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Reduced form of Yang-Mills equation of $SU(3)$ on $R^{4,0}$

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Abstract. The Yang-Mills equation of $SU(3)$ on $R^{4,0}$ is studied in this note. An explicit expression of a differential linear transformation is given. The Yang-Mills equation of $SU(3)$ on Euclidean four-dimensional flat space becomes a reduced form via this differential linear transformation. And then, some solutions of the Yang-Mills equation could to be obtained very easily.

1. Notations

Let $R^{4,0}$ be the Euclidean four-dimensional flat space and $x = (x^1, x^2, x^3, x^4)$ be the coordinates. The matrix $ds^2 = \sum g^{ij}dx^i dx^j$ of $R^{4,0}$ is given by $g^{ii} = 1,\ \ i = 1, 2, 3, 4,\ \ g^{ij} = 0,\ i \neq j$. Let $gl(3)$ be the Lie algebra of group $SU(3)$. The anti-Hermitian representations of a basis of $gl(3)$ are taken as follows

$$X_1 = \begin{pmatrix} \mathbf{i} & 0 & 0 \\ 0 & -\mathbf{i} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & \mathbf{i} & 0 \\ \mathbf{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & \mathbf{i} \\ 0 & 0 & 0 \\ \mathbf{i} & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{i} \\ 0 & \mathbf{i} & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$X_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{i} & 0 \\ 0 & 0 & -\mathbf{i} \end{pmatrix},$$

in which $\mathbf{i} = \sqrt{-1}$. There are several choices of the base X_8. For convenient to compute by using computer, we set X_8 as above.

There are 32 unknown functions $A^{ij}(x),\ i = 1, 2, \cdots, 8,\ j = 1, 2, 3, 4$. Their connections are given by

$$G_j = \sum_{i=1}^{8} A^{ij}(x) X_i,\ j = 1, 2, 3, 4.$$

That is

$$G_j = \begin{pmatrix} \mathbf{i}\,A^{1j} & \mathbf{i}\,A^{2j} - A^{5j} & \mathbf{i}\,A^{3j} - A^{6j} \\ \mathbf{i}\,A^{2j} + A^{5j} & -\mathbf{i}\,A^{1j} + \mathbf{i}\,A^{8j} & \mathbf{i}\,A^{4j} - A^{7j} \\ \mathbf{i}\,A^{3j} + A^{6j} & \mathbf{i}\,A^{4j} + A^{7j} & -\mathbf{i}\,A^{8j} \end{pmatrix},$$

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$$j = 1, 2, 3, 4.$$

The curvatures are given by

$$F_{jk} = \partial_k G_j - \partial_j G_k + G_j G_k - G_k G_j,\ j, k = 1, 2, 3, 4,$$

in which the notations $\partial_j = \partial/\partial x^j,\ j = 1, 2, 3, 4.$

The Yang-Mills equation of $SU(3)$ on $R^{4,0}$ is

$$\sum_{k=1}^{4} g^{kk}(\partial_k F_{jk} + G_k F_{jk} - F_{jk} G_k) = 0,\ j = 1, 2, 3, 4. \tag{1.1}$$

Note that the Yang-Mills equation in formula (1.1)is a 3×3 matrix:

$$\begin{pmatrix} Ym_{j1} & Ym_{j2} - Ym_{j3} & Ym_{j4} - Ym_{j5} \\ Ym_{j2} + Ym_{j3} & Ym_{j1} + YM_{j9} & Ym_{j7} - Ym_{j8} \\ Ym_{j4} + Ym_{j5} & Ym_{j7} + Ym_{j8} & Ym_{j9} \end{pmatrix}.$$

2. Differential linear transformation

Using the differential liner transformation, we obtain the reduced form of Yang -Mills equations of $SU(2)$ on Euclidean space $R^{4,0}$ (see [6]), Mincovski space $R^{3,1}$(see [4]) and Artin space $R^{2,2}$(see [5]). The goal of this note is to give the reduced form of Yang-Mills equation of $SU(3)$ on Euclidean flat space $R^{4,0}$. Similarly to [6], we have

Definition a transformation $S : u(x) \longmapsto v(x), v = C(u)$ between functions $u(x)$ and $v(x)$ is called a **differential linear transformation** if C(u) consists of linear differential operators only.

