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Spherical Conformal Geometry with Geometric Algebra

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1. Introduction

The study of spheres dates back to the first century in the book *Sphaerica* of Menelaus. Spherical trigonometry was thoroughly developed in modern form by Euler in his 1782 paper [?]. Spherical geometry in n-dimensions was first studied by Schläfli in his 1852 treatise, which was published posthumously in 1901 [?]. The most important transformation in spherical geometry, Möbius transformation, was considered by Möbius in his 1855 paper [?].

The first person who studied spherical trigonometry with vectors was Hamilton [?]. In his 1987 book [?], Hestenes studied spherical trigonometry with *Geometric Algebra*, which laid down foundations for later study with Geometric Algebra.

This chapter is a continuation of the previous chapter. In this chapter, we consider the homogeneous model of the spherical space, which is similar to that of the Euclidean space. We establish conformal geometry of spherical space in this model, and discuss several typical conformal transformations.

Although it is well known that the conformal groups of n-dimensional Euclidean and spherical spaces are isometric to each other, and are all isometric to the group of isometries of hyperbolic $(n+1)$-space [?], [?], spherical conformal geometry on one hand has its unique conformal transformations, on the other hand can provide good understanding for hyperbolic conformal geometry. It is an indispensible part of the unification of all conformal geometries in the homogeneous model, which is going to be addressed in the next chapter.

2. Homogeneous model of spherical space

In the previous chapter, we have seen that given a null vector $e \in \mathcal{R}^{n+1,1}$, the intersection of the null cone $\mathcal{N}^n$ of $\mathcal{R}^{n+1,1}$ with the hyperplane $\{x \in \mathcal{R}^{n+1,1} | x \cdot e = -1\}$ represents points in $\mathcal{R}^n$. This representation is established through the projective split of the null cone with respect to null vector e.

What if we replace the null vector e with any nonzero vector in $\mathcal{R}^{n+1,1}$? This section shows you that when e is replaced by a vector p_0 of negative signature, then when assuming $p_0^2 = -1$ the set

$$\mathcal{N}^n_{p_0} = \{x \in \mathcal{N}^n | x \cdot p_0 = -1\} \tag{1}$$

represents points in $\mathcal{S}^n$, the n-dimensional spherical space:

$$\mathcal{S}^n = \{x \in \mathcal{R}^{n+1} | x^2 = 1\}. \tag{2}$$

The space dual to p_0 corresponds to $\mathcal{R}^{n+1} = \tilde{p}_0$, an $(n+1)$-dimensional Euclidean space whose unit sphere is just $\mathcal{S}^n$.

Applying the orthogonal decomposition

$$x = P_{p_0}(x) + P_{\widetilde{p_0}}(x) \tag{3}$$

to vector $x \in \mathcal{N}^n_{p_0}$, we get

$$x = p_0 + \mathbf{x} \tag{4}$$

where $\mathbf{x} \in \mathcal{S}^n$. This defines a map $i_{p_0} : \mathbf{x} \in \mathcal{S}^n \longrightarrow x \in \mathcal{N}^n_{p_0}$, which is one-to-one and onto. Its inverse map is $P^{\perp}_{p_0} = P_{\tilde{p}_0}$.

Theorem 1.

$$\mathcal{N}^n_{p_0} \simeq \mathcal{S}^n. \tag{5}$$

We call $\mathcal{N}^n_{p_0}$ the *homogeneous model* of $\mathcal{S}^n$. Its elements are called *homogeneous points.*

Distances

For two points $\mathbf{a}, \mathbf{b} \in \mathcal{S}^n$, their *spherical distance* $d(\mathbf{a}, \mathbf{b})$ is defined as a number in the range $[0, \pi]$ such that

$$\cos d(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}. \tag{6}$$

We can define several other distances that are equivalent to the spherical distance. Two distances d_1, d_2 are said to be equivalent if for any two pairs of points $\mathbf{a}_1, \mathbf{b}_1$ and $\mathbf{a}_2$, $\mathbf{b}_2$, $d_1(\mathbf{a}_1, \mathbf{b}_1) = d_1(\mathbf{a}_2, \mathbf{b}_2)$ if and only if $d_2(\mathbf{a}_1, \mathbf{b}_1) = d_2(\mathbf{a}_2, \mathbf{b}_2)$. The following distance is called *chord distance*, as it measures the length of the chord between $\mathbf{a}, \mathbf{b}$:

$$d_c(\mathbf{a}, \mathbf{b}) = |\mathbf{a} - \mathbf{b}|. \tag{7}$$

The following distance is called *normal distance*:

$$d_n(\mathbf{a}, \mathbf{b}) = 1 - \mathbf{a} \cdot \mathbf{b}. \tag{8}$$

It equals the distance between points $\mathbf{a}, \mathbf{b}'$, where $\mathbf{b}'$ is the projection of $\mathbf{b}$ onto $\mathbf{a}$. The following distance is called *stereographic distance*, as it measures the distance between the origin o and $\mathbf{a}'$, the intersection of of the line connecting $-\mathbf{a}$ and $\mathbf{b}$ with the hyperspace of $\mathcal{R}^{n+1}$ parallel with the tangent hyperplane of $\mathcal{S}^n$ at $\mathbf{a}$:

$$d_s(\mathbf{a}, \mathbf{b}) = \frac{|\mathbf{a} \wedge \mathbf{b}|}{1 + \mathbf{a} \cdot \mathbf{b}}. \tag{9}$$

