MM Research Preprints, 166–187
No. 18, Dec. 1999. Beijing

Rational Solutions of Riccati-like Partial Differential Equations[1)]

Ziming Li
MMRC, Institute of Systems Science
Academia Sinica, Beijing 100080, China
Email: zmli@mmrc.iss.ac.cn

Fritz Schwarz
GMD, Institut SCAI
53754 Sankt Augustin, Germany
Email: fritz.schwarz@gmd.de

Abstract. When factoring linear partial differential systems with a finite-dimensional solution space or analyzing symmetries of nonlinear ode's, we need to look for rational solutions of certain nonlinear pde's. The nonlinear pde's are called Riccati-like because they arise in a similar way as Riccati ode's. In this paper we describe the structure of rational solutions of a Riccati-like system, and an algorithm for computing them. The algorithm is also applicable to finding all rational solutions of Lie's system $\{\partial_x u + u^2 + a_1u+a_2v+a_3,\ \partial_y u+uv+b_1u+b_2v+b_3,\ \partial_x v+uv+c_1u+c_2v+c_3,\ \partial_y v+v^2+d_1u+d_2v+d_3\}$, where $a_1 \ldots d_3$ are rational functions of x and y.

1. Introduction

Riccati's equation is one of the first examples of a differential equation that has been considered extensively in the literature, shortly after Leibniz and Newton had introduced the concept of the derivative of a function at the end of the 17th century. Riccati equations occur in many problems of mathematical physics and pure mathematics. A good survey is given in the book by Reid (1972).

Of particular importance is the relation between Riccati's equation and a linear ode $y''+ay'+by=0$ with $a, b \in \mathbb{C}(x)$, where $\mathbb{C}$ is the field of the complex numbers. For example, solutions h with the property that the quotient $p = h'/h \in \mathbb{C}(x)$ may be represented as $h = \exp\left(\int p\,\mathrm{d}x\right)$ if p satisfies the first-order Riccati equation $p'+p^2+ap+b=0$. Equivalently, this linear ode allows the first-order right factor $y'-qy$ over $\mathbb{C}(x)$ if q obeys the same equation as p. In general, finding the first-order right rational factors of a linear homogeneous ode is equivalent to finding the rational solutions of its associated Riccati equation (see e.g. Singer 1991).

It turns out that this correspondence carries over to systems of linear homogeneous partial differential equations with a finite-dimensional solution space. Systems of this kind occur

[1)] The work is done when the first author visited GMD from September, 1997 to December, 2000 in Lie's symmetry theory for solving nonlinear ode's and related equivalence problems. For example, Lie studied the coherent nonlinear system

$$\begin{aligned}\{\partial_x u + u^2 + a_1 u + a_2 v + a_3, \quad \partial_y u + uv + b_1 u + b_2 v + b_3,\\ \partial_x v + uv + c_1 u + c_2 v + c_3, \quad \partial_y v + v^2 + d_1 u + d_2 v + d_3\}\end{aligned} \tag{1}$$

for the first time in connection with the symmetry analysis of second-order ode's with projective symmetry group (see, e.g. Lie 1873, page 365). It is suggested therefore to call (**??**) *Lie's system*. We will show that Lie's system may be transformed to the Riccati-like system $\mathcal{R}_3^{(2)}$ given in Example **??**.

In this paper the following problem will be considered. Given a set $\mathcal{L}$ of linear homogeneous pde's in one unknown function $z(x, y)$ whose coefficients are in $\mathbb{C}(x, y)$ and whose solution space is of finite dimension, find a solution of $\mathcal{L}$ in the form $\exp\left(\int u\,\mathrm{d}x + v\,\mathrm{d}y\right)$, where u and v are in $\mathbb{C}(x, y)$ with $\partial_y u = \partial_x v$. In other words, we want to find a linear differential ideal over $\mathbb{C}(x, y)$ with one-dimensional solution space containing $\mathcal{L}$. As shown in Section **??**, this problem is equivalent to finding a rational solution of the Riccati-like system associated with $\mathcal{L}$. We describe an algorithm, implemented in the ALLTYPE system Schwarz (1998b), that computes all rational solutions of an associated Riccati-like system. The algorithm is also applicable to finding rational solutions of Lie's system and hyperexponential solutions of linear homogeneous pde's with finite-dimensional solution space in several unknowns.

The paper is organized as follows. Section **??** contains necessary preliminaries. Section **??** presents the main results: the structure of rational solutions of a Riccati-like system and an algorithm for computing them. Applications are given in Section **??**.

2. Preliminaries

We denote by $\mathbb{K}$ the field of rational functions $\mathbb{C}(x, y)$, and by ∂_x and ∂_y the usual partial differential operators acting on $\mathbb{K}$. Let Ω be the field of of all complex meromorphic functions in the space of complex variables of x and y. Note that both Ω and $\mathbb{C}$ have the same constant field $\mathbb{C}$. In this paper we are concerned with solutions in Ω. An element a of Ω is called an x-constant (resp. y-constant) if $\partial_x a = 0$ (resp. $\partial_y a = 0$). An x-derivative (resp. y-derivative) of a means $\partial_x^k a$ (resp. $\partial_y^k a$) for some nonnegative integer k.

By a system we mean a finite subset of some differential polynomial ring over $\mathbb{K}$. Basic notions related to differential polynomials are used such as: rankings, leaders, autoreduced sets, differential remainders, and characteristic sets for differential ideals. For their definitions, the reader is referred to Ritt (1950), Rosenfeld (1959), and Kolchin (1973). The differential ideal generated by a subset $\mathcal{P}$ of some differential polynomial ring is denoted by $[\mathcal{P}]$.

In the rest of this section, we describe three useful notions: the elimination property of linear differential ideals with finite linear dimension, integrable pairs, and coherent orthonomic systems,

2.1. Linear systems with finite linear dimension

Let z be a differential indeterminate over $\mathbb{K}$ and fix an orderly ranking on the differential polynomial ring $\mathbb{K}\{z\}$. We denote by $\mathbb{L}$ the $\mathbb{K}$-linear space consisting of all linear homogeneous polynomials in $\mathbb{K}\{z\}$.

Given a linear system $\mathcal{L} \subset \mathbb{L}$, the linear dimension of $[\mathcal{L}]$ in $\mathbb{K}\{z\}$ is the codimension of $\mathbb{L} \cap [\mathcal{L}]$ in $\mathbb{L}$ (Kolchin 1973, page 151). The general solution of $\mathcal{L}$ depends on a finite number of unspecified constants if and only if the linear dimension of $[\mathcal{L}]$ is finite (Kolchin 1973, page 152). To check if the linear dimension of $[\mathcal{L}]$ is finite, we compute a coherent autoreduced set $\mathcal{A}$ (Rosenfeld, 1959) such that $[\mathcal{L}] = [\mathcal{A}]$. This computation can be done by various methods such as: Janet bases (Janet 1920, Schwarz 1998a) the characteristic set method (Wu 1989, Li and Wang 1999), and Gröbner bases for differential operators (Kandri-Rody and Weispfennig) . The linear dimension of $[\mathcal{L}]$ is finite if and only if an x-derivative and a y-derivative of z appear as leaders of some elements of $\mathcal{A}$.

We denote by $\mathbb{L}_x$ (resp. $\mathbb{L}_y$) the subset of $\mathbb{L}$ consisting of differential polynomials involving only x-derivatives (resp. y-derivatives) of z. The following elimination property is well-known.

Lemma 2.1 *Let $\mathcal{L}$ be a finite subset of $\mathbb{L}$. If $[\mathcal{L}]$ is of finite linear dimension, then there is an algorithm for computing two nonzero elements $[\mathcal{L}] \cap \mathbb{L}_x$ and $[\mathcal{L}] \cap \mathbb{L}_y$, respectively.*

Proof. Compute a coherent autoreduced set $\mathcal{A}$ in $\mathbb{L}$ such that $[\mathcal{L}] = [\mathcal{A}]$. Assume that $\mathcal{A}$ is not $\{1\}$, for, otherwise, $[\mathcal{L}]$ is trivial. A finite base B of the $\mathbb{K}$-linear space $V = \mathbb{L}/(\mathbb{L} \cap [\mathcal{L}])$ consists of all differential monomials $\partial_x^i \partial_y^j z$ that cannot be reduced w.r.t. $\mathcal{A}$. Using $\mathcal{A}$, we can express any element of V as a $\mathbb{K}$-linear combination of elements of B. Hence, we can find smallest integer n such that z, $\partial_x z$, $\ldots$, $\partial_x^n z$ are $\mathbb{K}$-linearly dependent, that is, we can find $a_0, a_1, \ldots, a_{n-1} \in \mathbb{K}$ such that

$$L_x = a_0 z + a_1 \partial_x z + \cdots + a_{n-1} \partial_x^{n-1} z + \partial_x^n z \in [\mathcal{L}].$$

Similarly, we can find a nonzero linear differential polynomial in $[\mathcal{L}] \cap \mathbb{L}_y$. □

2.2. Integrable pairs

To describe the structure of rational solutions in Section **??**, we define a pair of rational functions $(f, g) \in \mathbb{K} \times \mathbb{K}$ to be *integrable* if $\partial_y f = \partial_x g$. For an integrable pair (f, g), the expression $H = \exp\left(\int f \, \mathrm{d}x + g \, \mathrm{d}y\right)$ defines an element of Ω unique up to a nonzero multiplicative constant. We regard H as an element of Ω when the value of the multiplicative constant is irrelevant to our discussion. Note that $\partial_x H/H = f$ and $\partial_y H/H = g$.

Two integrable pairs (f, g) and (p, q) are said to be *equivalent* if there exists a nonzero h in $\mathbb{K}$ such that $f - p = \partial_x h/h$ and $g - q = \partial_y h/h$. Write $(f, g) \sim (p, q)$ when they are equivalent. Since $(f, g) \sim (p, q)$ if and only if the ratio of $\exp\left(\int f \, \mathrm{d}x + g \, \mathrm{d}y\right)$ and $\exp\left(\int p \, \mathrm{d}x + q \, \mathrm{d}y\right)$ is a rational function, $\sim$ is an equivalence relation on the set of integrable pairs. It is easy to verify that, for integrable pairs (f_1, g_1), (p_1, q_1), (f_2, g_2), (p_2, q_2) and integers m, n, $(f_1, g_1) \sim (p_1, q_1)$ and $(f_2, g_2) \sim (p_2, q_2)$ implies $(mf_1 + nf_2,\, mg_1 + ng_2) \sim (mp_1 + np_2,\, mq_1 + nq_2)$.