In [7], Maxwell equation is studied by using differential linear transformation. Now we set

$$A^{ij}(x) = \sum_{k=1}^{4} s_{jk} \partial_k u^i(x), \quad i = 1, 2, \cdots, 8, \quad j = 1, 2, 3, 4, \tag{1.2}$$

in which $u^i(x)$ are eight new functions and $s_{jk},\ j, k = 1, 2, 3, 4$ are undetermined constants. We denote them by a matrix

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}.$$

3. Reduced form of Yang-Mills equation

For $R^{4,0}$, we substitute $A^{ij}(x),\ i = 1, 2, \cdots, 8,\ j = 1, 2, 3, 4$ which are given by formula (1.2) in Yang-Mills equation (1.1) and obtain certain product terms such as

$$\partial_j u^s(x) \partial_k u^t(x) \partial_l u^w(x),\ j, k, l = 1, 2, 3, 4, \quad s, t, w = 1, 2, \cdots, 6$$

The coefficients of these terms are polynomials of undetermined constants $s_{jk},\ j,k=1,2,3,4$. We set these polynomials equal to zeros and then obtain a system of polynomial equations. In paper [6] by using Wu elimination [1,2], we get the matrix as following

$$S_{40}=s\begin{pmatrix} 1 & \mathbf{i} & -1 & \mathbf{i} \\ -\mathbf{i} & 1 & -\mathbf{i} & -1 \\ 1 & \mathbf{i} & 1 & -\mathbf{i} \\ -\mathbf{i} & 1 & \mathbf{i} & 1 \end{pmatrix} \tag{1.3}$$

in which s is a parameter. The main conclusion of this note is

Theorem By the differential linear transformation given by formula (1.2) and (1.3), the Yang-Mills equation (1.1) turns out to be a reduced form

$$YM_{ij}=\sum_{k=1}^{8} L_{ijk}D_{40}(u^k),\quad i=1,2,3,4,\quad j=1,2,\cdots,8 \tag{1.4}$$

in which

$$D_{40}=\partial_1^2+\partial_2^2+\partial_3^2+\partial_4^2$$

is the Laplace operator and $L_{ijk}\quad i=1,2,3,4,\quad j=1,2,\cdots,8,\ k=1,2,\cdots,8$ are certain linear differential operators. For example, for $i=1$ we have

$$\begin{aligned} YM_{11}= \ & s(-\mathbf{i}\partial_2+\partial_3-\mathbf{i}\partial_4)D_{40}(u^1)\\ & -2s^2\,\mathbf{i}\,L_{40}(u^5)D_{40}(u^2)-2s^2\,\mathbf{i}\,L_{40}(u^6)D_{40}(u^3)\\ & +2s^2\,\mathbf{i}\,L_{40}(u^2)D_{40}(u^5)+2s^2\,\mathbf{i}\,L_{40}(u^3)D_{40}(u^6) \end{aligned}$$

$$\begin{aligned} YM_{12}= \ & s(-\mathbf{i}\partial_2+\partial_3-\mathbf{i}\partial_4)D_{40}(u^2)\\ & -2s^2L_{40}(u^5)D_{40}(u^1)+s^2L_{40}(u^7)D_{40}(u^3)\\ & +s^2L_{40}(u^6)D_{40}(u^4)-s^2L_{40}(u^4)D_{40}(u^6)\\ & -s^2L_{40}(u^3)D_{40}(u^7)+s^2L_{40}(u^5)D_{40}(u^8)\\ & +s^2(\,2\,L_{40}(u^1)-L_{40}(u^8))D_{40}(u^5) \end{aligned}$$