Some relations among these distances are:

$$\begin{array}{rcl} d_c(\mathbf{a}, \mathbf{b}) & = & 2 \sin \dfrac{d(\mathbf{a}, \mathbf{b})}{2}, \\ d_n(\mathbf{a}, \mathbf{b}) & = & 1 - \cos d(\mathbf{a}, \mathbf{b}), \\ d_s(\mathbf{a}, \mathbf{b}) & = & \tan \dfrac{d(\mathbf{a}, \mathbf{b})}{2}, \\ d_s^2(\mathbf{a}, \mathbf{b}) & = & \dfrac{d_n(\mathbf{a}, \mathbf{b})}{2 - d_n(\mathbf{a}, \mathbf{b})}. \end{array} \tag{10}$$

For two points $\mathbf{a}, \mathbf{b}$ in $\mathcal{S}^n$, we have

$$a \cdot b = \mathbf{a} \cdot \mathbf{b} - 1 = -d_n(\mathbf{a}, \mathbf{b}). \tag{11}$$

Therefore the inner product of two homogeneous points "in" $\mathcal{S}^n$ characterizes the normal distance between two points.

Spheres and hyperplanes

A sphere in $\mathcal{S}^n$ is understood to be a set of points having equal distances with a fixed point in $\mathcal{S}^n$. A sphere is said to be *great*, or *unit*, if it has normal radius 1. In this chapter, we call a great sphere a *hyperplane*, or an $(n-1)$-*plane*, of $\mathcal{S}^n$, and a non-great one a *sphere*, or an $(n-1)$-*sphere*.

The intersection of a sphere with a hyperplane is an $(n-2)$-dimensional sphere, called $(n-2)$-*sphere*; the intersection of two hyperplanes is an $(n-2)$-dimensional plane, called $(n-2)$-*plane*. In general, for $1 \leq r \leq n-1$, the intersection of a hyperplane with an r-sphere is called an $(r-1)$-*sphere*; the intersection of a hyperplane with an r-plane is called an $(r-1)$-*plane*. A 0-plane is a pair of antipodal points, and a 0-sphere is a pair of non-antipodal ones.

We require that the normal radius of a sphere must be less than 1. In this way a sphere has only one center. The *interior* of a sphere is the region of $\mathcal{S}^n$ bordered by the sphere and containing the center of the sphere; the *exterior* of the sphere is the other region of $\mathcal{S}^n$. For a sphere with center $\mathbf{c}$ and normal radius ρ, its interior is

$$\{\mathbf{x} \in \mathcal{S}^n | d_n(\mathbf{x}, \mathbf{c}) < \rho\}; \tag{12}$$

its exterior is

$$\{\mathbf{x} \in \mathcal{S}^n | \rho < d_n(\mathbf{x}, \mathbf{c}) \leq 2\}. \tag{13}$$

A sphere with center $\mathbf{c}$ and normal radius ρ is characterized by the vector

$$s = c - \rho p_0 \tag{14}$$

of positive signature: a point $\mathbf{x}$ is on the sphere if and only if $x \cdot c = -\rho$, or equivalently,

$$x \wedge \tilde{s} = 0. \tag{15}$$

(**??**) is called the *standard form* of a sphere.

A hyperplane is characterized by its *normal vector* $\mathbf{n}$: a point $\mathbf{x}$ is on the hyperplane if and only if $\mathbf{x} \cdot \mathbf{n} = 0$, or equivalently,

$$x \wedge \tilde{\mathbf{n}} = 0. \tag{16}$$

Theorem 2. *The intersection of any Minkowski hyperspace$\tilde{s}$ with $\mathcal{N}^n_{p_0}$ is a sphere or hyperplane in $\mathcal{S}^n$, and every sphere or hyperplane of $\mathcal{S}^n$ can be obtained in this way. Vector s has the standard form*

$$s = \mathbf{c} + \lambda p_0, \tag{17}$$

where $0 \leq \lambda < 1$. It represents a hyperplane if and only if $\lambda = 0$.

The dual theorem is:

Theorem 3. *Given homogeneous points $a_0, \dots, a_n$ such that*

$$\tilde{s} = a_0 \wedge \cdots \wedge a_n, \tag{18}$$

then the $(n+1)$*-blade* $\tilde{s}$ *represents a sphere in* $\mathcal{S}^n$ *if*

$$p_0 \wedge \tilde{s} \neq 0, \tag{19}$$

a hyperplane if

$$p_0 \wedge \tilde{s} = 0. \tag{20}$$

The above two theorems also provide an approach to compute the center and radius of a sphere in $\mathcal{S}^n$: Let $\tilde{s} = a_0 \wedge \cdots \wedge a_n \neq 0$, then it represents the sphere or hyperplane passing through points $\mathbf{a}_0, \ldots, \mathbf{a}_n$. When it represents a sphere, let $(-1)^\epsilon$ be the sign of $s \cdot p_0$. Then the center of the sphere is

$$(-1)^{\epsilon+1} \frac{P^\perp_{p_0}(s)}{|P^\perp_{p_0}(s)|}, \tag{21}$$

the normal radius is

$$1 - \frac{|s \cdot p_0|}{|s \wedge p_0|}. \tag{22}$$

3. Relation between two spheres or hyperplanes

Let $\tilde{s}_1, \tilde{s}_2$ be two distinct spheres or hyperplanes in $\mathcal{S}^n$. The signature of the blade

$$s_1 \wedge s_2 = (\tilde{s}_1 \vee \tilde{s}_2)^\sim \tag{23}$$

characterizes the relation between the two spheres or hyperplanes:

Theorem 4. *Two spheres or hyperplanes* $\tilde{s}_1, \tilde{s}_2$ *intersect, are tangent, or do not intersect if and only if* $(s_1 \wedge s_2)^2 < 0, = 0, > 0$ *respectively.*

There are the following cases:
Case 1. For two hyperplanes represented by $\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2$, since $\mathbf{n}_2 \wedge \mathbf{n}_2$ has Euclidean signature, the two hyperplanes always intersect. The intersection is an $(n-2)$-plane, and is normal to both $P^\perp_{\mathbf{n}_1}(\mathbf{n}_2)$ and $P^\perp_{\mathbf{n}_2}(\mathbf{n}_1)$.
Case 2. For a hyperplane $\tilde{\mathbf{n}}$ and a sphere $(\mathbf{c} + \lambda p_0)^\sim$, since

$$(\mathbf{n} \wedge (\mathbf{c} + \lambda p_0))^2 = (\lambda + |\mathbf{c} \wedge \mathbf{n}|)(\lambda - |\mathbf{c} \wedge \mathbf{n}|), \tag{24}$$

- if $\lambda < |\mathbf{c} \wedge \mathbf{n}|$, they intersect. The intersection is an $(n-2)$-sphere with center $\dfrac{P^\perp_{\mathbf{n}}(\mathbf{c})}{|P^\perp_{\mathbf{n}}(\mathbf{c})|}$ and normal radius $1 - \dfrac{\lambda}{|\mathbf{c} \wedge \mathbf{n}|}$.
- if $\lambda = |\mathbf{c} \wedge \mathbf{n}|$, they are tangent at point $\dfrac{P^\perp_{\mathbf{n}}(\mathbf{c})}{|P^\perp_{\mathbf{n}}(\mathbf{c})|}$.
- if $\lambda > |\mathbf{c} \wedge \mathbf{n}|$, they do not intersect. There is a pair of points in $\mathcal{S}^n$ which are inversive with respect to the sphere, at the same time symmetric with respect to the hyperplane. They are $\dfrac{P^\perp_{\mathbf{n}}(\mathbf{c}) \pm \mu \mathbf{n}}{\lambda}$, where $\mu = \sqrt{\lambda^2 + (\mathbf{c} \wedge \mathbf{n})^2}$.

Case 3. For two spheres $(\mathbf{c}_i + \lambda_i p_0)^\sim$, $i = 1, 2$, since

$$((\mathbf{c}_1 + \lambda_1 p_0) \wedge (\mathbf{c}_2 + \lambda_2 p_0))^2 = (\mathbf{c}_1 \wedge \mathbf{c}_2)^2 + (\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2)^2, \tag{25}$$

- if $|\mathbf{c}_1 \wedge \mathbf{c}_2| > |\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2|$, they intersect. The intersection is an $(n-2)$-sphere on the hyperplane of $\mathcal{S}^n$ represented by

$$(\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2)^\sim. \tag{26}$$

  The intersection has center

$$\frac{\lambda_1 P^\perp_{\mathbf{c}_2}(\mathbf{c}_1) + \lambda_2 P^\perp_{\mathbf{c}_1}(\mathbf{c}_2)}{|\lambda_1 \mathbf{c}_2 - \lambda_2 \mathbf{c}_1||\mathbf{c}_1 \wedge \mathbf{c}_2|} \tag{27}$$

  and normal radius

$$1 - \frac{|\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2|}{|\mathbf{c}_1 \wedge \mathbf{c}_2|}. \tag{28}$$

- if $|\mathbf{c}_1 \wedge \mathbf{c}_2| = |\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2|$, they are tangent at point

$$\frac{\lambda_1 P^\perp_{\mathbf{c}_2}(\mathbf{c}_1) + \lambda_2 P^\perp_{\mathbf{c}_1}(\mathbf{c}_2)}{|\mathbf{c}_1 \wedge \mathbf{c}_2|^2}. \tag{29}$$

- if $|\mathbf{c}_1 \wedge \mathbf{c}_2| < |\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2|$, they do not intersect. There is a pair of points in $\mathcal{S}^n$ which are inversive with respect to both spheres. They are

$$\frac{\lambda_1 P^\perp_{\mathbf{c}_2}(\mathbf{c}_1) + \lambda_2 P^\perp_{\mathbf{c}_1}(\mathbf{c}_2) \pm \mu(\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2)}{(\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2)^2}, \tag{30}$$

  where $\mu = \sqrt{(\mathbf{c}_1 \wedge \mathbf{c}_2)^2 + (\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2)^2}$. The two points are called the *Poncelet points* of the spheres.