An element a in Ω is said to be a *hyperexponential* if $\partial_x a/a$ and $\partial_y a/a$ belong to $\mathbb{K}$. If a is a hyperexponential, $(\partial_x a/a, \partial_y a/a)$ is an integrable pair. Conversely, for an integrable pair (f, g), we may construct a hyperexponential $\exp\left(\int f \, \mathrm{d}x + g \, \mathrm{d}y\right)$ in Ω. For an integrable pair (f, g), define $E^{(f,g)}$ to be the $\mathbb{C}$-linear space $\{h \exp\left(\int f \, \mathrm{d}x + g \, \mathrm{d}y\right) \mid h \in \mathbb{K}\}$. A direct calculations shows that $E^{(f,g)}$ consists of all hyperexponentials a such that $(\partial_x a/a, \partial_y a/a) \sim (f, g)$. The following lemma is used to group rational solutions of a Riccati-like system.

Lemma 2.2 *If (f_1, g_1), (f_2, g_2), $\ldots$, (f_n, g_n) are mutually inequivalent integrable pairs, the sum of $\mathbb{C}$-linear spaces $E^{(f_1,g_1)}$, $E^{(f_2,g_2)}$, $\ldots$, $E^{(f_n,g_n)}$ is direct.*

Proof. We proceed by induction. For $n = 2$, if $E^{(f_1,g_1)} \cap E^{(f_2,g_2)}$ contains a nonzero element a, $(\partial_x a/a, \partial_y a/a) \sim (f_i, g_i)$, for $i = 1, 2$. Hence, $(f_1, g_1) \sim (f_2, g_2)$, a contradiction. The sum of $E^{(f_1,g_1)}$ and $E^{(f_2,g_2)}$ is direct.

Assume that the result is proved for lower values of n. If the sum of $E^{(f_1,g_1)}$, $E^{(f_2,g_2)}$, $\ldots$, $E^{(f_n,g_n)}$ is not direct, there are nonzero $z_1 \in E^{(f_2,g_2)}$, $z_2 \in E^{(f_2,g_2)}$, $\ldots$, $z_n \in E^{(f_n,g_n)}$ which are $\mathbb{C}$-linearly dependent. By a possible rearrangement of indexes, we have

$$z_n = c_1 z_1 + c_2 z_2 + \cdots + c_{n-1} z_{n-1} \tag{2}$$

for some $c_1, c_2, \ldots, c_{n-1} \in \mathbb{C}$. Since $z_1, z_2, \ldots, z_{n-1}$ are linearly independent over $\mathbb{C}$ by the induction hypothesis, Theorem 1 in Kolchin (1973, page 86) implies that there exist derivatives $\theta_1, \theta_2, \ldots, \theta_{n-1}$ such that $W = \det(\theta_i z_j)$ is nonzero, where $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$. Since the z_i's are hyperexponentials, there exists $r_{ij} \in \mathbb{K}$ such that $\theta_j z_i = r_{ji} z_i$. Applying $\theta_1, \theta_2, \ldots, \theta_{n-1}$ to (**??**) then yields a linear system

$$\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1,n-1} \\ r_{21} & r_{22} & \cdots & r_{2,n-1} \\ \cdot & \cdot & \cdots & \cdot \\ r_{n-1,1} & r_{n-1,2} & \cdots & r_{n-1,n-1} \end{pmatrix} \begin{pmatrix} c_1 z_1 \\ c_2 z_2 \\ \vdots \\ c_{n-1} z_{n-1} \end{pmatrix} = \begin{pmatrix} r_{1n} z_n \\ r_{2n} z_n \\ \vdots \\ r_{n,n-1} z_n \end{pmatrix}$$

whose coefficient matrix (r_{ji}) is of full rank because $W \neq 0$. Solving this system, we get $c_i z_i = s_i z_n$, where $s_i \in \mathbb{K}$. Since $c_k \neq 0$ for some k with $1 \leq k \leq n-1$, z_k/z_n is rational, so $(f_k, g_k) \sim (f_n, g_n)$, a contradiction. □

2.3. Coherent orthonomic systems

A system $\mathcal{P}$ in a differential polynomial ring $\mathbb{D}$ is orthonomic if it is an autoreduced set and any element of $\mathcal{P}$ is linear w.r.t. its leader and monic, or, equivalently, both initial and separant of any element in $\mathcal{P}$ is one. A linear autoreduced set can be considered as an orthonomic system. System (**??**) is orthonomic w.r.t. any orderly ranking on u and v. Systems studied in this paper are always orthonomic with respect to certain orderly ranking on their differential indeterminates and derivatives.

Let $\mathcal{P}$ be an orthonomic system in $\mathbb{D}$. We recall the notions of Δ-polynomials and coherence in Rosenfeld (1959) for $\mathcal{P}$. Note that these two notions are originally defined for autoreduced sets and the name "Δ-polynomial" is due to Boulier et. al (1995). Let P_1 and P_2 belong to $\mathcal{P}$ with the respective leaders $\theta_1 z$ and $\theta_2 z$, where z is a differential indeterminate, and θ_1, θ_2 are derivatives. Then there exist derivatives ϕ_1 and ϕ_2 with minimal orders $\phi_1\theta_1 = \phi_2\theta_2$. The Δ-polynomial of P_1 and P_2, denoted by $\Delta(P_1, P_2)$, is $\phi_1 P_1 - \phi_2 P_2$. Note that $\Delta(P_1, P_2)$ is well-defined provided the leaders of P_1 and P_2 are derivatives of the same indeterminate. The system $\mathcal{P}$ is coherent if, for every such pair P_1, P_2 in $\mathcal{P}$, $\Delta(P_1, P_2)$ can be written as a $\mathbb{D}$-linear combination of derivatives of elements of $\mathcal{P}$, in which each derivative has its leader lower than $\phi_1\theta_1 z$ (in a preselected ranking).

To study orthonomic systems, we need to make sure that they cannot be formally reduced to non-orthonomic or trivial ones. Corollary 3 in Chapter I of Rosenfeld (1959) asserts that an orthonomic system $\mathcal{P}$ is a characteristic set of $[\mathcal{P}]$ if and only if $\mathcal{P}$ is coherent. Hence, $\mathcal{P}$ cannot be formally reduced any further if it is reduced. By the same corollary, one can easily show

Lemma 2.3 *An orthonomic system $\mathcal{P}$ is coherent if and only if all Δ-polynomials (possibly) formed by elements of $\mathcal{P}$ have zero as their differential remainders w.r.t. $\mathcal{P}$.*

Hence, we are able to decide algorithmically whether an orthonomic system is coherent.

3. Rational solutions of associated Riccati-like systems

Given a linear differential system $\mathcal{L} \subset \mathbb{L}$ with finite linear dimension, we want to compute its hyperexponential solutions. The substitution:

$$z \leftarrow \exp\left(\int u\,\mathrm{d}x + v\,\mathrm{d}y\right) \quad \text{where } \partial_y u = \partial_x v \tag{3}$$

transforms $\mathcal{L}$ to a nonlinear system in $\mathbb{K}\{u, v\}$. The union of this system and $\{\partial_y u - \partial_x v\}$ is called the *Riccati-like system associated with* $\mathcal{L}$, or, simply, an *associated Riccati-like system.* Conversely, the substitution: $u \leftarrow \partial_x z/z$, $v \leftarrow \partial_y z/z$ transforms an associated Riccati-like system in $\mathbb{K}\{u, v\}$ to a system in $\mathbb{L}$. As in the ordinary case, computing hyperexponential solutions of $\mathcal{L}$ is equivalent to computing rational solutions of its associated Riccati-like system.

Example 3.1 Let $\mathcal{L}_2 = \{\partial_x^2 z + a_1\partial_x z + a_2 z,\ \partial_y z + b_1\partial_x z + b_2 z\}$ be coherent. Then the linear dimension of $\mathcal{L}$ is 2. The Riccati-like system $\mathcal{R}_2$ associated with $\mathcal{L}$ is $\{\partial_x u + u^2 + a_1 u + a_2,\ v + b_1 u + b_2,\ \partial_y u - \partial_x v\}$. Coherent linear systems with dimension 3 may be either $\mathcal{L}_3^{(1)}$ equal to $\{\partial_x^3 z + a_1\partial_x^2 z + a_2\partial_x z + a_3 z,\ \partial_y z + b_1\partial_x^2 z + b_2\partial_x z + b_3 z\}$, or $\mathcal{L}_3^{(2)}$ equal to

$$\{\partial_x^2 z + a_1\partial_x z + a_2\partial_y z + a_3 z,\ \partial_y\partial_x z + b_1\partial_x z + b_2\partial_y z + b_3 z,\ \partial_y^2 z + c_1\partial_x z + c_2\partial_y z + c_3 z\}.$$

Their respective associated Riccati-like systems are $\mathcal{R}_3^{(1)}$ equal to

$$\{\partial_x^2 u + 3u\partial_x u + u^3 + a_1(\partial_x u + u^2) + a_2 u + a_3,\ v + b_1(\partial_x u + u^2) + b_2 u + b_3,\ \partial_y u - \partial_x v\}$$

and $\mathcal{R}_3^{(2)}$ equal to

$$\{\partial_x u + u^2 + a_1 u + a_2 v + a_3,\ \partial_y u + uv + b_1 u + b_2 v + b_3,\ \partial_y v + v^2 + c_1 u + c_2 v + c_3,\ \partial_y u - \partial_x v\}.$$

In the rest of this section, let $\mathcal{L}$ be a system in $\mathbb{L}$ with finite linear dimension d. The Riccati-like system associated with $\mathcal{L}$ is denoted by $\mathcal{R}$. The set of rational solutions of $\mathcal{R}$ is denoted by $\mathbb{S}$. All elements of $\mathbb{S}$ are integrable pairs, because $\partial_y u - \partial_x v$ is contained in $\mathcal{R}$. We describe the structure of $\mathbb{S}$ and present an algorithm for computing $\mathbb{S}$.