$$\begin{aligned} YM_{13}= \ & s(-\mathbf{i}\partial_2+\partial_3-\mathbf{i}\partial_4)D_{40}(u^3)\\ & -s^2L_{40}(u^6)D_{40}(u^1)-s^2L_{40}(u^7)D_{40}(u^2)\\ & +s^2L_{40}(u^5)D_{40}(u^4)-s^2L_{40}(u^4)D_{40}(u^5)\\ & +s^2L_{40}(u^2)D_{40}(u^7)-s^2L_{40}(u^6)D_{40}(u^8)\\ & +s^2(L_{40}(u^1)+L_{40}(u^8))D_{40}(u^6) \end{aligned}$$

$$\begin{aligned} YM_{14}= \ & s(-\mathbf{i}\partial_2+\partial_3-\mathbf{i}\partial_4)D_{40}(u^4)\\ & +s^2L_{40}(u^7)D_{40}(u^1)-s^2L_{40}(u^6)D_{40}(u^2)\\ & -s^2L_{40}(u^5)D_{40}(u^3)+s^2L_{40}(u^3)D_{40}(u^5)\\ & +s^2L_{40}(u^2)D_{40}(u^6)-2s^2L_{40}(u^7)D_{40}(u^8)\\ & +s^2(2\,L_{40}(u^8)-L_{40}(u^1))D_{40}(u^7) \end{aligned}$$

$$\begin{aligned} YM_{15}= \ & s(-\mathbf{i}\partial_2+\partial_3-\mathbf{i}\partial_4)D_{40}(u^5)\\ & +2s^2L_{40}(u^2)D_{40}(u^1)+s^2L_{40}(u^4)D_{40}(u^3)\\ & -s^2L_{40}(u^3)D_{40}(u^4)+s^2L_{40}(u^7)D_{40}(u^6)\\ & -s^2L_{40}(u^6)D_{40}(u^7)-s^2L_{40}(u^2)D_{40}(u^8)\\ & +s^2(\,2\,L_{40}(u^1)-L_{40}(u^8))D_{40}(u^2) \end{aligned}$$

$$\begin{aligned} YM_{16} = \ & s(-\mathbf{i}\partial_2 + \partial_3 - \mathbf{i}\partial_4)D_{40}(u^6) \\ & + s^2 L_{40}(u^3)D_{40}(u^1) + s^2 L_{40}(u^4)D_{40}(u^2) \\ & - s^2 L_{40}(u^2)D_{40}(u^4) - s^2 L_{40}(u^7)D_{40}(u^5) \\ & + s^2 L_{40}(u^5)D_{40}(u^7) + s^2 L_{40}(u^3)D_{40}(u^8) \\ & - s^2(L_{40}(u^1) + L_{40}(u^8))D_{40}(u^3) \end{aligned}$$

$$\begin{aligned} YM_{17} = \ & s(-\mathbf{i}\partial_2 + \partial_3 - \mathbf{i}\partial_4)D_{40}(u^7) \\ & + s^2 L_{40}(u^4)D_{40}(u^1) - s^2 L_{40}(u^3)D_{40}(u^2) \\ & + s^2 L_{40}(u^2)D_{40}(u^3) - s^2 L_{40}(u^6)D_{40}(u^5) \\ & + s^2 L_{40}(u^5)D_{40}(u^6) - 2s^2 L_{40}(u^4)D_{40}(u^8) \\ & + s^2(2\, L_{40}(u^8) - L_{40}(u^1))D_{40}(u^4) \end{aligned}$$

$$\begin{aligned} YM_{18} = \ & s(-\mathbf{i}\partial_2 + \partial_3 - \mathbf{i}\partial_4)D_{40}(u^8) \\ & + 2s^2\, \mathbf{i}\, L_{40}(u^6)D_{40}(u^3) + 2s^2\, \mathbf{i}\, L_{40}(u^7)D_{40}(u^4) \\ & - 2s^2\, \mathbf{i}\, L_{40}(u^3)D_{40}(u^6) - 2s^2\, \mathbf{i}\, L_{40}(u^4)D_{40}(u^7) \end{aligned}$$

where $L_{40} = -\mathbf{i}\partial_1 + \partial_2 + \mathbf{i}\partial_3 + \partial_4$, a linear differential operator. Note that $\mathbf{i}\, L_{40} = \partial_1 + \mathbf{i}\partial_2 - \partial_3 + \mathbf{i}\partial_4$.