The scalar

$$s_1 * s_2 = \frac{s_1 \cdot s_2}{|s_1||s_2|} \tag{31}$$

is called the *inversive product* of vectors s_1 and s_2. Obviously, it is invariant under orthogonal transformations in $\mathcal{R}^{n+1,1}$. We have the following conclusion for the geometric interpretation of the inversive product:

Theorem 5. *When the two spheres or hyperplanes $\tilde{s}_1$ and $\tilde{s}_2$ intersect, let $\mathbf{a}$ be a point at the intersection, m_i, $i = 1, 2$, be the respective outward unit normal vector at $\mathbf{a}$ of $\tilde{s}_i$ if it is a sphere, or $s_i/|s_i|$ if it is a hyperplane, then*

$$s_1 * s_2 = m_1 \cdot m_2. \tag{32}$$

Proof. s_i has the standard form $\mathbf{c}_i + \lambda_i p_0$. When $\tilde{s}_i$ is a sphere, its outward unit normal vector at point $\mathbf{a}$ is

$$\mathbf{m}_i = \frac{\mathbf{a}(\mathbf{a} \wedge \mathbf{c}_i)}{|\mathbf{a} \wedge \mathbf{c}_i|}, \tag{33}$$

which equals $\mathbf{c}_i$ when $\tilde{s}_i$ is a hyperplane. That point $\mathbf{a}$ is on both $\tilde{s}_1$ and $\tilde{s}_2$ gives

$$\mathbf{a} \cdot \mathbf{c}_i = \lambda_i, \text{ for } i = 1, 2. \tag{34}$$

So

$$\mathbf{m}_1 \cdot \mathbf{m}_2 = \frac{\mathbf{c}_1 - \mathbf{a} \cdot \mathbf{c}_1 \mathbf{a}}{\sqrt{1 - (\mathbf{a} \cdot \mathbf{c}_1)^2}} \cdot \frac{\mathbf{c}_2 - \mathbf{a} \cdot \mathbf{c}_2 \mathbf{a}}{\sqrt{1 - (\mathbf{a} \cdot \mathbf{c}_2)^2}} = \frac{\mathbf{c}_1 \cdot \mathbf{c}_2 - \lambda_1 \lambda_2}{\sqrt{(1 - \lambda_1^2)(1 - \lambda_2^2)}}. \tag{35}$$

On the other hand,

$$s_1 * s_2 = \frac{(\mathbf{c}_1 + \lambda_1 p_0) \cdot (\mathbf{c}_2 + \lambda_2 p_0)}{|\mathbf{c}_1 + \lambda_1 p_0||\mathbf{c}_2 + \lambda_2 p_0|} = \frac{\mathbf{c}_1 \cdot \mathbf{c}_2 - \lambda_1 \lambda_2}{\sqrt{(1 - \lambda_1^2)(1 - \lambda_2^2)}}. \tag{36}$$

□

An immediate corollary is that any orthogonal transformation in $\mathcal{R}^{n+1,1}$ induces an angle-preserving transformation in $\mathcal{S}^n$. This is conformal transformation, which will be discussed in the last section.

4. r-dimensional Spheres and planes

We have the following conclusion parallel to that in Euclidean geometry:

Theorem 6. *For $2 \leq r \leq n+1$, every r-blade A_r of Minkowski signature in $\mathcal{R}^{n+1,1}$ represents an $(r-2)$-dimensional sphere or plane in $\mathcal{S}^n$.*

Corollary 1. *The $(r-2)$-dimensional sphere passing through r points $\mathbf{a}_1, \ldots, \mathbf{a}_r$ in $\mathcal{S}^n$ is represented by $a_1 \wedge \cdots \wedge a_r$; the $(r-2)$-plane passing through $r-1$ points $\mathbf{a}_1, \ldots, \mathbf{a}_{r-1}$ in $\mathcal{S}^n$ is represented by $p_0 \wedge a_1 \wedge \cdots \wedge a_{r-1}$.*

There are two possibilities:
Case 1. When $p_0 \wedge A_r = 0$, A_r represents an $(r-2)$-plane in $\mathcal{S}^n$. After normalization, the *standard form* of the $(r-2)$-plane is

$$p_0 \wedge \mathbf{I}_{r-1}, \tag{37}$$

where $\mathbf{I}_{r-1}$ is a unit $(r-1)$-blade of $\mathcal{G}(\mathcal{R}^{n+1})$ representing the minimal space in $\mathcal{R}^{n+1}$ supporting the $(r-2)$-plane of $\mathcal{S}^n$.
Case 2. A_r represents an $(r-2)$-dimensional sphere if

$$A_{r+1} = p_0 \wedge A_r \neq 0. \tag{38}$$

The vector

$$s = A_r A_{r+1}^{-1} \tag{39}$$

has positive square and $p_0 \cdot s \neq 0$, so its dual $\tilde{s}$ represents an $(n-1)$-dimensional sphere. According to Case 1, A_{r+1} represents an $(r-1)$-dimensional plane in $\mathcal{S}^n$. Therefore

$$A_r = sA_{r+1} = (-1)^{\epsilon} \tilde{s} \vee A_{r+1}, \tag{40}$$

where $\epsilon = \frac{(n+2)(n+1)}{2} + 1$, represents the intersection of $(n-1)$-sphere $\tilde{s}$ with $(r-1)$-plane A_{r+1}.

