MM Research Preprints, 188–192
No. 18, Dec. 1999. Beijing

Cohomology of special 128-groups [1)]

Yujie Ma[2)]

Abstract. Using the Carlson's functions on calculating the cohomology of p-groups, we obtain the cohomology rings of the special 128-groups.

1. Introduction

Let G denote a finite group. One of the basic tools for understanding G is its expression as a group extension

$$1 \to N \to G \to K$$

where N is a normal subgroup in G and K is the quotient G/N . An obvious problem is to try to understand all extensions of this type and indeed determine what different groups can arise in this way. Assume that N is abelian; if we choose a section $\phi : K \to G, k \mapsto g_k$, then the group structure on G implies that for $k, l \in K$ there must exist $n_{k.l} \in N$ satisfying certain "cocycle properties". Group cohomology arises when trying to understand equivalence classes of these functions in a functorial way. In particular, isomorphism classes of extensions as described above are in one-to-one correspondence with these equivalence classes of 2-cocyles, which can be regarded as elements in a "cohomology group" denoted by $H^2(K, N)$. It was, however, the impetus from topology that eventually gave rise to a global definition of group cohomology using machinery from homological algebra, generalizing specific low-dimensional information.

Let us review the essential elements in the general definition. Let Z denote the integers with the trivial action of a finite group G. It is evident that we can map a copy of the group ring ZG onto it with a finitely generated kernel IG (this is, in fact, the augmentation ideal). Now taking generators for IG as a ZG-module, we can map a free ZG -module of finite rank onto IG. Continuing in this way, we obtain a sequence of free ZG-modules and ZG-maps

$$\cdots \longrightarrow F_n \xrightarrow{\partial_n} F_{n-1} \longrightarrow \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

such that at each stage the image of the homomorphism coming in is equal to the kernel of the homomorphism going out (it is said to be exact), and $F_0/\text{image}(\partial_1) = Z$; i.e., F_0 maps onto Z. Such an object is called a free resolution for Z, and an obvious analogue can be constructed for any finitely generated ZG-module taking the place of Z.

Now let A denote any ZG-module. We can consider all G-homomorphisms from the F_1 to A and put these together to obtain a cochain complex.

$$\cdots \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

[1)] This work was supported in part by the NSF Grant CCR-9420857 and by the Chinese National Science Foundation.

[2)] Systems Science Institute, Academia Sinica, 100080, Beijing China

The cohomology of G with coefficients in A is defined as the cohomology of this cochain complex; that is . $H^i(G,A) = H^i(Hom_G(F_*,A))$. Note that the exactness of $F_1 \to F_0 \to Z$ implies the exactness of $0 \to Hom_Z(Z,A) \to Hom_Z(F_0,A) \to Hom_Z(F_1,A)$, whence we see that $H^0(G,A) = Hom_G(Z,Z) = A^G$, the submodule of invariants. Furthermore, if A has a trivial action, then a similar argument shows that $H^1(G,A) = Hom_G(IG,A)$, where IG is the augmentation ideal in ZG. This group is isomorphic to the group of homomorphisms from G to A. A slightly more complicated argument can be used to show that in face $H^2(G,A)$ will recover the isomorphism classes of extensions described previously. If R is any commutative ring with 1, then we can define $H^i(G,M)$ for any finitely generated RG-module M using RG-resolutions of the trivial module R. It turns out that any two resolutions will give rise to the same cohomology groups.

Let us fix as coefficients a field F_p of characteristic p which divides the order of G. If G happens to be a finite p-group, then it is possible to construct minimal resolutions P_* of the trivial module F_p which are directly related to the cohomology date. In fact, we have that in this situation $H^*(G,F_p) \cong Hom_G(P_*,F_P)$, i.e., the cochain complex has zero coboundary maps. from this it is evident that computing minimal resolutions is an important aspect of group cohomology. Indeed , computer-assisted calculations of minimal resolutions can provide substantial information on the low-dimensional cohomology of a finite p-group of reasonable size.

2. Programs

All of the calculations were performed on an SUN. The machine has 512M. of RAM and approximately 10G. of hard drive. All of the programs are written in MAGMA code and run on the MAGMA platform. Most of the functions we adapted are due to Jon Carlson. In Appendix I, we give the program , which we used to do the calculation. In Appendix II, we give a sample of the results. Now, we give the notes of the details of the calculation.

The first step is the computation of a standard free module for the group algebra. In particular, matrices of the action of a minimal set of generators for G and also for a power commutator set of generators for G on the free module kG are fixed. In this way the programs can rely almost entirely on linear algebra. That is, the more time consuming module theoretic machinery is used only at the beginning. Next a minimal projective resolution of the trivial module is computed and stored. Because the rate of growth(complexity) of the projective resolution is related to the 2-rank of G the number of steps that were computed for each resolution varied according to the 2-rank of G. This can be proved using norm maps.

Once the projective resolution is obtained the computer begins constructing chain maps for the cohomology generators. This is a careful process and no excess generators are computed. That is, the cup product of any two cohomology elements is obtained as the composition of the corresponding chain maps. The program computes some of the products as it goes and hence the only generators for which the chain maps are calculated are those which are not polynomials in the other generators. The printed output of this stage of the program is a list of the generators and a list of their degrees.

Following the chain map calculation we get the relations among the generators. That is, for a degree n, within the range of the computation, all possible monomials of degree n in the generators are compute. Each such monomial is a k-homomorphism from the n^{th} term, modulo its radical, in the projective resolution into k. Thus the set of relations among the monomials is the null space of the matrix whose rows are the matrices of these homomorphisms. Hence we have determined all possible relations of degree n in the generators. Of course these information is too massive to be helpful. What we really want is a minimal set of generators for the ideal of relations. To get the ideal we need to call upon the Gröbner basis machinery. The notation is taken directly from the output from MAGMA. Hence "$*$" means multiplication and "^{n}" means n^{th} power.