Proof Formula (1.4) can be obtained via direct computation.

From this reduced form, some solutions can be obtained easily. Especially, any solution $u^i(x)$ of

$$D_{40}(u^i) = 0$$

provide a solution of Yang-Mills equation of $SU(3)$ on $R^{4,0}$. Thus we have

Proposition. By using differential linear transformation given by (1.2) and (1.3), a lot of solutions of Yang-Mills equation of $SU(3)$ on R^{40} can be obtained from the reduced form of Yang-Mills equation

Appendix

The explicit form of the reduced form of Yang-Mills equations $YM_{ij},\ i = 2,3,4,\ j = 1,2,\cdots,8$ are given in this appendix.

$i = 2$ the reduced form of Yang-Mills equation is :

$$\begin{aligned} YM_{21} = \ & s(\mathbf{i}\partial_1 + \mathbf{i}\partial_3 + \partial_4)D_{40}(u^1) \\ & + 2s^2 L_{40}(u^5)D_{40}(u^2) + 2s^2 L_{40}(u^6)D_{40}(u^3) \\ & - 2s^2 L_{40}(u^2)D_{40}(u^5) - 2s^2 L_{40}(u^3)D_{40}(u^6) \end{aligned}$$

$$\begin{aligned} YM_{22} = \ & s(\mathbf{i}\partial_1 + \mathbf{i}\partial_3 + \partial_4)D_{40}(u^2) \\ & - 2\,\mathbf{i}\, s^2 L_{40}(u^5)D_{40}(u^1) + \mathbf{i}\, s^2 L_{40}(u^7)D_{40}(u^3) \\ & + \mathbf{i}\, s^2 L_{40}(u^6)D_{40}(u^4) - \mathbf{i}\, s^2 L_{40}(u^4)D_{40}(u^6) \\ & - \mathbf{i}\, s^2 L_{40}(u^3)D_{40}(u^7) + \mathbf{i}\, s^2 L_{40}(u^5)D_{40}(u^8) \\ & + \mathbf{i}\, s^2(\, 2\, L_{40}(u^1) - L_{40}(u^8))D_{40}(u^5) \end{aligned}$$

$$
\begin{aligned}
YM_{23} = \;& s(\mathbf{i}\partial_1 + \mathbf{i}\partial_3 + \partial_4)D_{40}(u^3) \\
& -\mathbf{i}\, s^2 L_{40}(u^6)D_{40}(u^1) - \mathbf{i}\, s^2 L_{40}(u^7)D_{40}(u^2) \\
& +\mathbf{i}\, s^2 L_{40}(u^5)D_{40}(u^4) - \mathbf{i}\, s^2 L_{40}(u^4)D_{40}(u^5) \\
& +\mathbf{i}\, s^2 L_{40}(u^2)D_{40}(u^7) - \mathbf{i}\, s^2 L_{40}(u^6)D_{40}(u^8) \\
& +\mathbf{i}\, s^2 (L_{40}(u^1) + L_{40}(u^8))D_{40}(u^6)
\end{aligned}
$$

$$
\begin{aligned}
YM_{24} = \;& s(\mathbf{i}\partial_1 + \mathbf{i}\partial_3 + \partial_4)D_{40}(u^4) \\
& +\mathbf{i}\, s^2 L_{40}(u^7)D_{40}(u^1) - \mathbf{i}\, s^2 L_{40}(u^6)D_{40}(u^2) \\
& -\mathbf{i}\, s^2 L_{40}(u^5)D_{40}(u^3) + \mathbf{i}\, s^2 L_{40}(u^3)D_{40}(u^5) \\
& +\mathbf{i}\, s^2 L_{40}(u^2)D_{40}(u^6) - 2\mathbf{i}\, s^2 L_{40}(u^7)D_{40}(u^8) \\
& +\mathbf{i}\, s^2 (2\, L_{40}(u^8) - L_{40}(u^1))D_{40}(u^7)
\end{aligned}
$$