With suitable normalization, we can write $s = c - \rho p_0$. Since $s \wedge A_{r+1} = p_0 \wedge A_{r+1} = 0$, the sphere A_r is also centered at c and has normal radius ρ. Accordingly we can represent an $(r-2)$-dimensional sphere in the *standard form*

$$(c - \rho p_0)\ (p_0 \wedge \mathbf{I}_r), \tag{41}$$

where $\mathbf{I}_r$ is a unit r-blade of $\mathcal{G}(\mathcal{R}^{n+1})$ representing the minimal space in $\mathcal{R}^{n+1}$ supporting the $(r-2)$-sphere of $\mathcal{S}^n$.

This completes our classification of standard representations for spheres and planes in $\mathcal{S}^n$.

Expanded form

For $r+1$ homogeneous points $a_0, \ldots, a_r$ "in" $\mathcal{S}^n$, where $0 \leq r \leq n+1$, we have

$$A_{r+1} = a_0 \wedge \cdots \wedge a_r = \mathbf{A}_{r+1} + p_0 \wedge \mathbf{A}_r, \tag{42}$$

where

$$\begin{array}{lcl} \mathbf{A}_{r+1} & = & \mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_r, \\ \mathbf{A}_r & = & \partial\!\!\!/ \mathbf{A}_{r+1}. \end{array} \tag{43}$$

When $\mathbf{A}_{r+1} = 0$, A_{r+1} represents an $(r-1)$-plane, otherwise it represents an $(r-1)$-sphere. In the latter case, $p_0 \wedge \mathbf{A}_{r+1} = p_0 \wedge A_{r+1}$ represents the support plane of the $(r-1)$-sphere in $\mathcal{S}^n$, and $p_0 \wedge \mathbf{A}_r$ represents the $(r-1)$-plane normal to the center of the $(r-1)$-sphere in the support plane. The center of the $(r-1)$-sphere is

$$\frac{\mathbf{A}_r \mathbf{A}_{r+1}^{\dagger}}{|\mathbf{A}_r||\mathbf{A}_{r+1}|}, \tag{44}$$

the normal radius is

$$1 - \frac{|\mathbf{A}_{r+1}|}{|\mathbf{A}_r|}. \tag{45}$$

Since

$$A_{r+1}^{\dagger} \cdot A_{r+1} = \det(a_i \cdot a_j)_{(r+1)\times(r+1)} = (-\frac{1}{2})^{r+1} \det(|\mathbf{a}_i - \mathbf{a}_j|^2)_{(r+1)\times(r+1)}, \tag{46}$$

when $r = n+1$, we obtain Ptolemy's Theorem for spherical geometry:

Theorem 7 (Ptolemy's Theorem)**.** *Let* $\mathbf{a}_0, \cdots, \mathbf{a}_{n+1}$ *be points in* $\mathcal{S}^n$*, then they are on a sphere or hyperplane of* $\mathcal{S}^n$ *if and only if* $\det(|\mathbf{a}_i - \mathbf{a}_j|^2)_{(n+2)\times(n+2)} = 0$.

5. Stereographic projection

In the homogeneous model of $\mathcal{S}^n$, let $\mathbf{a}_0$ be a fixed point on $\mathcal{S}^n$. The space $\mathcal{R}^n = (\mathbf{a}_0 \wedge p_0)^{\sim}$, which is parallel to the tangent spaces of $\mathcal{S}^n$ at points $\pm\mathbf{a}_0$, is Euclidean. By the stereographic projection of $\mathcal{S}^n$ from point $\mathbf{a}_0$ to the space $\mathcal{R}^n$, every ray starting from $\mathbf{a}_0$ towards $\mathbf{a} \in \mathcal{S}^n$ intersects the space at point

$$j_{SR}(\mathbf{a}) = \frac{\mathbf{a}_0(\mathbf{a}_0 \wedge \mathbf{a})}{1 - \mathbf{a}_0 \cdot \mathbf{a}} = 2(\mathbf{a} - \mathbf{a}_0)^{-1} + \mathbf{a}_0. \tag{47}$$

The following are some plain facts about the stereographic projection:

1. A hyperplane passing through $\mathbf{a}_0$ and normal to $\mathbf{n}$ is mapped to the hyperspace in $\mathcal{R}^n$ normal to $\mathbf{n}$.

2. A hyperplane normal to $\mathbf{n}$ but not passing through $\mathbf{a}_0$ is mapped to the sphere in $\mathcal{R}^n$ with center $c = \mathbf{n} - \frac{\mathbf{a}_0}{\mathbf{n}\cdot\mathbf{a}_0}$ and radius $\rho = \frac{1}{\sqrt{|\mathbf{n}\cdot\mathbf{a}_0|}}$. Such a sphere has the feature that
$$\rho^2 = 1 + c^2, \tag{48}$$
i.e., its intersection with the unit sphere of $\mathcal{R}^n$ is a unit $(n-2)$-dimensional sphere. Conversely, given a point c in $\mathcal{R}^n$, we can find a unique hyperplane in $\mathcal{S}^n$ whose stereographic projection is the sphere in $\mathcal{R}^n$ with center c and radius $\sqrt{1+c^2}$. It is the hyperplane normal to $a_0 - c$.