3. Appendix I

```
// Input the Carlson's cohomology-ring-functions
load "ProjectiveResolutionNP1";
load "AllChainMapsNP1";
load "RelationsNP1";
load "Automorphisms";
load "ChainMap";
load "General";
load "MaxEle";
load "RingOps";
load "ResolutionAccess";

// Input the library of 128-groups
// Choose a special128-group to calculate
load gen2;
load twogp128;

// Record the results of the calculation
SetLogFile("special.log");
    for i:=1 to 2328 do
G:=gen2(gps[i], 0); if
IsSpecial(G) then
k:=i;
print "THE FOLLOWING IS THE COHOMOLOGY OF No.[", k, "]";
print gps[k];
G:=gen2(gps[k],0);
gg:=CosetImage(G,sub¡G —Id(G)¿);
ff:=GF(2);
Inn:=ResolutionInput(gg,ff);
kk:=TrivialModule(gg,ff);
prop:=ProjectiveResolution(kk,Inn,4);
all:=AllChainMaps(prop);
rel:=CohomologyRelations(prop,all);
rel;
end if;
end for;
```

4. Appendix II

THE FOLLOWING IS THE COHOMOLOGY OF No.[1136]
$[4, 7, 2, 4398050732032]$
Graded Polynomial ring of rank 10 over GF(2)
Graded Reverse Lexicographical Order
Variables: $z, y, x, w, v, u, t, s, r, q$
Variable weights: $1, 1, 1, 1, 2, 2, 2, 2, 3, 4$
Graded Polynomial ring of rank 10 over $GF(2)$
Graded Reverse Lexicographical Order
Variables: $z, y, x, w, v, u, t, s, r, q$
Variable weights: $1, 1, 1, 1, 2, 2, 2, 2, 3, 4$, Ideal of Graded Polynomial ring of rank 10 over GF(2)
Graded Reverse Lexicographical Order
Variables: $z, y, x, w, v, u, t, s, r, q$
Variable weights: $1, 1, 1, 1, 2, 2, 2, 2, 3, 4$
Groebner basis:

$$
\begin{aligned}
&[u^3t^2 + ut^4 + yvt^2r,\\
&z^3ws + y^3ws + z^2us + y^2ts + ywts + u^3 + ut^2 + yvr + ysr,\\
&z^3s^2 + y^3s^2 + y^2ws^2 + zu^2t + zt^3 + zu^2s + zt^2s,\\
&z^2us^2 + y^2ts^2 + ywts^2 + u^3t + ut^3 + u^3s + ut^2s + yvtr + yvsr + ys^2r,\\
&yt^3r + utr^2,\\
&y^2w^2, z^3t + y^3t + y^2wt + z^3s + y^3s + y^2ws + zu^2 + zt^2,\\
&yw^2t + ywr,\\
&z^2ut + y^2t^2 + ywt^2 + z^2us + y^2ts + ywts + u^3 + ut^2 + yvr + ytr + ysr,\\
&z^2t^2 + y^2t^2, zut^2 + yt^3 + utr,\\
&ywtr + yr^2, ut^2r + yvr^2, zw^2,\\
&y^2v + z^2t + z^2s + u^2,\\
&ywv + z^2t + y^2t + ywt + z^2s + y^2s + zws + u^2 + ut + yr,\\
&w^2v + z^2t + y^2t + z^2s + y^2s + u^2 + t^2 + xr,\\
&zwt + ywt + yr,\\
&y^2r, u^2r, zy + y^2 + zw,\\
&zx, yx, yu + zt, xu, wu + zt + yt, zr]
\end{aligned}
$$

$$
\begin{aligned}
&[zy + y^2 + zw, zx, yx, zw^2, yu + zt, xu,\\
&wu + zt + yt, y^2v + z^2t + z^2s + u^2,\\
&ywv + z^2t + y^2t + ywt + z^2s + y^2s + zws + u^2 + ut + yr,\\
&w^2v + z^2t + y^2t + z^2s + y^2s + u^2 + t^2 + xr,\\
&zwt + ywt + yr, zr]
\end{aligned}
$$

References

[1] A.Adem, Recent developments in the cohomology of finite groups, *Notices of the AMS*, Vol. 44., No.7 (1997) 806-812.

[2] D.J. Benson, *Modular Representation Theory: New Trends and Methods,*d Lecture Notes in Mathematics, Vol. 1081 (Springer, Berlin, 1984.)

[3] J.F. Carlson, *Modules and Group Algebra,* ETH Lecture Notes, Birkhäuser Verlag, 1996.

[4] T. Diethelm, On the cohomology of groups of p-length 1, *Comment, Math. Helv.* **58** (1983), 379-387.

[5] L. Evens, *Cohomology of Groups,* Oxford Univ. Press, 1992.

[6] P.J. Hilton and U. Stammbach, *A Course in Homological Algebra*, GTM **4**, 1971.

[7] R.J. Milgram, *The cohomology of the Mathieu group M_{23},* Preprint, 1993.

[8] D. Quillen, The spectrum of an equivariant cohomology ring I, *Ann. of Math.* **94** (1971), 549-572.

[9] J.Y. Whitt, *Computing the cohomology of groups with trivial intersection Sylow p-subgroups,* Dissertation of Univ. of Georgia, 1998.