$$
\begin{aligned}
YM_{25} = \;& s(\mathbf{i}\partial_1 + \mathbf{i}\partial_3 + \partial_4)D_{40}(u^5) \\
& +2\mathbf{i}\, s^2 L_{40}(u^2)D_{40}(u^1) + \mathbf{i}\, s^2 L_{40}(u^4)D_{40}(u^3) \\
& -\mathbf{i}\, s^2 L_{40}(u^3)D_{40}(u^4) + \mathbf{i}\, s^2 L_{40}(u^7)D_{40}(u^6) \\
& -\mathbf{i}\, s^2 L_{40}(u^6)D_{40}(u^7) - \mathbf{i}\, s^2 L_{40}(u^2)D_{40}(u^8) \\
& -\mathbf{i}\, s^2 (\, 2\, L_{40}(u^1) - L_{40}(u^8))D_{40}(u^2)
\end{aligned}
$$

$$
\begin{aligned}
YM_{26} = \;& s(\mathbf{i}\partial_1 + \mathbf{i}\partial_3 + \partial_4)D_{40}(u^6) \\
& +\mathbf{i}\, s^2 L_{40}(u^3)D_{40}(u^1) + \mathbf{i}\, s^2 L_{40}(u^4)D_{40}(u^2) \\
& -\mathbf{i}\, s^2 L_{40}(u^2)D_{40}(u^4) - \mathbf{i}\, s^2 L_{40}(u^7)D_{40}(u^5) \\
& +\mathbf{i}\, s^2 L_{40}(u^5)D_{40}(u^7) + \mathbf{i}\, s^2 L_{40}(u^3)D_{40}(u^8) \\
& -\mathbf{i}\, s^2 (L_{40}(u^1) + L_{40}(u^8))D_{40}(u^3)
\end{aligned}
$$

$$
\begin{aligned}
YM_{27} = \;& s(\mathbf{i}\partial_1 + \mathbf{i}\partial_3 + \partial_4)D_{40}(u^7) \\
& -\mathbf{i}\, s^2 L_{40}(u^4)D_{40}(u^1) + \mathbf{i}\, s^2 L_{40}(u^3)D_{40}(u^2) \\
& -\mathbf{i}\, s^2 L_{40}(u^2)D_{40}(u^3) + \mathbf{i}\, s^2 L_{40}(u^6)D_{40}(u^5) \\
& -\mathbf{i}\, s^2 L_{40}(u^5)D_{40}(u^6) + 2\mathbf{i}\, s^2 L_{40}(u^4)D_{40}(u^8) \\
& -\mathbf{i}\, s^2 (2\, L_{40}(u^8) - L_{40}(u^1))D_{40}(u^4)
\end{aligned}
$$

$$
\begin{aligned}
YM_{28} = \;& s(\mathbf{i}\partial_1 + \mathbf{i}\partial_3 + \partial_4)D_{40}(u^8) \\
& +2s^2 L_{40}(u^6)D_{40}(u^3) + 2s^2 L_{40}(u^7)D_{40}(u^4) \\
& -2s^2 L_{40}(u^3)D_{40}(u^6) - 2s^2 L_{40}(u^4)D_{40}(u^7)
\end{aligned}
$$

where $L_{40} = -\mathbf{i}\partial_1 + \partial_2 - \mathbf{i}\partial_3 - \partial_4$.