3.1. Standard representations of rational solutions

To describe rational solutions of Riccati ode's and Riccati-like systems precisely, we introduce some notation. Let F be a subfield of $\mathbb{K}$. By an *F-linearly independent set*, we mean a finite subset of $\mathbb{C}[x, y]$ whose elements are F-linearly independent. Let a, b be in $\mathbb{K}$. Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ be $\mathbb{C}(y)$ and $\mathbb{C}(x)$-independent sets, respectively. We define

$$X_A^a = \left\{ a + \frac{\partial_x\left(\sum_{i=1}^k c_i a_i\right)}{\sum_{i=1}^k c_i a_i} \mid c_1, \ldots, c_k \text{ are } x\text{-constants} \right\} \bigcap \mathbb{K}$$

and

$$Y_B^b = \left\{ b + \frac{\partial_y \left(\sum_{i=1}^{l} c_i b_i\right)}{\sum_{i=1}^{l} c_i b_i} \mid c_1, \ldots, c_l \text{ are } y\text{-constants} \right\} \bigcap \mathbb{K}.$$

If, moreover, (a, b) is integrable and $H = \{h_1, \ldots, h_m\}$ is a $\mathbb{C}$-linearly independent set, we define

$$S_H^{(a,b)} = \left\{ \left(a + \frac{\partial_x \left(\sum_{i=1}^{m} c_i h_i\right)}{\sum_{i=1}^{m} c_i h_i},\ b + \frac{\partial_y \left(\sum_{i=1}^{m} c_i h_i\right)}{\sum_{i=1}^{m} c_i h_i} \right) \mid c_1, \ldots, c_m \in \mathbb{C} \right\}.$$

Lemma 3.1 *Let $r_1, r_2, \ldots, r_k$ belong to $\mathbb{C}(y)[x]$ and $c_1, c_2, \ldots, c_k$ be x-constants of Ω. If*

$$f = \frac{c_1 \partial_x r_1 + c_2 \partial_x r_2 + \cdots + c_k \partial_x r_k}{c_1 r_1 + c_2 r_2 + \cdots + c_k r_k}$$

belongs to $\mathbb{K}$, then there exist $s_1, s_2, \ldots, s_k \in \mathbb{C}[x, y]$ and $d_1, d_2, \ldots, d_k \in \mathbb{C}[y]$ such that

$$f = \frac{d_1 \partial_x s_1 + d_2 \partial_x s_2 + \cdots + d_k \partial_x s_k}{d_1 s_1 + d_2 s_2 + \cdots + d_k s_k}.$$

Proof. Assume that $f = f_1/f_2$, where $f_1, f_2 \in \mathbb{C}[x, y]$. Equating the coefficients of the like powers of x in

$$(c_1 \partial_x r_1 + c_2 \partial_x r_2 + \cdots + c_k \partial_x r_k) f_2 = (c_1 r_1 + c_2 r_2 + \cdots + c_k r_k) f_1,$$

we find that $c_1, c_2, \ldots, c_k$ satisfy a linear system over $\mathbb{C}(y)$. Hence, $c_1, c_2, \ldots, c_k$ can be chosen as elements in $\mathbb{C}(y)$. Let h be the common denominator of all the c_i and the r_i. Then h is in $\mathbb{C}[y]$. We derive that

$$f = \frac{h(c_1 \partial_x r_1 + c_2 \partial_x r_2 + \cdots + c_k \partial_x r_k)}{h(c_1 r_1 + c_2 r_2 + \cdots + c_k r_k)} = \frac{d_1 \partial_x s_1 + d_2 \partial_x s_2 + \cdots + d_k \partial_x s_k}{d_1 s_1 + d_2 s_2 + \cdots + d_k s_k}$$

where $d_1, d_2, \ldots, d_k \in \mathbb{C}[y]$ and $s_1, s_2, \ldots, s_k \in \mathbb{C}[x, y]$. □

By Lemma **??** the c_i's in X_A^a and Y_B^b can be chosen as elements in $\mathbb{C}[y]$ and $\mathbb{C}[x]$, respectively. The next theorem describes the structure of $\mathbb{S}$.

Theorem 3.2 *There exist mutually inequivalent integrable pairs $(a_1, b_1), \ldots, (a_m, b_m)$, and $\mathbb{C}$-linearly independent sets $H_1, \ldots, H_m$ such that $\mathbb{S}$ is the (disjoint) union of $S_{H_i}^{(a_i,b_i)}$, $i = 1$, $\ldots$, m. Moreover, $\sum_{i=1}^{m} |H_i|$ is no greater than d.*

Proof. Let V be the solution space of $\mathcal{L}$ and (f_1, g_1), $\ldots$, (f_m, g_m) in $\mathbb{S}$, none of which is equivalent to the other. Then $E^{(f_1,g_1)} \cap V$, $\ldots$, $E^{(f_m,g_m)} \cap V$ are all nontrivial, and form a direct sum Lemma **??**. Thus, m is no greater than d. We may further assume that there are only m equivalence classes (w.r.t. $\sim$) in $\mathbb{S}$.

Let (f, g) be one of the (f_i, g_i)'s. Assume that the intersection of V and $E^{(f,g)}$ is of dimension k over $\mathbb{C}$. Then there exist $\mathbb{C}$-linearly independent rational functions r_1, $\ldots$, r_k such that $r_1 \exp\left(\int f \, \mathrm{d}x + g \, \mathrm{d}y\right)$, $\ldots$, $r_k \exp\left(\int f \, \mathrm{d}x + g \, \mathrm{d}y\right)$ form a basis for $E^{(f,g)} \cap V$. Let $S_{(f,g)}$ be the subset of $\mathbb{S}$ consisting of integrable pairs equivalent to (f, g). Then

$$S_{(f,g)} = \left\{ \left(f + \frac{c_1 \partial_x r_1 + \cdots + c_k \partial_x r_k}{c_1 r_1 + \cdots + c_k r_k},\ g + \frac{c_1 \partial_y r_1 + \cdots + c_k \partial_y r_k}{c_1 r_1 + \cdots + c_k r_k} \right) \mid c_1, \ldots, c_k \in \mathbb{C} \right\}.$$

Without loss of generality, we may assume that $r_j = h_j/h$ where $h, h_j \in \mathbb{C}[x,y]$, for $j = 1, \ldots, k$. Letting $(a,b) = (f - \partial_x h/h,\, g - \partial_y h/h)$ and $H = \{h_1, \ldots, h_k\}$, we find $S_{(f,g)} = S_H^{(a,b)}$. Since $\mathbb{S}$ is the (disjoint) union of $S_{(f_i,g_i)}$, $i = 1, \ldots, m$, there exist integrable pairs (a_i, b_i), with $(a_i, b_i) \sim (f_i, g_i)$, and $\mathbb{C}$-linearly independent sets H_i such that $\mathbb{S}$ is the (disjoint) union of $S_{H_i}^{(a_i,b_i)}$, $i = 1, \ldots, m$. Moreover, $\sum_{i=1}^m |H_i| \le d$, because $\dim\left(E^{(a_i,b_i)} \cap V\right) = |H_i|$. □

We call the set $\left\{S_{H_i}^{(a_i,b_i)} \mid i = 1, \ldots, m\right\}$ a *standard representation* of $\mathbb{S}$.

Theorem **??** can be specialized to describe the rational solutions of an ordinary Riccati equation. More precisely, define two elements f and g of $\mathbb{K}$ are equivalent w.r.t. x if $f - g = \partial_x h/h$ for some $h \in \mathbb{K}$, and denote this relation by $\sim_x$. Likewise, we define the relation $\sim_y$ on $\mathbb{K}$. Both $\sim_x$ and $\sim_y$ are equivalence relations. Remark that, for $f, g, p, q \in \mathbb{K}$, $f \sim_x p$ and $g \sim_y q$ do not imply $(f,g) \sim (p,q)$, because (f,g) and (p,q) may not be integrable. Let R_x be an ordinary Riccati equation w.r.t. ∂_x associated with a d_xth order linear ordinary differential equation L_x whose coefficient field is $\mathbb{K}$. Denote by $\mathbb{S}_x$ the set of rational solutions of R_x in $\mathbb{K}$. Along the same line, one can show

Corollary 3.3 *There exists $a_1, \ldots, a_m$ in $\mathbb{K}$, none of which is equivalent to the other w.r.t. x, and $\mathbb{C}(y)$-linearly independent sets $A_1, \ldots, A_m$ such that $\mathbb{S}_x$ is the (disjoint) union of $X_{A_i}^{a_i}$, $i = 1, \ldots, m$. Moreover, $\sum_{i=1}^m |A_i|$ is no greater than d_x.*

We also call the set $\{X_{A_i}^{a_i} \mid i = 1, \ldots, m\}$ *a standard representation* of $\mathbb{S}_x$.

Algorithms for computing $\mathbb{S}_x$ are found in Singer (1991) and Bronstein (1992). Actually, their algorithms can compute all solutions of R_x in $\overline{\mathbb{C}(y)}(x)$. But we only need $\mathbb{S}_x$, a subset of $\mathbb{K}$. Their algorithms output a finite list of sets $X_{H_i}^{b_i}$ whose union is $\mathbb{S}_x$. We modify the output to give a standard representation of $\mathbb{S}_x$. If none of the b_i's is equivalent to the other, then we are done. Otherwise, the next lemma is applied.

Lemma 3.4 *If b_1 and b_2 are equivalent w.r.t. x, we can compute $b \in \mathbb{K}$ and a $\mathbb{C}(y)$-linearly independent set H such that $X_{H_1}^{b_1} \cup X_{H_2}^{b_2} \subset X_H^b \subset \mathbb{S}_x$.*

Proof. Let H_i be $\{h_{i1}, \ldots, h_{ik_i}\}$ for $k = 1, 2$. Assume that $b_1 = b_2 + \partial_x g/g$ for some $g \in \mathbb{K}$. Then

$$X_{H_1}^{b_1} = \left\{ b_2 + \frac{c_{11}\partial_x(h_{11}g) + \cdots + c_{1k_1}(\partial_x h_{1k_1} g)}{c_{11}(h_{11}g) + \cdots + c_{1k_1}(h_{1k_1}g)} \mid c_{11}, \ldots c_{1k_1} \in \mathbb{C}(y) \right\}.$$

From the set $G = \{h_{11}g, \ldots, h_{1k_1}g, h_{21}, \ldots, h_{2k_2}\}$, we pick up a maximally $\mathbb{C}(y)$-linearly independent set

$$B = \left\{ \frac{h_1}{h}, \ldots, \frac{h_k}{h} \mid h, h_1, \ldots, h_k, \in \mathbb{C}(y)[x] \right\}$$

Setting $b = b_2 - \partial_x h/h$ and $H = \{h_1, \ldots, h_k\}$, we obtain X_H^b, which contains both $X_{H_1}^{b_1}$ and $X_{H_2}^{b_2}$ because each element of G is a $\mathbb{C}(y)$-linear combination of some elements of B. Note that, for all $c_1, \ldots, c_k \in \mathbb{C}(y)$,

$$(c_1 h_1 + \cdots + c_k h_k) \exp\left(\int b \,\mathrm{d}x\right) = \left(c_1 \frac{h_1}{h} + \cdots + c_k \frac{h_k}{h}\right) \exp\left(\int b_2 \,\mathrm{d}x\right),$$

which is a solution of L_x. Hence, X_H^b is contained in $\mathbb{S}_x$. □

Example 3.2 The ordinary Riccati equation $x^2(\partial_x u + u^2) + xu - 1 = 0$ has rational solutions $u = 1/x$ and $-1/x + 2cx/(1 + cx^2)$, where c is an x-constant. The set of rational solutions are the union of $X_{\{1\}}^{\frac{1}{x}}$ and $X_{\{1,x^2\}}^{\frac{-1}{x}}$. Note that $(1/x) \sim_x (-1/x)$ because

$$\frac{1}{x} - \frac{-1}{x} = \frac{2}{x} = \frac{\partial_x x^2}{x^2}.$$

Applying Lemma **??** to $X_{\{1\}}^{\frac{1}{x}}$ and $X_{\{1,x^2\}}^{\frac{-1}{x}}$, we find that $X_{\{1\}}^{\frac{1}{x}}$ is contained in $X_{\{1,x^2\}}^{\frac{-1}{x}}$.

