the $(r-1)$-plane represented by $a_0 \wedge A^{\sim}_{n-r+2}$, or (2) whose representations pass through the space A_{n-r+2}, or (3) passing through the $(n-r)$-plane represented by A_{n-r+2}, if the blade A_{n-r+2} is (1) Euclidean, or (2) degenerate, or (3) Minkowskii, respectively.

4. Conformal transformations

It is a well-known fact that the orthogonal group $O(n+1,1)$ is a double covering of the conformal group $M(n)$ of $\mathcal{D}^n$: $M(n) = O(n+1,1)/\{\pm 1\}$.

In Clifford algebra, $O(n+1,1)$ is isomorphic to the projective pin group $Pin(n+1,1)$, which is the quotient of the versor group of $\mathcal{R}^{n+1,1}$ by $\mathcal{R}-\{0\}$. The group $Pin(n+1,1)$ has four connected components:

$E_+(n+1,1)$, the set of versors which are geometric products of even number of positive-signatured vectors and even number of negative-signatured vectors. It is a subgroup of $Pin(n+1,1)$ which is isomorphic to the proper Lorentz group $Lor^+(n+1,1)$.

$E_-(n+1,1)$, the set of versors which are geometric products of odd number of positive-signatured vectors and odd number of negative-signatured vectors. $E_-(n+1,1)$ and $E_+(n+1,1)$ form a subgroup of $Pin(n+1,1)$ which is isomorphic to the special orthogonal group $SO(n+1,1)$.

$O_+(n+1,1)$, the set of versors which are geometric products of odd number of positive-signatured vectors and even number of negative-signatured vectors. $O_+(n+1,1)$ and $E_+(n+1,1)$ for a subgroup of $Pin(n+1,1)$ which is isomorphic to the Lorentz group $Lor(n+1,1)$.

$O_-(n+1,1)$, the set of versors which are geometric products of even number of positive-signatured vectors and odd number of negative-signatured vectors. $O_-(n+1,1)$ and $E_+(n+1,1)$ for a subgroup of $Pin(n+1,1)$ which is isomorphic to the skew Lorentz group $Lor^-(n+1,1)$.

Let $I_{n+1,1}$ be a unit pseudoscalar. Then the versor action of $I_{n+1,1}$ maps x to $-x$ for any x in $\mathcal{R}^{n+1,1}$. Therefore

$$M(n) = Pin(n+1,1)/\{I_{n+1,1}\}, \tag{4..19}$$

which serves as the second main theorem on the homogeneous model:

Theorem 4.1. Any conformal transformation in $\mathcal{D}^n$ can be realized through the versor action of a versor in $\mathcal{G}_{n+1,1}$, and vice versa. Any two versors realize the same conformal transformation if and only they are equal up to a nonzero scalar or pseudoscalar factor.

The pseudoscalar $I_{n+1,1}$ induces a duality in $\mathcal{G}_{n+1,1}$, under which $O_+(n+1,1)$ and $O_-(n+1,1)$ are interchanged when n is even, and $O_+(n+1,1)$ and $E_-(n+1,1)$ are interchanged when n is odd. Therefore

$$M(n+1,1) = \begin{cases} Lor(n+1,1) = Lor^-(n+1,1), & \text{when } n \text{ is even;} \\ Lor(n+1,1) = SO(n+1,1), & \text{when } n \text{ is odd.} \end{cases} \tag{4..20}$$

Now we use the versor representation to study a conformal transformation which is similar to dilation in Euclidean space. This is tidal transformation, whose corresponding versor is $1+\lambda a_0 c$, where $\lambda \in \mathcal{R}$, $c \in \mathcal{R}^{n+1,1}$ and $c \cdot a_0 = 0$.

Under this transformation, the concentric pencil $(a_0 \wedge c)^\sim$ is invariant. When c is a point or end, the set $\{c, -c\}$ is invariant; when c is a direction, the hyperplane $c^\sim$ is not invariant, but its sphere at infinity is.

Assume that p is a fixed point in $\mathcal{D}^n$, and is transformed to a point or point at infinity q. It can be proved that parameter λ is a function of q on line $c \wedge p$. Below we list some results on this function.

1. When c is a point,
   (a) for any point or point at infinity q on line $c \wedge p$, let $C_c(q) = -c^{-1}qc$, then $\lambda(-C_c(q)) = \dfrac{1}{\lambda(q)}$.

(b) for any point q on line $c \wedge p$, $\lambda(q) = \dfrac{(q-p)^2}{(q-c)^2 - (p-c)^2}$.

(c) $\lambda(p - e^{\pm d(p,c)}c) = e^{\pm d(p,c)}$.

2. When c is an end,

(a) for any point q on line $c \wedge p$, $\lambda(q) = \frac{1}{2}(\frac{1}{q \cdot c} - \frac{1}{p \cdot c})$.

(b) if q is the end of line $c \wedge p$ other than c, then $\lambda(q) = -\dfrac{1}{2p \cdot c}$.

3. When c is a direction,

(a) for any point or point at infinity q on line $c \wedge p$, $\lambda(-C_c(q)) = -\dfrac{1}{\lambda(q)}$.

(b) for any point q on line $c \wedge p$, $\lambda(q) = \dfrac{(q-p)^2}{(q-c)^2 - (p-c)^2}$.

(c) assume that $p \cdot c < 0$ and let $d(p,c)$ be the hyperbolic distance from p to the intersection t of line $c \wedge p$ with hyperplane $c^{\sim}$ on the branch of $\mathcal{D}^n$ containing p, then

$$\lambda(C_c(p)) = -\sinh d(p,c), \quad \lambda(t) = -\tanh \frac{d(p,c)}{2},$$
$$\lambda(p + e^{d(p,c)}c) = -e^{d(p,c)}, \quad \lambda(p + e^{-d(p,c)}c) = e^{-d(p,c)}.$$

(d) assume that $p \cdot c = 0$ and q is on the branch of $\mathcal{D}^n$ containing p, then $\lambda(q) = -\epsilon \tanh \dfrac{d(p,q)}{2}$, where ϵ is the sign of $q \cdot c$.

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