$i = 3$ the reduced form of Yang-Mills equation is :

$$
\begin{aligned}
YM_{31} = \;& s(\partial_1 + \mathbf{i}\partial_2 - \mathbf{i}\partial_4)D_{40}(u^1) \\
& -2\mathbf{i}\, s^2 L_{40}(u^5)D_{40}(u^2) - 2\mathbf{i}\, s^2 L_{40}(u^6)D_{40}(u^3) \\
& +2\mathbf{i}\, s^2 L_{40}(u^2)D_{40}(u^5) + 2\mathbf{i}\, s^2 L_{40}(u^3)D_{40}(u^6)
\end{aligned}
$$

$$
\begin{aligned}
YM_{32} = \;& s(\partial_1 + \mathbf{i}\partial_2 - \mathbf{i}\partial_4)D_{40}(u^2) \\
& -2s^2 L_{40}(u^5)D_{40}(u^1) + s^2 L_{40}(u^7)D_{40}(u^3) \\
& +s^2 L_{40}(u^6)D_{40}(u^4) - s^2 L_{40}(u^4)D_{40}(u^6) \\
& -s^2 L_{40}(u^3)D_{40}(u^7) + s^2 L_{40}(u^5)D_{40}(u^8) \\
& +s^2 (\, 2\, L_{40}(u^1) - L_{40}(u^8))D_{40}(u^5)
\end{aligned}
$$

$$\begin{aligned} YM_{33} = \ & s(\partial_1 + \mathbf{i}\partial_2 - \mathbf{i}\partial_4)D_{40}(u^3) \\ & -s^2 L_{40}(u^6)D_{40}(u^1) - s^2 L_{40}(u^7)D_{40}(u^2) \\ & +s^2 L_{40}(u^5)D_{40}(u^4) - s^2 L_{40}(u^4)D_{40}(u^5) \\ & +s^2 L_{40}(u^2)D_{40}(u^7) - s^2 L_{40}(u^6)D_{40}(u^8) \\ & +s^2(L_{40}(u^1) + L_{40}(u^8))D_{40}(u^6) \end{aligned}$$

$$\begin{aligned} YM_{34} = \ & s(\partial_1 + \mathbf{i}\partial_2 - \mathbf{i}\partial_4)D_{40}(u^4) \\ & +\mathbf{i}\, s^2 L_{40}(u^7)D_{40}(u^1) - \mathbf{i}\, s^2 L_{40}(u^6)D_{40}(u^2) \\ & -\mathbf{i}\, s^2 L_{40}(u^5)D_{40}(u^3) + \mathbf{i}\, s^2 L_{40}(u^3)D_{40}(u^5) \\ & +\mathbf{i}\, s^2 L_{40}(u^2)D_{40}(u^6) - 2\mathbf{i}\, s^2 L_{40}(u^7)D_{40}(u^8) \\ & +\mathbf{i}\, s^2(2\, L_{40}(u^8) - L_{40}(u^1))D_{40}(u^7) \end{aligned}$$

$$\begin{aligned} YM_{35} = \ & s(\partial_1 + \mathbf{i}\partial_3 - \mathbf{i}\partial_4)D_{40}(u^5) \\ & +2s^2 L_{40}(u^2)D_{40}(u^1) + s^2 L_{40}(u^4)D_{40}(u^3) \\ & -s^2 L_{40}(u^3)D_{40}(u^4) + s^2 L_{40}(u^7)D_{40}(u^6) \\ & -s^2 L_{40}(u^6)D_{40}(u^7) - s^2 L_{40}(u^2)D_{40}(u^8) \\ & -s^2(\, 2\, L_{40}(u^1) - L_{40}(u^8))D_{40}(u^2) \end{aligned}$$

$$\begin{aligned} YM_{36} = \ & s(\partial_1 + \mathbf{i}\partial_2 + \mathbf{i}\partial_4)D_{40}(u^6) \\ & +s^2 L_{40}(u^3)D_{40}(u^1) + s^2 L_{40}(u^4)D_{40}(u^2) \\ & -s^2 L_{40}(u^2)D_{40}(u^4) - s^2 L_{40}(u^7)D_{40}(u^5) \\ & +s^2 L_{40}(u^5)D_{40}(u^7) + s^2 L_{40}(u^3)D_{40}(u^8) \\ & -s^2(L_{40}(u^1) + L_{40}(u^8))D_{40}(u^3) \end{aligned}$$