3.2. Computing standard representations

Our idea for computing a standard representation of $\mathbb{S}$ consists of four steps. First, compute two nonzero ode's $L_x(z) \in [\mathcal{L}] \cap \mathbb{L}_x$ and $L_y(z) \in [\mathcal{L}] \cap \mathbb{L}_y$ (with lowest order). Second, translate $L_x(z)$ to the Riccati ode $R_x(u)$, and $L_y(z)$ to $R_y(v)$ by (**??**) and compute respective standard representations of rational solutions of the Riccati ode's in $\mathbb{K}$. In the rest of this section, we denote by $\mathcal{X}$ and $\mathcal{Y}$ the standard representations of $R_x(u)$ and $R_y(v)$, respectively. Third, from $\mathcal{X}$ and $\mathcal{Y}$, construct a finite number of mutually inequivalent integrable pairs such that an element of $\mathbb{S}$ is equivalent to such a pair. At last, for each pair obtained from the third step, compute elements of $\mathbb{S}$ that are equivalent to it. The first step is carried out by Lemma **??**, and the second by known algorithms. Before describing the last two steps, we remind the reader of a useful identity

$$\partial_y \left(\frac{\partial_x a}{a} \right) = \partial_x \left(\frac{\partial_y a}{a} \right) \quad \text{for all } a \in \Omega. \tag{4}$$

Lemma 3.5 *If (f, g) belongs to $\mathbb{S}$, then there exist unique $X_A^a \in \mathcal{X}$ and $Y_B^b \in \mathcal{Y}$ such that all rational solutions of $\mathcal{R}$ equivalent to (f, g) are contained in $X_A^a \times Y_B^b$.*

Proof. Since $(f, g) \in \mathbb{S}$, f and g are rational solutions of $R_x(u)$ and $R_y(v)$, respectively. There then exist unique $X_A^a \in \mathcal{X}$ and unique $Y_B^b \in \mathcal{Y}$ such that (f, g) is in $X_A^a \times Y_B^b$. If (p, q) is in $\mathbb{S}$ and equivalent to (f, g), $p \in X_A^a$ and $q \in Y_B^b$ by Corollary **??**. □

Lemma **??** reduces our task to computing $\mathbb{S} \cap \left(X_A^a \times Y_B^b \right)$, for all $X_A^a \in \mathcal{X}$ and $Y_B^b \in \mathcal{Y}$. Now, we search for elements in $\mathcal{X} \times \mathcal{Y}$ that have possibly nonempty intersection with $\mathbb{S}$. We call an element $X_A^a \times Y_B^b$ of $\mathcal{X} \times \mathcal{Y}$ a *candidate* if (a, b) is integrable, and try to transform other elements in $\mathcal{X} \times \mathcal{Y}$ to candidates by the following lemma.

Lemma 3.6 *Let a and b belong to $\mathbb{K}$. There exist two polynomials $p, q \in \mathbb{C}[x, y]$ such that $(a + \partial_x p/p,\ b + \partial_y q/q)$ is integrable if and only if*

$$\partial_x \partial_y (\log z) = \partial_y a - \partial_x b \tag{5}$$

has a solution in $\mathbb{K}$.

Proof. Let r be $\partial_y a - \partial_x b$. Assume that there exist such p and q. Then

$$\begin{aligned} r &= \partial_x \left(\frac{\partial_y q}{q} \right) - \partial_y \left(\frac{\partial_x p}{p} \right) = \partial_x \left(\frac{\partial_y q}{q} \right) - \partial_x \left(\frac{\partial_y p}{p} \right) \quad \text{(by (\textbf{??}))} \\ &= \partial_x \left(\partial_y \log q - \partial_y \log p \right) = \partial_x \partial_y \left(\log \frac{q}{p} \right). \end{aligned}$$

Conversely, if $z = q/p$ is a solution of (**??**), reversing the above calculation shows that

$$\left(a + \frac{\partial_x p}{p},\, b + \frac{\partial_y q}{q}\right)$$

is integrable. □

Now, we present an algorithm for computing such p and q.

Algorithm IntegrablePair (*Find an integrable pair*). Given a, b in $\mathbb{K}$, the algorithm finds p, q in $\mathbb{C}[x, y]$ such that $(a + \partial_x p/p,\, b + \partial_y q/q)$ is integrable, or determines that no such p and q exist.

I1. [Initialize.] Set $r \leftarrow \partial_y a - \partial_x b$. If $r = 0$, set $p \leftarrow 1, q \leftarrow 1$ and exit.

I2. [Hermite's reduction.] Apply Hermite's reduction (w.r.t. x) to get $f, h \in \mathbb{K}$ such that

$$r = \partial_x f + h \tag{6}$$

If h is nonzero, the algorithm terminates; no such p and q exist.

I3. [Partial fraction.] Compute the irreducible partial fraction decomposition of f w.r.t. y over its coefficient field. If the decomposition is

$$\sum_i m_i \frac{\partial_y q_i}{q_i} - \sum_j n_j \frac{\partial_y p_j}{p_j} + g \tag{7}$$

where $p_i, q_j \in \mathbb{C}[x, y]$, $m_i, n_j \in \mathbb{Z}^+ \cup \{0\}$, and $g \in \mathbb{C}(y)$, set $p \leftarrow \prod_j p_j^{n_j}$, $q \leftarrow \prod_i q_i^{m_i}$. Otherwise, no such p and q exist. □

Step I1 is clear. If $h \neq 0$, then $\int r \, \mathrm{d}x$ is not rational by Hermite's reduction (Geddes, et. al 1992, Bronstein 1997). Thus, (**??**) has no rational solution, and such p and q do not exist by Lemma **??**. Suppose now $h = 0$. Equations (**??**) and (**??**) imply that

$$\frac{\partial_y z}{z} = f + w \tag{8}$$

where w is an x-constant. Assume that the irreducible partial fraction decomposition of f w.r.t. y is

$$f = \underbrace{\sum_j n_j \frac{\partial_y q_j}{q_j} - \sum_i m_i \frac{\partial_y p_i}{p_i}}_{G} + \underbrace{\sum_k \frac{s_k}{t_k} + r}_{g}, \tag{9}$$

where $p_i, q_i, s_k, t_k, r \in \mathbb{C}[x, y]$, p_i, q_i, t_k are irreducible over $\mathbb{C}(x)[y]$ and relatively prime to each other. If z is rational, then the partial decomposition of $\partial_y z/z$ is of form G, so that g must be an x-constant because of (**??**), (**??**) and the uniqueness of partial fraction decomposition. Suppose now that g belongs to $\mathbb{C}(y)$. We compute

$$\begin{aligned}
\partial_y\left(a + \frac{\partial_x p}{p}\right) - \partial_x\left(b + \frac{\partial_y q}{q}\right) &= r + \partial_y\left(\frac{\partial_x p}{p}\right) - \partial_x\left(\frac{\partial_y q}{q}\right) \\
&\overset{(??)}{=} r + \partial_x\left(\frac{\partial_y p}{p} - \frac{\partial_y q}{q}\right) \\
&\overset{(??)}{=} \partial_x\left(f + \frac{\partial_y p}{p} - \frac{\partial_y q}{q}\right) \overset{(??)}{=} \partial_x g = 0.
\end{aligned}$$

IntegrablePair then returns p and q, as desired.

Example 3.3 Given

$$a = \frac{y^2x^3 + x^3 + xy^2 + y - x^2y - x}{yx^3 + y^2x^4 - x^5y - x^4} \quad \text{and} \quad b = \frac{1}{x-y},$$

IntegrablePair proceeds as follows.

I1. $r = 1/(1+xy)^2$.

I2. $f = -1/(xy^2 + y)$, $h = 0$.

I3. $f = x/(1+xy) - 1/y$, $p = 1$, $q = 1 + xy$.

Hence, $\left(a,\, b + \frac{x}{1+xy}\right)$ is integrable.

Example 3.4 Apply **IntegrablePair** to

$$a = \frac{y^3 + xy^2 + 2y - x^2y - x}{xy^3 + y^2 - x^2y^2 - xy} \quad \text{and} \quad b = \frac{1}{x-y}.$$

We find that

$$r = -\frac{x^2y^2 + 2xy + 1 - y^2}{x^2y^4 + 2xy^3 + x^2y^4 + y^2} \quad \text{and} \quad f = -\frac{x^2y + x + y}{y^2(xy+1)}.$$

In step I3 f decomposes into

$$\frac{x}{1+xy} - \frac{1}{y} - \frac{x}{y^2}.$$

Since $g = -x/y^2$ is not in $\mathbb{C}(y)$, no such p and q exist.

In what follows, by "given $X_A^a \times Y_B^b$", we mean that we are given $a, b \in \mathbb{K}$, a $\mathbb{C}(y)$-linearly independent set A, and a $\mathbb{C}(x)$-linearly independent set B.

The set $X_A^a \times Y_B^b$ contains no rational solutions of $\mathcal{R}$ if **IntegrablePair** confirms that no $p, q \in \mathbb{C}[x, y]$ are such that $(a + \partial_x p/p,\, b + \partial_y q/q)$ is integrable, because a rational solution of $\mathcal{R}$ must be integrable. Otherwise, we construct a candidate described below.

Algorithm Candidate (*Find a solution candidate*). Given $X_A^a \times Y_B^b$, the algorithm finds $X_F^f \times Y_G^g$ such that (f, g) is an integrable pair and $X_F^f \times Y_G^g = X_A^a \times Y_B^b$, or determines that $X_A^a \times Y_B^b$ contains no integrable pairs.

C1. [construct f and g.] If **IntegrablePair**(a, b) returns $p, q \in \mathbb{C}[x, y]$, set

$$f \leftarrow a - \frac{\partial_x q}{q},\; g \leftarrow b - \frac{\partial_y p}{p}.$$

Otherwise, the algorithm terminates; $X_A^a \times Y_B^b$ contains no integrable pairs.