3. A sphere passing through $\mathbf{a}_0$, with center $\mathbf{c}$ and normal radius ρ, is mapped to the hyperplane in $\mathcal{R}^n$ normal to $P^{\perp}_{\mathbf{a}_0}(\mathbf{c})$ and with $\frac{1-\rho}{\sqrt{1-(1-\rho)^2}}$ as the sign distance from the origin.

4. A sphere not passing through $\mathbf{a}_0$, with center $\mathbf{c}$ and normal radius ρ, is mapped to the sphere in $\mathcal{R}^n$ with center $\frac{(1-\rho)\mathbf{p}_0 + P^{\perp}_{\mathbf{a}_0}(\mathbf{c})}{d_n(\mathbf{c},\mathbf{a}_0)-\rho}$ and radius $\frac{\sqrt{1-(1-\rho)^2}}{|d_n(\mathbf{c},\mathbf{a}_0)-\rho|}$.

It is a classical result that the map j_{SR} is a conformal map from $\mathcal{S}^n$ to $\mathcal{R}^n$. The conformal coefficient λ is defined by
$$|j_{SR}(\mathbf{a}) - j_{SR}(\mathbf{b})| = \lambda(\mathbf{a},\mathbf{b})|\mathbf{a}-\mathbf{b}|, \text{ for } \mathbf{a},\mathbf{b}\in\mathcal{S}^n. \tag{49}$$
We have
$$\lambda(\mathbf{a},\mathbf{b}) = \frac{1}{\sqrt{(1-\mathbf{a}_0\cdot\mathbf{a})(1-\mathbf{a}_0\cdot\mathbf{b})}}. \tag{50}$$
Below we show that using the null cone of $\mathcal{R}^{n+1,1}$ we can construct the conformal map j_{SR} trivially: it is nothing but a rescaling of null vectors.

Let
$$e = \mathbf{a}_0 + p_0, \quad e_0 = \frac{-\mathbf{a}_0 + p_0}{2}, \quad E = e\wedge e_0. \tag{51}$$
Then for $\mathcal{R}^n = (e\wedge e_0)^{\sim} = (\mathbf{a}_0\wedge p_0)^{\sim}$, the map $i_E : x\in\mathcal{R}^n \mapsto e_0 + x + \frac{x^2}{2}e \in \mathcal{N}^n_e$ defines a homogeneous model for the Euclidean space.

For any null vector h in $\mathcal{R}^{n+1,1}$, in the homogeneous model of $\mathcal{S}^n$, it represents a point in $\mathcal{S}^n$, while in the homogeneous model of $\mathcal{R}^n$ it represents a point or point at infinity of $\mathcal{R}^n$. The rescaling transformation $k_R : \mathcal{N}^n \longrightarrow \mathcal{N}^n_e$ defined by
$$k_R(h) = -\frac{h}{h\cdot e}, \text{ for } h\in\mathcal{N}^n, \tag{52}$$
induces the conformal map j_{SR} through the following commutative diagram:

$$\begin{array}{ccc} \mathbf{a}+p_0 \in \mathcal{N}^n_{p_0} & \overset{k_R}{-----\rightarrow} & \dfrac{\mathbf{a}+p_0}{1-\mathbf{a}\cdot\mathbf{a}_0} \in \mathcal{N}^n_e \\ i_{p_0} \uparrow & & \downarrow P^{\perp}_E \\ \mathbf{a} \in \mathcal{S}^n & \overset{j_{SR}}{-----\rightarrow} & \dfrac{\mathbf{a}_0(\mathbf{a}_0 \wedge \mathbf{a})}{1-\mathbf{a}\cdot\mathbf{a}_0} \in \mathcal{R}^n \end{array} \tag{53}$$

i.e., $j_{SR} = P^{\perp}_E \circ k_R \circ i_{p_0}$. The conformal coefficient λ can be derived from the following identity: for any vector x and null vectors h_1, h_2,

$$\left| -\frac{h_1}{h_1 \cdot x} + \frac{h_2}{h_2 \cdot x} \right| = \frac{|h_1 - h_2|}{\sqrt{|(h_1 \cdot x)(h_2 \cdot x)|}}. \tag{54}$$

The inverse of the map j_{SR}, denoted by j_{RS}, is

$$j_{RS}(u) = \frac{(u^2-1)\mathbf{a}_0 + 2u}{u^2+1} = 2(u-\mathbf{a}_0)^{-1} + \mathbf{a}_0, \text{ for } u \in \mathcal{R}^n. \tag{55}$$

According to [?], (**??**) can also be written as

$$j_{RS}(u) = -(u-\mathbf{a}_0)^{-1}\mathbf{a}_0(u-\mathbf{a}_0). \tag{56}$$

From the above algebraic construction of the stereographic projection, we see that the null vectors in $\mathcal{R}^{n+1,1}$ can be interpreted geometrically in both $\mathcal{S}^n$ and $\mathcal{R}^n$, so are the Minkowski blades of $\mathcal{R}^{n+1,1}$. Every vector in $\mathcal{R}^{n+1,1}$ of positive signature can be interpreted as a sphere or hyperplane in both spaces. We will discuss this further in the next chapter.