$$\begin{aligned} YM_{37} = \ & s(\partial_1 + \mathbf{i}\partial_3 - \mathbf{i}\partial_4)D_{40}(u^7) \\ & -s^2 L_{40}(u^4)D_{40}(u^1) + s^2 L_{40}(u^3)D_{40}(u^2) \\ & -s^2 L_{40}(u^2)D_{40}(u^3) + s^2 L_{40}(u^6)D_{40}(u^5) \\ & -s^2 L_{40}(u^5)D_{40}(u^6) + 2s^2 L_{40}(u^4)D_{40}(u^8) \\ & -s^2(2\, L_{40}(u^8) - L_{40}(u^1))D_{40}(u^4) \end{aligned}$$

$$\begin{aligned} YM_{38} = \ & s(\partial_1 + \mathbf{i}\partial_3 - \mathbf{i}\partial_4)D_{40}(u^8) \\ & -2\mathbf{i}\, s^2 L_{40}(u^6)D_{40}(u^3) - 2\mathbf{i}\, s^2 L_{40}(u^7)D_{40}(u^4) \\ & +2\mathbf{i}\, s^2 L_{40}(u^3)D_{40}(u^6) + 2\mathbf{i}\, s^2 L_{40}(u^4)D_{40}(u^7) \end{aligned}$$

where $L_{40} = -\mathbf{i}\partial_1 + \partial_2 - \mathbf{i}\partial_3 - \partial_4$.

$i = 4$ the reduced form of Yang-Mills equation is :

$$\begin{aligned} YM_{41} = \ & s(\mathbf{i}\partial_1 - \partial_2 - \mathbf{i}\partial_3)D_{40}(u^1) \\ & +2s^2 K_{40}(u^5)D_{40}(u^2) + 2s^2 K_{40}(u^6)D_{40}(u^3) \\ & -2s^2 K_{40}(u^2)D_{40}(u^5) - 2s^2 K_{40}(u^3)D_{40}(u^6) \end{aligned}$$

$$\begin{aligned} YM_{42} = \ & s(\mathbf{i}\partial_1 - \partial_2 - \mathbf{i}\partial_3)D_{40}(u^2) \\ & -2\,\mathbf{i}\, s^2 K_{40}(u^5)D_{40}(u^1) + \mathbf{i}\, s^2 K_{40}(u^7)D_{40}(u^3) \\ & +\mathbf{i}\, s^2 K_{40}(u^6)D_{40}(u^4) - \mathbf{i}\, s^2 K_{40}(u^4)D_{40}(u^6) \\ & -\mathbf{i}\, s^2 K_{40}(u^3)D_{40}(u^7) + \mathbf{i}\, s^2 K_{40}(u^5)D_{40}(u^8) \\ & +\mathbf{i}\, s^2(\, 2\, K_{40}(u^1) - K_{40}(u^8))D_{40}(u^5) \end{aligned}$$

$$
\begin{aligned}
YM_{43} = \;& s(\mathbf{i}\partial_1 - \partial_2 - \mathbf{i}\partial_3)D_{40}(u^3) \\
& -\mathbf{i}\, s^2 K_{40}(u^6)D_{40}(u^1) - \mathbf{i}\, s^2 K_{40}(u^7)D_{40}(u^2) \\
& +\mathbf{i}\, s^2 K_{40}(u^5)D_{40}(u^4) - \mathbf{i}\, s^2 K_{40}(u^4)D_{40}(u^5) \\
& +\mathbf{i}\, s^2 K_{40}(u^2)D_{40}(u^7) - \mathbf{i}\, s^2 K_{40}(u^6)D_{40}(u^8) \\
& +\mathbf{i}\, s^2 (K_{40}(u^1) + K_{40}(u^8))D_{40}(u^6)
\end{aligned}
$$