C2. [construct F and G.] Set $F \leftarrow \{qh \mid h \in A\}$, $G \leftarrow \{ph \mid h \in B\}$. $\square$

In Step C1 we set the integrable pair (f, g) to be $(a - \partial_x q/q,\ b - \partial_y p/p)$ instead of $(a + \partial_x p/p,\ b + \partial_y q/q)$, because the former construction makes step C2 simpler. Equation (**??**) implies that (f, g) is integrable if $(a + \partial_x p/p,\ b + \partial_y q/q)$ is. To see $X_F^f \times Y_G^g = X_A^a \times Y_B^b$, we let $\alpha = \sum_{s \in A} c_s s$, where the c_s's are in $\mathbb{C}[y]$, not all zero. Since $a + \partial_x \alpha/\alpha = f + \partial_x(q\alpha)/q\alpha$, $X_F^f = X_A^a$. Likewise, $Y_G^g = Y_B^b$. The algorithm **Candidate** is correct.

Example 3.5 Let a and b be the same as those in Example **??**. Let $A = \{1, x\}$ and $B = \{1\}$. **IntegrablePair**(a, b) returns $p = 1$ and $q = 1 + xy$. **Candidate** then returns $f = a - y/(1 + xy)$, $g = b$, $F = \{1 + xy,\ x(1 + xy)\}$ and $G = B$.

Applying the above algorithm to each member of $\mathcal{X} \times \mathcal{Y}$, we obtain a set of disjoint candidates

$$\mathcal{C} = \left\{ X_{F_1}^{f_1} \times Y_{G_1}^{g_1}, \ldots, X_{F_k}^{f_k} \times Y_{G_k}^{g_k} \right\}$$

such that $\mathbb{S}$ is contained in $\left(X_{F_1}^{f_1} \times Y_{G_1}^{g_1} \right) \bigcup \cdots \bigcup \left(X_{F_k}^{f_k} \times Y_{G_k}^{g_k} \right)$, and a rational solution of $\mathcal{R}$ can belong to only one member in $\mathcal{C}$.

Lemma 3.7 *Let $X_F^f \times Y_G^g$ be one of the sets in $\mathcal{C}$. Let*

$$e_x = \max_{p \in F} \deg_x p \quad \text{and} \quad e_y = \max_{q \in G} \deg_y q.$$

If an integrable pair (a, b) belongs to $X_F^f \times Y_G^g$, then there exists a polynomial $h \in \mathbb{C}[x, y]$ with $\deg_x h \le e_x$ and $\deg_y h \le e_y$ such that

$$(a, b) = \left(f + \frac{\partial_x h}{h},\ g + \frac{\partial_y h}{h} \right). \tag{10}$$

Proof. Let $(a, b) = (f + \partial_x s/s,\ g + \partial_y t/t)$, where s and t are, respectively, $\mathbb{C}(y)$- and $\mathbb{C}(x)$-linear combinations of elements in F and G. Since (a, b) and (f, g) are integrable pairs, so is $(\partial_x s/s, \partial_y t/t)$. The function

$$h = \exp\left(\int \frac{\partial_x s}{s}\,\mathrm{d}x + \frac{\partial_y t}{t}\,\mathrm{d}y \right)$$

is well-defined and has the property $\partial_x h/h = \partial_x s/s$ and $\partial_y h/h = \partial_y t/t$. It follows that

$$s = ah \quad \text{and} \quad t = bh \tag{11}$$

for some x-constant a and y-constant b. Consequently, $sb = ta$. Let α be an element of $\mathbb{C}$ such that $t(\alpha, y) \neq 0$. Then $a = s(\alpha, y)b(\alpha)/t(\alpha, y)$, which implies that a belongs to $\mathbb{C}(y)$, and, therefore, h is in $\mathbb{C}(y)[x]$ by (**??**). Likewise, h is in $\mathbb{C}(x)[y]$. Hence, h is in $\mathbb{C}[x, y]$. Accordingly, $\deg_x h = \deg_x s \le e_x$ and $\deg_y h = \deg_y t \le e_y$ by (**??**). □

Suppose that (a, b) is a rational solution of $\mathcal{R}$ contained in $X_F^f \times Y_G^g$. By Lemma **??** there exists $h \in \mathbb{C}[x, y]$ such that (**??**) holds. Moreover, respective degree bounds for h in x and y are known. The next algorithm **PolynomialPart** determines h.

Algorithm PolynomialPart (*Find polynomial part*). Given an associated Riccati-like system $\mathcal{R}$, and a candidate $X_F^f \times Y_G^g$, the algorithm computes a $\mathbb{C}$-linearly independent set H such that the set of rational solutions of $\mathcal{R}$ equivalent to (f, g) is equal to $S_H^{(f,g)}$. If H is empty, then such rational solutions do not exist.

P1. [bound degrees.] Set $e_x \leftarrow \max_{p\in F} \deg_x p$, $e_y \leftarrow \max_{q\in G} \deg_y q$.

P2. [$e_x = e_y = 0$.] If both e_x and e_y are zero, check whether (f, g) satisfies all the equations in $\mathcal{R}$; if the answer is affirmative, set $H \leftarrow \{1\}$, otherwise, set $H \leftarrow \emptyset$; the algorithm terminates.

P3. [form a linear algebraic system.] Set $h \leftarrow \sum_{i=0}^{e_x} \sum_{j=0}^{e_y} c_{ij} x^i y^j$, where the c_{ij} are unspecified constants. Substitute $a + \partial_x h/h$ and $b + \partial_y h/h$ for u and v in each equation of $\mathcal{R}$, respectively. Set L be the result, which is a linear homogeneous algebraic system in the c_{ij}'s.

P4. [compute H.] Calculate a basis B for the solution space of L. If B consists of only zero vector, then set $H \leftarrow \emptyset$. Otherwise, set H to be the set consisting of polynomials corresponding to vectors of B. $\square$

In Step P3 substituting $u \leftarrow a+\partial_x h/h$, $v \leftarrow b+\partial_y h/h$ into $\mathcal{R}$ is equivalent to substituting $z \leftarrow h \exp(f\,\mathrm{d}x + g\,\mathrm{d}y)$ into $\mathcal{L}$; the latter yields a linear homogeneous system in the unspecified constants c_{ij}'s, because $\exp(f\,\mathrm{d}x + g\,\mathrm{d}y)$ is a nonzero overall factor. Thus, L obtained in step P3 is a linear homogeneous algebraic system. The correctness of **PolynomialPart** then follows from Lemma **??**.

Example 3.6 Determine the rational solutions of

$$\mathcal{R} = \left\{ \partial_x u + u^2 - \frac{2}{x} u + \frac{2y^2 - 2y}{x^2} v + \frac{2}{x^2},\ \partial_y u + uv,\ \partial_y v + v^2 + \frac{2}{y-1} v,\ \partial_y u - \partial_x v \right\}.$$

in $S_1 = X^{0}_{\{1,\,x\,x^2\}} \times Y^{\frac{-1}{y-1}}_{\{y-1,\,1\}}$ and $S_2 = X^{y}_{\{1,\,x\}} \times Y^{x}_{\{1,\,y\}}$, respectively. Applying **PolynomialPart** to S_1 yields

P1. $e_x = 2$ and $e_y = 1$.

P2. Skipped.

P3. $h = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 x^2 y$, $u \leftarrow \partial_x h/h$, $v \leftarrow -1/(2y) + \partial_y h/h$, $L = \{c_0, c_4 + c_1, c_5 + c_3\}$.

P4. $H = \{y,\ x - xy,\ x^2 - x^2 y\}$.

The rational solutions in $X^{0}_{\{1,\,x\,x^2\}} \times Y^{\frac{-1}{y-1}}_{\{y-1,\,1\}}$ are

$$\left(\frac{c_2(1-y) + 2c_3 x(1-y)}{c_1 y + c_2(x - xy) + c_3(x^2 - x^2 y)},\ -\frac{1}{y-1} + \frac{c_1 - c_2 x - c_3 x^2}{c_1 y + c_2(x - xy) + c_3(x^2 - x^2 y)} \right)$$

where $c_1, c_2, c_3 \in \mathbb{C}$.

Applying **PolynomialPart** to S_2 yields

P1. $e_x = 1$ and $e_y = 1$.

P2. Skipped.

P3. $h = c_0 + c_1x + c_2y + c_3xy$, $u \leftarrow y + \partial_x h/h$, $v \leftarrow x + \partial_y h/h$, $L = \{c_0, c_1, c_2, c_3\}$.

P4. $H = \emptyset$.

Now, we present the main algorithm.

Algorithm RationalSolution (*Find rational solutions of an associated Riccati-like system*). Given an associated Riccati-like system $\mathcal{R}$, the algorithm computes a standard representation of all rational solutions of $\mathcal{R}$.

R1. [Compute a coherent autoreduced set.] Transform $\mathcal{R}$ to the linear system $\mathcal{L}$ by the substitution $u \leftarrow \partial_x z/z$, $v \leftarrow \partial_y z/z$. Compute a coherent autoreduced set $\mathcal{A}$ such that $[\mathcal{A}] = [\mathcal{L}]$. If $\mathcal{A} = \{1\}$, the algorithm terminates; no solution exists.

R2. [Eliminate.] Use $\mathcal{A}$ and Lemma **??** to compute two linear ode's $L_x(z)$ (w.r.t. x) and $L_y(z)$ (w.r.t. y). Transform $L_x(z)$ and $L_y(z)$ to their associated Riccati ode's $R_x(u)$ and $R_y(v)$, respectively.

R3. [Find rational solutions of the Riccati ode's.] Compute respective standard representations $\mathcal{X}$ and $\mathcal{Y}$ of rational solutions of $R_x(u)$ and $R_y(v)$. If either $\mathcal{X}$ or $\mathcal{Y}$ is empty, the algorithm terminates; no rational solution exists.

R4. [Construct candidates.] Apply **Candidate** to each member of $\mathcal{X} \times \mathcal{Y}$ to get a set of candidates $\mathcal{C} = \left\{X_{F_1}^{f_1} \times Y_{G_1}^{g_1}, \ldots, X_{F_k}^{f_k} \times Y_{G_k}^{g_k}\right\}$. If $\mathcal{C}$ is empty, the algorithm terminates; no rational solution exists.

R5. [Compute polynomial parts.] Apply **PolynomialPart** to each member of $\mathcal{C}$ and collect all nonempty results to construct a standard representation of $\mathbb{S}$. □

A few words need to be said about the correctness of **RationalSolution**. The set $\mathbb{S}$ is the set of rational solutions of the Riccati-like system associated with $\mathcal{A}$ because $[\mathcal{L}] = [\mathcal{A}]$. The set $\mathbb{S}$ is contained in the union of members of $\mathcal{X} \times \mathcal{Y}$ by step R3, and in the union of members in $\mathcal{C}$ by step R4. Hence, **RationalSolution** returns a standard representation of $\mathbb{S}$ in step R5.