6. Conformal transformations

In this section we present some results on the conformal transformations in $\mathcal{S}^n$. We know that the conformal group of $\mathcal{S}^n$ is isomorphic with the Lorentz group of $\mathcal{R}^{n+1,1}$. On the other hand, a Lorentz transformation in $\mathcal{R}^{n+1,1}$ is the product of at most $n+2$ reflections with respect to vectors of positive signature. We first analyze the induced conformal transformation in $\mathcal{S}^n$ of such a reflection in $\mathcal{R}^{n+1,1}$.

6.1. Inversions and reflections

After normalization, any vector in $\mathcal{R}^{n+1,1}$ of positive signature can be written as $s = \mathbf{c} + \lambda p_0$, where $0 \le \lambda < 1$. For any point $\mathbf{a}$ in $\mathcal{S}^n$, the reflection of a with respect to s is

$$\frac{1+\lambda^2 - 2\lambda\mathbf{c}\cdot\mathbf{a}}{1-\lambda^2} b, \tag{57}$$

where

$$\mathbf{b} = \frac{(1-\lambda^2)\mathbf{a} + 2(\lambda - \mathbf{c}\cdot\mathbf{a})\mathbf{c}}{1+\lambda^2 - 2\lambda\mathbf{c}\cdot\mathbf{a}}. \tag{58}$$

If $\lambda = 0$, i.e., if $\tilde{s}$ represents a hyperplane of $\mathcal{S}^n$, then (**??**) gives

$$\mathbf{b} = \mathbf{a} - 2\mathbf{c}\cdot\mathbf{a}\,\mathbf{c}, \tag{59}$$

i.e., $\mathbf{b}$ is the reflection of $\mathbf{a}$ with respect to the hyperplane $\tilde{\mathbf{c}}$ of $\mathcal{S}^n$.

If $\lambda \neq 0$, let $\lambda = 1 - \rho$. Then from (**??**) we get

$$\left(\frac{\mathbf{c} \wedge \mathbf{a}}{1 + \mathbf{c} \cdot \mathbf{a}}\right)^{\dagger} \left(\frac{\mathbf{c} \wedge \mathbf{b}}{1 + \mathbf{c} \cdot \mathbf{b}}\right) = \frac{\rho}{2 - \rho}. \tag{60}$$

Since the right-hand side of (**??**) is positive, $\mathbf{c}, \mathbf{a}, \mathbf{b}, -\mathbf{c}$ are on a half great circle of $\mathcal{S}^n$. Using (**??**), (**??**), we can write (**??**) as

$$d_s(\mathbf{a}, \mathbf{c}) d_s(\mathbf{b}, \mathbf{c}) = \rho_s^2, \tag{61}$$

where ρ_s is the stereographic distance corresponding to the normal distance ρ. We say $\mathbf{a}, \mathbf{b}$ are *inversive* with respect to the sphere with center $\mathbf{c}$ and stereographic radius ρ_s. This conformal transformation is called an *inversion* in $\mathcal{S}^n$.

In the language of Geometric Algebra, an inversion can be easily described. The two inversive homogeneous points a, b correspond to the null directions in the 2-dimensional space $a \wedge (c - \rho p_0)$, which is degenerate when a is on the sphere represented by $(c - \rho p_0)^{\sim}$ and Minkowski otherwise.

Any conforml transformation in $\mathcal{S}^n$ is generated by inversions with respect to spheres and reflections with respect to hyperplanes.

6.2. Other typical conformal transformations

Antipodal transformation

By antipodal transformation a point $\mathbf{a}$ of $\mathcal{S}^n$ is mapped to point $-\mathbf{a}$. This transformation is induced by the versor p_0.

Rotations

A rotation in $\mathcal{S}^n$ is just a rotation in $\mathcal{R}^{n+1}$. Any rotation in $\mathcal{S}^n$ can be induced by a spinor in $\mathcal{G}(\mathcal{R}^{n+1})$, which is a geometric product of factors of the form $e^{\mathbf{I}_2\theta/2}$.

Given a unit 2-blade $\mathbf{I}_2$ in $\mathcal{G}(\mathcal{R}^{n+1})$ and $0 < \theta < 2\pi$, the spinor $e^{\mathbf{I}_2\theta/2}$ induces a rotation in $\mathcal{S}^n$. Using the orthogonal decomposition

$$\mathbf{x} = P_{\mathbf{I}_2}(\mathbf{x}) + P_{\mathbf{I}_2}^{\perp}(\mathbf{x}), \text{ for } \mathbf{x} \in \mathcal{S}^n, \tag{62}$$

we get

$$e^{-\mathbf{I}_2\theta/2}\mathbf{x}e^{\mathbf{I}_2\theta/2} = P_{\mathbf{I}_2}(\mathbf{x})e^{\mathbf{I}_2\theta} + P_{\tilde{\mathbf{I}}_2}(\mathbf{x}). \tag{63}$$

Therefore when $n > 1$, the set of fixed points by this rotation is the $(n-2)$-plane in $\mathcal{S}^n$ represented by $\tilde{\mathbf{I}}_2$. It is called the *axis* of the rotation. θ is the angle of rotation for the points on the line of $\mathcal{S}^n$ represented by $p_0 \wedge \mathbf{I}_2$. This line is called the *line of rotation*.