$$
\begin{aligned}
YM_{44} = \;& s(\mathbf{i}\partial_1 - \partial_2 - \mathbf{i}\partial_3)D_{40}(u^4) \\
& +\mathbf{i}\, s^2 K_{40}(u^7)D_{40}(u^1) - \mathbf{i}\, s^2 K_{40}(u^6)D_{40}(u^2) \\
& -\mathbf{i}\, s^2 K_{40}(u^5)D_{40}(u^3) + \mathbf{i}\, s^2 K_{40}(u^3)D_{40}(u^5) \\
& +\mathbf{i}\, s^2 K_{40}(u^2)D_{40}(u^6) - 2\mathbf{i}\, s^2 K_{40}(u^7)D_{40}(u^8) \\
& +\mathbf{i}\, s^2 (2\, K_{40}(u^8) - K_{40}(u^1))D_{40}(u^7)
\end{aligned}
$$

$$
\begin{aligned}
YM_{45} = \;& s(\mathbf{i}\partial_1 - \partial_2 - \mathbf{i}\partial_3)D_{40}(u^5) \\
& +2\mathbf{i}\, s^2 K_{40}(u^2)D_{40}(u^1) + \mathbf{i}\, s^2 K_{40}(u^4)D_{40}(u^3) \\
& -\mathbf{i}\, s^2 K_{40}(u^3)D_{40}(u^4) + \mathbf{i}\, s^2 K_{40}(u^7)D_{40}(u^6) \\
& -\mathbf{i}\, s^2 K_{40}(u^6)D_{40}(u^7) - \mathbf{i}\, s^2 K_{40}(u^2)D_{40}(u^8) \\
& -\mathbf{i}\, s^2 (\, 2\, K_{40}(u^1) - K_{40}(u^8))D_{40}(u^2)
\end{aligned}
$$

$$
\begin{aligned}
YM_{46} = \;& s(\mathbf{i}\partial_1 - \partial_2 - \mathbf{i}\partial_3)D_{40}(u^6) \\
& +\mathbf{i}\, s^2 K_{40}(u^3)D_{40}(u^1) + \mathbf{i}\, s^2 K_{40}(u^4)D_{40}(u^2) \\
& -\mathbf{i}\, s^2 K_{40}(u^2)D_{40}(u^4) - \mathbf{i}\, s^2 K_{40}(u^7)D_{40}(u^5) \\
& +\mathbf{i}\, s^2 K_{40}(u^5)D_{40}(u^7) + \mathbf{i}\, s^2 K_{40}(u^3)D_{40}(u^8) \\
& -\mathbf{i}\, s^2 (K_{40}(u^1) + K_{40}(u^8))D_{40}(u^3)
\end{aligned}
$$

$$
\begin{aligned}
YM_{47} = \;& s(\mathbf{i}\partial_1 - \partial_2 - \mathbf{i}\partial_3)D_{40}(u^7) \\
& -\mathbf{i}\, s^2 K_{40}(u^4)D_{40}(u^1) + \mathbf{i}\, s^2 K_{40}(u^3)D_{40}(u^2) \\
& -\mathbf{i}\, s^2 K_{40}(u^2)D_{40}(u^3) + \mathbf{i}\, s^2 K_{40}(u^6)D_{40}(u^5) \\
& -\mathbf{i}\, s^2 K_{40}(u^5)D_{40}(u^6) + 2\mathbf{i}\, s^2 K_{40}(u^4)D_{40}(u^8) \\
& -\mathbf{i}\, s^2 (2\, K_{40}(u^8) - K_{40}(u^1))D_{40}(u^4)
\end{aligned}
$$

$$
\begin{aligned}
YM_{48} = \;& s(\mathbf{i}\partial_1 - \partial_2 - \mathbf{i}\partial_3)D_{40}(u^8) \\
& +2s^2 K_{40}(u^6)D_{40}(u^3) + 2s^2 K_{40}(u^7)D_{40}(u^4) \\
& -2s^2 K_{40}(u^3)D_{40}(u^6) - 2s^2 K_{40}(u^4)D_{40}(u^7)
\end{aligned}
$$

where $K_{40} = -\mathbf{i}\partial_1 + \partial_2 + \mathbf{i}\partial_3 + \partial_4$.

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