Step R1 is a necessary preparation for step R2, because the operations given in Lemma **??** are based on a characteristic set (Janet basis) of $[\mathcal{L}]$. Step R1 may be omitted if $R_x(u)$ and $R_y(v)$ can be easily read off from $\mathcal{R}$. If the coefficients of equations in $\mathcal{R}$ belong to $\mathbb{Q}(x, y)$, step R3 may require to introduce a finite algebraic extension of $\mathbb{Q}$. Nevertheless, steps R4 and R5 can be carried out.

A few examples illustrate how **RationalSolution** works.

Example 3.7 Find rational solutions of the Riccati-like system

$$\mathcal{R} = \left\{\partial_x u + u^2 - \frac{2}{x}u + \frac{2y^2 - 2y}{x^2}v + \frac{2}{x^2},\ \partial_y u + uv,\ \partial_y v + v^2 + \frac{2}{y-1}v,\ \partial_y u - \partial_x v\right\}.$$

Apply **RationalSolution** to get

R1. $\mathcal{L} = \mathcal{A} = \left\{\partial_x^2 z - \frac{2}{x}\partial_x z + \frac{2y^2-2y}{x^2}\partial_y z + \frac{2}{x^2}z,\ \partial_x\partial_y z,\ \partial_y^2 z + \frac{2}{y-1}\partial_y z\right\}$.

R2. $L_x(z) = \partial_x^3 z,\ L_y(z) = \partial_y^2 z + 2\partial_y z/(y-1),\ R_x(u) = \partial_x^2 u + 3u\partial_x u + u^3,\ R_y(v) = \partial_y v + v^2 + 2v/(y-1)$.

R3. $\mathcal{X} = \{X_A^a \mid a = 0,\, A = \{1,\, x,\, x^2\}\},\ \mathcal{Y} = \left\{Y_B^b \mid b = -1/(y-1),\, B = \{y-1,\, 1\}\right\}$.

R4. $\mathcal{C} = \left\{X_A^a \times Y_B^b\right\}$

R5. A standard representation of $\mathbb{S}$ is $\left\{S_H^{(a,b)} \mid H = \{y,\, x - xy,\, x^2 - x^2 y\}\right\}$ (see Example **??**).

In other words, the hyperexponential solutions of $\mathcal{L}$ are

$$\left(c_1 y + c_2(x - xy) + c_3(x^2 - x^2 y)\right) \exp\left(\int 0\,\mathrm{d}x - \frac{1}{y-1}\,\mathrm{d}y\right),$$

where c_1, c_2, c_3 are arbitrary elements of $\mathbb{C}$.

Example 3.8 Compute rational solutions of the Riccati-like system

$$\begin{aligned}\mathcal{R} \;=\; &\Big\{\partial_x^2 u + 3u\partial_x u + u^3 + \tfrac{6x^2-6xy+y^2}{x^2(2x-y)}(\partial_x u + u^2),\\ &\partial_y^2 v + 3v\partial_y v + v^3 + \tfrac{2y-3x}{x(y-x)}(\partial_y v + v^2) + \tfrac{y-2x}{x^2(y-x)}v\Big\}.\end{aligned}$$

We skip steps R1 and R2 because the first and second equations in $\mathcal{R}$ can be used as $R_x(u)$ and $R_y(v)$, respectively. In step R3 we compute $\mathcal{X} = \left\{X_{A_1}^{a_1}, X_{A_2}^{a_2}\right\}$ and $\mathcal{Y} = \left\{Y_{B_1}^{b_1}, Y_{B_2}^{b_2}\right\}$, where $a_1 = 0,\ A_1 = \{1, x\},\ a_2 = y/x^2,\ A_2 = \{1\},\ b_1 = 0,\ B_1 = \{1\},\ b_2 = -1/x$, and $B_2 = \{1, y^2\}$. Step R4 finds two candidates $X_{A_1}^{a_1} \times Y_{B_1}^{b_1}$ and $X_{A_2}^{a_2} \times Y_{B_2}^{b_2}$. Step R5 finds a standard representation of $\mathbb{S}$ as the union of

$$S_{\{1,x\}}^{(0,0)} = \left\{\left(\frac{c_2}{c_1 + c_2 x},\, 0\right) \mid c_1, c_2 \in \mathbb{C}\right\}$$

and

$$S_{\{1,\,y^2\}}^{(a_2,b_2)} = \left\{\left(\frac{y}{x^2},\, -\frac{1}{x} + \frac{2c_4 y}{c_3 + c_4 y^2}\right) \mid c_3, c_4 \in \mathbb{C}\right\}.$$

The linear system with which $\mathcal{R}$ is associated, is

$$\left\{\partial_x^3 z + \frac{6x^2 - 6xy + y^2}{x^2(2x-y)}\partial_x^2 z,\ \partial_y^3 z + \frac{2y - 3x}{x(y-x)}\partial_y^2 z + \frac{y - 2x}{x^2(y-x)}\partial_y z\right\}.$$

Its hyperexponential solutions are $c_1 + c_2 x$ and $(c_3 + c_4 y)\exp\left(\int \frac{y}{x^2}\,\mathrm{d}x - \frac{1}{x}\,\mathrm{d}y\right)$ where c_1, c_2, c_3, c_4 are arbitrary elements of $\mathbb{C}$.

4. Applications

In this section we apply the algorithm **RationalSolution** to finding rational solutions of Lie's system and hyperexponential solutions of linear homogeneous differential systems with finite linear dimension in several unknowns.

Lie's system (**??**) occurred originally in his investigation of certain second-order ode's that were based on its symmetries (Lie 1873). It turned out to be almost as ubiquitous as the Riccati ode's, e.g. in decomposing systems of linear pde's into smaller components.

To extend the applicability of the algorithm, we consider the following more general orthonomic system

$$\begin{array}{ll}\{P_1 = \partial_x u + a_0 u^2 + a_1 u + a_2 v + a_3, & P_2 = \partial_y u + b_0 uv + b_1 u + b_2 v + b_3, \\ P_3 = \partial_x v + c_0 uv + c_1 u + c_2 v + c_3, & P_4 = \partial_y v + d_0 v^2 + d_1 u + d_2 v + d_3\}\end{array} \tag{12}$$

in $\mathbb{K}\{u, v\}$ w.r.t. the ranking $1 < v < u < \partial_y v < \partial_x v < \partial_y u < \partial_x u < \cdots$. Moreover, we assume that both a_0 and d_0 are nonzero.

The differential remainders of $\Delta(P_1, P_2)$ and $\Delta(P_3, P_4)$ are, respectively,

$$R_{12} = b_0(c_0 - a_0)vu^2 + a_2(b_0 - d_0)v^2 + \text{terms involving } u^2, uv, u, v, 1$$

and

$$R_{34} = c_0(d_0 - b_0)v^2 u + d_1(a_0 - c_0)u^2 + \text{terms involving } v^2, uv, u, v, 1.$$

Observe that R_{12} and R_{34} are in $\mathbb{K}[u, v]$. Hence, if either R_{12} or R_{34} is nonzero, all solutions of (**??**) would be solutions of some polynomials in $\mathbb{K}[u, v]$. We will not consider this degenerating case.

Lemma 4.1 *If (**??**) is coherent, either $a_0 = c_0$ and $b_0 = d_0$, or $b_0 = c_0 = a_2 = d_1 = 0$.*

Proof. If (**??**) is coherent, $R_{12} = R_{34} = 0$ by Lemma **??**. Hence

$$b_0(c_0 - a_0) = a_2(b_0 - d_0) = c_0(d_0 - b_0) = d_1(a_0 - c_0) = 0.$$

Since $a_0 d_0 \neq 0$ in (**??**), either $a_0 = c_0$ and $b_0 = d_0$, or $b_0 = c_0 = a_2 = d_1 = 0$. □ Lemma **??** splits (**??**) into two systems

$$\begin{array}{ll}\{F_1 = \partial_x u + a_0 u^2 + a_1 u + a_2 v + a_3, & F_2 = \partial_y u + d_0 uv + b_1 u + b_2 v + b_3, \\ F_3 = \partial_x v + a_0 uv + c_1 u + c_2 v + c_3, & F_4 = \partial_y v + d_0 v^2 + d_1 u + d_2 v + d_3\}\end{array} \tag{13}$$

and

$$\begin{array}{ll}\{G_1 = \partial_x u + a_0 u^2 + a_1 u + a_3, & G_2 = \partial_y u + b_1 u + b_2 v + b_3, \\ G_3 = \partial_x v + c_1 u + c_2 v + c_3, & G_4 = \partial_y v + d_0 v^2 + d_2 v + d_3.\}\end{array} \tag{14}$$

Note that Lie's system is a special case of (**??**). Clearly, the coherence of (**??**) implies the coherence of (**??**) and (**??**). We solve (**??**) and (**??**) separately.

Theorem 4.2 *If (**??**) is coherent, then the substitution*

$$u \leftarrow \frac{1}{a_0}U - \frac{1}{3a_0}\left(a_1 + c_2 - \frac{\partial_x(a_0 d_0)}{a_0 d_0}\right), \quad v \leftarrow \frac{1}{d_0}V - \frac{1}{3d_0}\left(b_1 + d_2 - \frac{\partial_y(a_0 d_0)}{a_0 d_0}\right) \tag{15}$$

*transforms (**??**) into an associated Riccati-like system (in U and V) of type $\mathcal{R}_3^{(2)}$ in Example **??***