For example, the spinor $\mathbf{a}(\mathbf{a}+\mathbf{b})$ induces a rotation from point $\mathbf{a}$ to point $\mathbf{b}$, with $p_0 \wedge \mathbf{a} \wedge \mathbf{b}$ as the line of rotation. The spinor $(\mathbf{c} \wedge \mathbf{a})(\mathbf{c} \wedge (\mathbf{a} + \mathbf{b}))$, where $\mathbf{a}, \mathbf{b}$ have equal distances from $\mathbf{c}$, induces a rotation from point $\mathbf{a}$ to point $\mathbf{b}$, with $p_0 \wedge P_{\mathbf{c}}^{\perp}(\mathbf{a}) \wedge P_{\mathbf{c}}^{\perp}(\mathbf{b})$ as the line of rotation.

Rotations belong to the orthogonal group $O(\mathcal{S}^n)$. A versor in $\mathcal{G}(\mathcal{R}^{n+1,1})$ induces an orthogonal transformation in $\mathcal{S}^n$ if and only if it leaves $\{\pm p_0\}$ invariant.

Tidal transformations

A tidal transformation, of coefficient λ where $\lambda \neq \pm 1$, and with respect to a point $\mathbf{c}$ in $\mathcal{S}^n$, is a conformal transformation induced by the spinor $1 + \lambda p_0 \wedge \mathbf{c}$. It changes a point $\mathbf{a}$ to point

$$\mathbf{b} = \frac{(1-\lambda^2)\mathbf{a} + 2\lambda(\lambda \mathbf{a}\cdot\mathbf{c} - 1)\mathbf{c}}{1 + \lambda^2 - 2\lambda \mathbf{a}\cdot\mathbf{c}}. \tag{64}$$

Points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are always on the same line. Conversely, from $\mathbf{a}, \mathbf{b}, \mathbf{c}$ we can obtain

$$\lambda = \frac{d_n^2(\mathbf{b}, \mathbf{a})}{d_n^2(\mathbf{b} - \mathbf{c}) - d_n^2(\mathbf{a} - \mathbf{c})}. \tag{65}$$

By this transformation, any line passing through point $\mathbf{c}$ is invariant, and any sphere with center $\mathbf{c}$ is transformed into a sphere with center $\mathbf{c}$ or $-\mathbf{c}$, or the hyperplane normal to $\mathbf{c}$. This is why we give it the name tidal transformation. Points $\pm\mathbf{c}$ are the source and influx of the tide.

Given points $\mathbf{a}, \mathbf{c}$ in $\mathcal{S}^n$, which are neither identical nor antipodal, let point $\mathbf{b}$ moves on line $\mathbf{ac}$ of $\mathcal{S}^n$, then $\lambda = \lambda(\mathbf{b})$ is determined by (**??**). This function has the following properties:

1. $\lambda \neq \pm 1$, i.e., $\mathbf{b} \neq \pm\mathbf{c}$. This is because when $\lambda = \pm 1$, then $1 + \lambda p_0 \wedge \mathbf{c}$ is no longer a spinor.
2. Let $\underline{\mathbf{c}}(\mathbf{a})$ be the reflection of $\mathbf{a}$ with respect to $\mathbf{c}$:
$$\underline{\mathbf{c}}(\mathbf{a}) = \mathbf{a} - 2\mathbf{a}\cdot\mathbf{c}\mathbf{c}^{-1}. \tag{66}$$
Then
$$\lambda(-\underline{\mathbf{c}}(\mathbf{a})) = \infty, \quad \lambda(\underline{\mathbf{c}}(\mathbf{a})) = \mathbf{a}\cdot\mathbf{c}. \tag{67}$$
3. When $\mathbf{b}$ moves from $-\underline{\mathbf{c}}(\mathbf{a})$ through $\mathbf{c}$, $\mathbf{a}$, $-\mathbf{c}$ back to $-\underline{\mathbf{c}}(\mathbf{a})$, λ increases strictly from $-\infty$ to ∞.
4. $$\lambda(-\underline{\mathbf{c}}(\mathbf{b})) = \frac{1}{\lambda(\mathbf{b})}. \tag{68}$$
5. When $\mathbf{c}\cdot\mathbf{a} = 0$ and $0 < \lambda < 1$, then $\mathbf{b}$ is between $\mathbf{a}$ and $-\mathbf{c}$, and
$$\lambda = d_s(\mathbf{a}, \mathbf{b}). \tag{69}$$
When $0 > \lambda > -1$, then $\mathbf{b}$ is between $\mathbf{a}$ and $\mathbf{c}$, and
$$\lambda = -d_s(\mathbf{a}, \mathbf{b}). \tag{70}$$

When $|\lambda| > 1$, a tidal transformation is the composition of an inversion with the antipodal transformation, because

$$1 + \lambda p_0 \wedge \mathbf{c} = -p_0(p_0 - \lambda\mathbf{c}). \tag{71}$$

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