Proof. Normalizing (**??**) by the substitution

$$u \leftarrow \frac{\bar{u}}{a_0},\ v \leftarrow \frac{\bar{v}}{d_0}, \tag{16}$$

we transform (**??**) into the coherent system

$$\begin{array}{ll}\{\bar{F}_1 = \partial_x \bar{u} + \bar{u}^2 + \bar{a}_1\bar{u} + \bar{a}_2\bar{v} + \bar{a}_3, & \bar{F}_2 = \partial_y \bar{u} + \bar{u}\bar{v} + \bar{b}_1\bar{u} + \bar{b}_2\bar{v} + \bar{b}_3, \\ \bar{F}_3 = \partial_x \bar{v} + \bar{u}\bar{v} + \bar{c}_1\bar{u} + \bar{c}_2\bar{v} + \bar{c}_3, & \bar{F}_4 = \partial_y \bar{v} + \bar{v}^2 + \bar{d}_1\bar{u} + \bar{d}_2\bar{v} + \bar{d}_3,\}\end{array} \tag{17}$$

where $\bar{a}_1, \ldots, \bar{d}_3 \in \mathbb{K}$. Since the differential remainders of $\Delta(\bar{F}_1, \bar{F}_2)$ and $\Delta(\bar{F}_3, \bar{F}_4)$ are, respectively,

$$\begin{aligned}\bar{R}_{12} = (\bar{c}_1 - \bar{b}_1)\bar{u}^2 + (\bar{c}_2 - \bar{b}_2)\bar{u}\bar{v} + \underbrace{(\bar{c}_3 + \bar{b}_2\bar{c}_1 + \partial_y \bar{a}_1 - \partial_x \bar{b}_1 - 2\bar{b}_3 - \bar{a}_2\bar{d}_1)}_{p}\,\bar{u} \\ + \text{ terms involving } \bar{v} \text{ and } 1,\end{aligned}$$

and

$$\begin{aligned}\bar{R}_{34} = (\bar{c}_1 - \bar{b}_1)\bar{u}\bar{v} + (\bar{c}_2 - \bar{b}_2)\bar{v}^2 + \underbrace{(2\bar{c}_3 - \bar{b}_2\bar{c}_1 - \bar{b}_3 + \bar{a}_2\bar{d}_1 - \partial_x \bar{d}_2 + \partial_y \bar{c}_2)}_{q}\,\bar{v} \\ + \text{ terms involving } \bar{u} \text{ and } 1,\end{aligned}$$

we deduce $\bar{b}_1 = \bar{c}_1$, $\bar{b}_2 = \bar{c}_2$ and $p = q = 0$. It follows that

$$p + q = 3\bar{c}_3 + \partial_y(\bar{a}_1 + \bar{c}_2) - 3\bar{b}_3 - \partial_x(\bar{b}_1 + \bar{d}_2) = 0. \tag{18}$$

Applying the substitution

$$\bar{u} \leftarrow U - s,\ \bar{v} \leftarrow V - t, \quad \text{where } s = \tfrac{1}{3}(\bar{a}_1 + \bar{c}_2) \text{ and } t = \tfrac{1}{3}(\bar{b}_1 + \bar{d}_2) \tag{19}$$

to (**??**), we get

$$\begin{array}{ll}\{f_1 = \partial_x U + U^2 + A_1 U + A_2 V + A_3, & f_2 = \partial_y U + UV + B_1 U + B_2 V + B_3, \\ f_3 = \partial_x V + UV + C_1 U + C_2 V + C_3, & f_4 = \partial_y V + V^2 + D_1 U + D_2 V + D_3\}\end{array} \tag{20}$$

where $A_1, \ldots, D_3 \in \mathbb{K}$. Since $\bar{b}_1 = \bar{c}_1$ and $\bar{b}_2 = \bar{c}_2$, $B_1 = C_1$ and $B_2 = C_2$. We compute

$$\begin{aligned}B_3 - C_3 &= (-\partial_y s - \bar{b}_1 s - \bar{b}_2 t + st + \bar{b}_3) - (-\partial_x t - \bar{c}_1 s - \bar{c}_2 t + st + \bar{c}_3) \\ &= \bar{b}_3 + \partial_x t - \bar{c}_3 - \partial_y s \quad (\text{since } \bar{b}_1 = \bar{c}_1 \text{ and } \bar{b}_2 = \bar{c}_2) \\ &= \frac{1}{3}\left(3\bar{b}_3 + \partial_x(\bar{b}_1 + \bar{d}_2) - 3\bar{c}_3 - \partial_y(\bar{a}_1 + \bar{c}_2)\right) \quad (\text{by } (\mathbf{??})) \\ &= 0 \quad (\text{by } (\mathbf{??})).\end{aligned}$$

Therefore, $f_2 - f_3 = \partial_y U - \partial_x V$. This implies that $\{f_1, f_2, f_2 - f_3, f_4\}$ is the associated Riccati-like system $\mathcal{R}_3^{(2)}$, so is (**??**). Since

$$\bar{a}_1 = a_1 - \partial_x a_0/a_0,\ \bar{c}_2 = c_2 - \partial_x d_0/d_0,\ \bar{b}_1 = b_1 - \partial_y a_0/a_0,\ \bar{d}_2 = d_2 - \partial_x d_0/d_0,$$

substitution (**??**) is the result of the composition of (**??**) and (**??**). □

Example 4.1 Consider the coherent Lie's system

$$\begin{aligned}
&\partial_x u + u^2 + \frac{y+x^2}{x(y-x^2)}u + \frac{4y(y+x^2)}{x^2(y-x^2)}v + \frac{6yx^2+x^4+4y^2}{x^2(y^2-2yx^2+x^4)} = 0,\\
&\partial_y u + uv + \frac{1}{y-x^2}u - \frac{2y+x^2}{x(y-x^2)}v - \frac{2(y+x^2)}{x(y^2-2yx^2+x^4)} = 0,\\
&\partial_x v + uv + \frac{1}{y-x^2}u - \frac{2y+x^2}{x(y-x^2)}v + \frac{x^2-2y}{x(y^2-2yx^2+x^4)} = 0,\\
&\partial_y v + v^2 + \frac{4}{y-x^2}v + \frac{2}{y^2-2yx^2+x^4} = 0.
\end{aligned}$$

By (**??**) we apply the transformation

$$u \leftarrow U + \frac{y}{3x(y-x^2)}, \quad v \leftarrow V - \frac{5}{3(y-x^2)}$$

to get

$$\begin{aligned}
&\partial_x U + U^2 + \frac{5y+3x^2}{3x(y-x^2)}U + \frac{4y(y+x^2)}{x^2(y-x^2)}V + \frac{9x^4+6yx^2-23y^2}{9x^2(y^2-2yx^2+x^4)} = 0,\\
&\partial_y U + UV - \frac{2}{3(y-x^2)}U - \frac{5y+3x^2}{3x(y-x^2)}V + \frac{2(5y-3x^2)}{9x(y^2-2yx^2+x^4)} = 0,\\
&\partial_x V + UV - \frac{2}{3(y-x^2)}U - \frac{5y+3x^2}{3x(y-x^2)}V + \frac{2(5y-3x^2)}{9x(y^2-2yx^2+x^4)} = 0,\\
&\partial_y V + V^2 + \frac{2}{3(y-x^2)}V - \frac{2}{9(y^2-2yx^2+x^4)} = 0,
\end{aligned}$$

which is equivalent to the Riccati-like system $\mathcal{R}_3^{(2)}$ in Example **??**, because the difference between the second and third equations is $\partial_y U - \partial_x V$. Apply **RationalSolution** to this system to find the rational solutions:

$$U = \frac{-1}{3x} + \frac{2x}{3(y-x^2)} + \frac{2c_2x}{c_1y+c_2x^2},\ V = \frac{-1}{3(y-x^2)} + \frac{c_1}{c_1y+c_2x^2}, \quad c_1, c_2 \in \mathbb{C}.$$

Hence, the rational solutions of the original system are

$$u = \frac{x}{y-x^2} + \frac{2c_2x}{c_1y+c_2x^2},\ v = \frac{-2}{y-x^2} + \frac{c_1}{c_1y+c_2x^2}, \quad c_1, c_2 \in \mathbb{C}.$$

We turn our attention to (**??**).

Theorem 4.3 *If (**??**) is coherent, it decouples into two individual coherent systems*

$$\{\partial_x u + a_0u^2 + a_1u + a_3, \partial_y u + b_1u + b_3\} \quad \{\partial_y v + d_0v^2 + d_2v + d_3, \partial_x v + c_2v + c_3\} \qquad (21)$$

for u and v, respectively.

Proof. The differential remainders of $\Delta(G_1, G_2)$ and $\Delta(G_3, G_4)$ are, respectively

$$R_{12} = -2b_2a_0uv \ + \ \text{ terms involving } u^2,\ u.\ v \text{ and } 1$$

and

$$R_{34} = 2c_1d_0uv \ + \ \text{ terms involving } v^2,\ u.\ v \text{ and } 1$$

Since (**??**) is coherent, $b_2 = c_1 = 0$. The theorem follows. □

The following system also appears frequently in symmetry analysis.

Theorem 4.4 *Let $a_1, \ldots, b_3 \in \mathbb{K}$ and $a_1b_1 \neq 0$. The first-order Riccati-like system*

$$\{F = \partial_x z + a_1z^2 + a_2z + a_3,\ G = \partial_y z + b_1z^2 + b_2z + b_3\} \tag{22}$$

*is coherent if and only if its general solution depends on a single constant. If (**??**) has a rational solution, one of the following alternatives applies.*

1. *The general solution is rational and has the form*

$$\frac{1}{a_1}\frac{\partial_x r}{r+c} + p = \frac{1}{b_1}\frac{\partial_y r}{r+c} + p$$

with $p, r \in \mathbb{K}$ and $c \in \mathbb{C} \cup \{\infty\}$.

2. *There are at most two special rational solutions not involving unspecified constants.*

Proof. Since the differential remainder of $\Delta(F, G)$ is an algebraic polynomial in $\mathbb{K}[z]$ of degree no greater than 2, (**??**) has at most two solutions if it is not coherent.

Assume that (**??**) is coherent. We show that it is the system $\mathcal{R}_2$ in disguise (see Example **??**). The substitution

$$z \leftarrow \frac{1}{a_1}u - \frac{1}{2a_1}\left(a_2 - \frac{\partial_x a_1}{a_1}\right) \tag{23}$$

transforms (**??**) into the coherent system

$$\{f = \partial_x u + u^2 + A_3,\ g = \partial_y u + B_1u^2 + B_2u + B_3\}, \tag{24}$$

where $A_3, B_1, B_2, B_3 \in \mathbb{K}$. Since the differential remainder of $\Delta(f, g)$ is zero,

$$B_2 = -\partial_x B_1,\ \partial_x B_2 = 2A_3B_1 - 2B_3,$$

which implies

$$g - B_1 f = \partial_y u - B_1\partial_x u - (\partial_x B_1)u + \frac{1}{2}\partial_x^2 B_1 = \partial_y u - \partial_x\left(B_1 u - \frac{1}{2}\partial_x B_1\right).$$

Hence, (**??**) is equivalent to the system

$$\left\{\partial_x u + u^2 + A_3,\ \partial_y u - \partial_x\left(B_1 u - \frac{1}{2}\partial_x B_1\right)\right\}.$$

Set $v = \left(B_1 w - \frac{1}{2}\partial_x B_1\right)$. The above system becomes

$$\left\{\partial_x u + u^2 + A_3,\ v - B_1 u + \frac{1}{2}\partial_x B_1,\ \partial_y u - \partial_x v\right\} \tag{25}$$

which is of type $\mathcal{R}_2$ in Example (**??**). It follows from (**??**) and the definition of v that (u, v) is a solution of (**??**) if and only if z given in (**??**) is a solution of (**??**). Since (**??**) is associated with a coherent linear system with linear dimension two, the general solution of (**??**) can be written as

$$(u, v) = \left(\frac{c_1\partial_x s_1 + c_2\partial_x s_2}{c_1 s_1 + c_2 s_2},\ \frac{c_1\partial_y s_1 + c_2\partial_y s_2}{c_1 s_1 + c_2 s_2}\right),$$

where s_1 and s_2 are in some differential extension $\mathbb{F}$ of $\mathbb{K}$, linearly independent over the constant field of $\mathbb{F}$, and c_1 and c_2 are in the same constant field. Therefore, (**??**) implies that the general solution of (**??**) can be written as

$$z = \frac{1}{a_1}\frac{c_1\partial_x s_1 + c_2\partial_x s_2}{c_1 s_1 + c_2 s_2} - \frac{1}{2a_1}\left(a_2 - \frac{\partial_x a_1}{a_1}\right).$$

Setting $c = c_1/c_2$, we prove that the general solution of (**??**) depends on one constant.

We now consider the rational solutions of (**??**). By (**??**) and the second equation in (**??**), z is a rational solution of (**??**) if and only if (u, v) is a rational solution of (**??**). By Theorem **??** (**??**) has either infinitely many rational solutions or at most two inequivalent rational solutions The former case corresponds to the first alternative, and the latter to the second. Assume that (**??**) has infinitely many rational solutions. Then, by Theorem **??**,

$$(u, v) = \left(\frac{c_1\partial_x h_1 + c_2\partial_x h_2}{c_1 h_1 + c_2 h_2} + a,\ \frac{c_1\partial_y h_1 + c_2\partial_y h_2}{c_1 h_1 + c_2 h_2} + b\right)$$

where $h_1, h_2, a, b \in \mathbb{K}$ and $c_1, c_2 \in \mathbb{C}$. Setting $r = h_2/h_1$, $c = c_1/c_2$, $f = \partial_x h_1/h_1 + a$ and $g = \partial_y h_1/h_1 + b$, we get

$$(u, v) = \left(\frac{\partial_x r}{c + r} + f,\ \frac{\partial_y r}{c + r} + g\right).$$

Transformation (**??**) implies that

$$z = \frac{1}{a_1}\frac{\partial_x r}{c + r} + p$$

for some $p \in \mathbb{K}$. Substituting $\partial_y r/(c + r) + g$ for v in the second equation of (**??**) yields

$$z = \frac{1}{a_1 B_1}\frac{\partial_y r}{c + r} + q$$

for some $q \in \mathbb{K}$. Hence, $p = q$ because we may set $c = \infty$, i.e., $c_2 = 0$. The theorem is then proved by noticing that $a_1 B_1 = b_1$. □ According to the proof of Theorem **??**, the rational solutions of (**??**) can be computed by the algorithm **RationalSolution**. We may also proceed as follows. Compute the rational solutions of F. If there are only a finite number of solutions, we need only check if they satisfy G. Otherwise, the rational solutions of F are given by

$$\frac{\partial_x r}{C + r} + f$$

where $r, f \in \mathbb{K}$ and $C \in \mathbb{C}(y)$. Substituting this expression for z in G yields

$$H = \partial_y C + B_1 C^2 + B_2 C + B_3,$$

for some $B_1, B_2, B_3 \in \mathbb{K}$. Collecting coefficients of H w.r.t. the powers of x yields a system in $\mathbb{C}(y)\{C\}$ consisting possibly of first-order Riccati ode's, first-order linear ode's and algebraic equations, whose rational solutions can be easily found.

Example 4.2 Compute the rational solutions of

$$\{\partial_x z + z^2,\ \partial_y z + (1 - x^2 - 2xy - y^2)z^2 + (2x + 2y)z - 1 = 0.\}$$

The rational solutions of the first equation are $1/(C(y) + x)$, where $C(y)$ is an x-constant. Substituting the expression into the second equation yields

$$\partial_y C(y) + C(y)^2 - 2yC(y) + y^2 - 1 = 0,$$

so that $C(y) = y + 1/(y + c)$, where c is a constant. This system has the rational solutions

$$z = \frac{1}{x + y + \frac{1}{y+c}} = \frac{\partial_x \left(\frac{xy+y^2+1}{x+y}\right)}{c + \frac{xy+y^2+1}{x+y}} + \frac{1}{x + y}.$$

At last, the following problem is considered. Let $\mathbb{D}$ be the differential polynomial ring $\mathbb{K}\{z_1, \ldots, z_n\}$. Given a linear system $\mathcal{L} \subset \mathbb{D}$ with finite linear dimension, find all hyperexponential solutions of $\mathcal{L}$. By general elimination procedures, we compute a linear characteristic set $\mathcal{L}_i$ for $[\mathcal{L}] \cap \mathbb{K}\{z_i\}$, for $i = 1, \ldots, n$. Since each $[\mathcal{L}_i]$ is also of finite linear dimension, the algorithm **RationalSolution** computes a standard representation $\mathbb{S}_i$ of the rational solutions of the Riccati-like system associated with $\mathcal{L}_i$. Assume that

$$\mathbb{S}_i = \left\{ S_{H_{ij}}^{(f_{ij}, g_{ij})} \mid j = 1, \ldots, m_i \right\}.$$

The hyperexponential solutions of $\mathcal{L}_i$ are then expressed as $E_i = \bigcup_{j=1}^{m_i} V_{ij}$, where

$$V_{ij} = \left\{ \left(\sum_{h \in H_{ij}} c_h h \right) \exp\left(\int f_{ij}\, \mathrm{d}x + g_{ij}\, \mathrm{d}y \right) \mid c_h \in \mathbb{C} \right\}.$$

The problem is thus reduced to computing hyperexponential solutions of $\mathcal{L}$ contained in $V_{1j_1} \times \cdots \times V_{nj_n}$ for $1 \le j_1 \le m_1, \ldots, 1 \le j_n \le m_n$. Substituting

$$\left(\sum_{h \in H_{ij_i}} c_h h \right) \exp\left(\int f_{ij_i}\, \mathrm{d}x + g_{ij_i}\, \mathrm{d}y \right)$$

for z_i in $\mathcal{L}$ yields a linear algebraic system $\mathcal{A}$ in the unspecified constants c's. Notice that the coefficients of $\mathcal{A}$ may be hyperexponential. Nevertheless, the constant solutions of $\mathcal{A}$ gives us the hyperexponential solutions of $\mathcal{L}$ in $V_{1j_1} \times \cdots \times V_{nj_n}$.

Example 4.3 Consider the system $\mathcal{L}$

$$\begin{array}{ll}\{ & x^2\partial_x z_2 - xy\partial_x z_1 + yz_1, \quad x^2\partial_x^2 z_1 - x\partial_x z_1 + z_1, \\ & y\partial_y z_1 - x\partial_x z_1 + z_1, \qquad xy\partial_y z_2 + xy\partial_x z_1 - xz_2 + yz_1\}.\end{array}$$

By elimination we get

$$\mathcal{L}_1 = \{y\partial_y z_1 - x\partial_x z_1 + z_1,\ x^2\partial_x^2 z_1 - x\partial_x z_1 + z_1\}$$

and

$$\mathcal{L}_2 = \{y\partial_y z_2 - xy^2\partial_x z_2 - z_2,\ x^2\partial_x^3 z_2 + 3x\partial_x^2 z_2 + \partial_x z_2\}.$$

By the algorithm **RationalSolution** we find that respective hyperexponential solutions of $\mathcal{L}_1$ and $\mathcal{L}_2$ are c_1x and c_2y, where $c_1, c_2 \in \mathbb{C}$. Substituting c_1x for z_1 and c_2y for z_2 into $\mathcal{L}$ yields the linear system $\{c_1 = 0\}$. Hence, the hyperexponential solutions of $\mathcal{L}$ are $(0, c_2y)$, where $c_2 \in \mathbb{C}$.

References

Boulier, F., Lazard, D., Ollivier, F., Petitot, M. (1995). Representation for the radical of a finitely generated differential ideal. In: Levelt, A.H.M. (ed.): *Proc. Int. Symp. on Symbolic and Algebraic Computation*, Montreal, Canada, ACM Press, 158–166.

Bronstein, M. (1992). Linear ordinary differential equations: breaking through the order 2 barrier. In: Wang. P. (ed.): *Proc. Int. Symp. on Symbolic and Algebraic Computation*, Berkeley, ACM Press, 42–48.

Bronstein, M. (1997). *Symbolic Integration I: Transcendental Functions.* Springer.

Geddes, K., Czapor, S., Labahn, G. (1992). *Algorithms for Computer Algebra.* Kluwer Academic Publishers.

Janet, M. (1920). Les systéms d'équations aux dérivées partielles. *Journal de mathématiques* **83**, 65–123.

Kandru-Rody, A. Weispfenning, V. (1990). Non-commutative Gröbner Bases in algebras of solvable type. *J. Symb. Comput.* **9**, 1–26.

Kolchin, E. (1973). *Differential Algebra and Algebraic Groups.* New York: Academic Press.

Li, Z., Wang, D.-m. (1999) Coherent, regular and simple systems in zero decompositions of partial differential systems. *Syst. Sci. Math. Sci.* **12** Suppl., 43–60.

Lie, S. Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten III. *Archiv for Mathematik* **VIII**, 371–458 (*Gesammelte Abhandlungen* **V**, 362–427).

Reid, W. T. (1972). *Riccati Differential Equations.* Academic Press, New York and London.

Ritt, J. F. (1950). *Differential Algebra.* New York: Amer. Math. Soc.

Rosenfeld, A. (1959). Specializations in differential algebra. *Trans. Amer. Math. Soc.* **90**, 394–407.

Schwarz, F. (1998a). Janet bases for symmetry groups. In: Buchberger, B. and Winkler, F. (eds.): *Gröbner Bases and Applications*, London Math. Soc. Lecture Notes Series **251**. Cambridge: Cambridge University Press, 221–234.

Schwarz, F. (1998b). ALLTYPES: An ALgebraic Language and Type System. In: Calmet J. and Plaza J. (eds.): *Artificial Intelligence and Symbolic Computation*, Lecture Notes in Artificial Intelligence **1476**. Springer, 270 – 283.

Singer, M. (1991). Liouillian solutions of linear differential equations with Liouillian coefficients. *J. Symb. Comput.* **11**, 251–273.

Wu, W.-t. (1989). On the foundation of algebraic differential geometry. *Syst. Sci. Math. Sci.* **2**, 289–312.