Blending Quadric Surfaces Via Base Curve Method

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Abstract. A method for blending surfaces (implicit or parametric) is introduced. The blending surface is defined by a collection of curves generated through the same base curve and has a parametric representation. Here the given surfaces are not restricted to any particular type of surface representation as long as they have a well-defined and continuous normal vector at each point of their blending boundaries. In this paper, we mainly discussed the blending problems of quadratic surfaces. In particular, we derive the uniform parametric blending surface for six quadratic surfaces with closed blending boundaries at the first time. We also use the method to solve n -way quadratic closed surfaces blending. The method is extensible to blend general surfaces, although we concentrate on quadratic surfaces.

1. Introduction

One of the fundamental tasks of CAGD is surface blending. There are several methods to solve the problem. For example, Hoffmann and Hopcroft proposed the potential method in 1986 (see [6]); Warren proposed the ideal theory method in 1989 (see [9]); Bloor and Wilson proposed PDE method in 1989 (see [2]); Bajaj and Ihm proposed Hermite interpolation method in 1992 (see [1]); Wu and Wang proposed Wu's method in 1994 (see [11]); Zhu and Jin proposed the generatrix method in 1998 (see [7]); Wu and Zhou in 1995 (see [10]); Hartmann in 1994 and 2001 (see [4, 5]); Rossignac and Requicha proposed rolling ball method in 1984 (see [8]), Chen et al.(see [3]) use piecewise algebraic surfaces to blend pipe surfaces, and so on. Hartmman([5]) introduced a method for constructing G^n -continuous transition surfaces between two given normal ringed surfaces based on a recent G^n -blending method for parametric curves. Here a ringed surface is a surface generated by sweeping a circle with non constant radius along a curve. The ringed surface is called normal if the circle is contained in the normal planes of the curve. But the method is only fit for a special kind of surfaces. Chen et al.([3]) presented a scheme to blend three cylinders with piecewise cubic algebraic surfaces. They used six algebraic surfaces to form the whole blending algebraic surface with degree three. But to get one part of the blending surface, one needs complicated computation. And it is not easy to get the range of the parameters of the blending surface we need. Zhu and Jin([7]) presented a method which was based on generatrix for blending round or elliptical tubes. The basic idea of it is to design a basic generatrix and then change the parameter of the generatrix to form the blending surface. Wu and Wang([11]) studied the blending problem of several quadrics by using Wu's method and gave some examples in transition of pipelines. In these examples, the method can be used to find all possible blending surfaces with given degree. However, one has to do complicated symbolic computation in this way. And one is

not sure which surface is a “good” surface that can be used in practice. When drawing it on computer, one has to seek a parametric representation of the implicit algebraic surface. And the blending surfaces are difficult(or not) to be adjusted. The problem still exists in Wu and Zhou’s method ([10]). However they reduced the problem of finding blending algebraic surfaces to one of solving a linear system. The major advantage of our method is that we can give the explicit equation for the blending surface, while most other methods only give the blending surface at certain conditions.

In this paper, we mainly discuss the smoothly joint problem of two quadric surfaces and derive the corresponding explicit formula. And we point out that the method can be expanded to n -way blending problems. The method is called the base curve method. It works as follows. We first construct a base curve connecting the two axes of the surfaces to be blended. Based on the curve, we construct a collection of curves. And the blending surface is defined by these curves. Examples are given in this paper shown that this method gives nice solution to the problem. To get the blending surfaces, we only need the normal vector of the given surfaces on each point of their blending boundaries. Here the boundary curves are regular and continuous, and the normal vector at each point of the boundaries are well-defined and continuous. That means the blending surfaces are only defined by the boundary conditions. This is an extrusive advantage of the method. Further more, we can adjust the shape of the blending surface by adjust the base curve. The method is easy to expand to solve other blending problems. And the blending surfaces having a parametric representation make them easy to be realized on computers or industrial applications. But the blending surfaces are non-rational.

2. The blending surface of two quadric surfaces

Definition Two C^1 continuous surfaces meet along a common boundary. The two surfaces are said to have G^1 **continuity** (or tangent plane continuity) if they have the same tangent plane at each point of the boundary and the unit normal vector is continuous along the common boundary.

A space curve(or surface(parametric or implicit). Here we only consider the parametric curve.) is called **regular** if the tangent vector(or normal vector) at every point on the curve(or surface) exists and is unique and nonzero. For example, if space curve $(P(t) = (x(t), y(t), z(t)), t \in [0, 1])$ has a tangent vector $Q(t) = (\frac{\partial x(t)}{\partial t}, \frac{\partial y(t)}{\partial t}, \frac{\partial z(t)}{\partial t})$ and $Q(t) \neq (0, 0, 0)$ for all t in $[0, 1]$, $P(t)$ is a regular curve. A regular space curve is called a base curve of a curve or a surface (**base curve** for short) if the curve or the surface is constructed through the space curve based on some given rules. For example, we can regard the X-axis as the base curve of the surface of revolution $(t, t^2 \cos \theta, t^2 \sin \theta)(t \in [1, 4], \theta \in [0, 2\pi])$ and the rule is revolving $y = t^2$ along the X-axis. It is like the spine curve of a canal surface.

Theorem. Let S_1, S_2 be regular surfaces, $C = C(t) = S_1 \cap S_2$ be a regular space curve,

$N = N(t)$ be the normal vector of S_1 at each point on C . $\forall P = C(t_0) \in C$, we assume there exists a regular space curve $C_2 = C_2(s) \subset S_2$ and $C_2(s_0) = P$. Further more,

$$\frac{C'(t_0) \times C'_2(s_0)}{\|C'(t_0) \times C'_2(s_0)\|} = \pm \frac{N(t_0)}{\|N(t_0)\|}. \quad (2.1)$$

Here $C'(t_0), C'_2(s_0)$ are the tangential vectors of the curves C, C_2 at point P respectively. Then S_1 and S_2 meet with G^1 continuity along C . $C'(t_0)$ is the first derivative of $C(t)$ when $t = t_0$. Here it denotes the tangent vector of curve $C(t)$ at $C(t_0)$ (Other similar denotations are the same meaning.).

Proof. The tangent plane of S_1 at P is $\{Q_1 | (Q_1 - P) \times N(t_0) = 0\}$. And the tangent plane of S_2 at P is $\{Q_2 | (Q_2 - P) \times (C'(t_0) \times C'_2(s_0)) = 0\}$. The two planes are obvious the same plane when (2.1) holds. As is shown in Figure 1. It means that the two surfaces are tangent plane continuity. So the theorem holds.

Corollary. Let S_1, S_2 be regular surfaces, $C = C(t) = S_1 \cap S_2$ be a regular space curve. $\forall P = C(t_0) \in C$, we assume there exists regular space curves $C_1 = C_1(s) \subset S_1, C_2 = C_2(s') \subset S_2$ and $C_1(s_0) = P, C_2(s'_0) = P$. Further more $\exists a \in \mathbf{R}/\{0\}, \forall b \in \mathbf{R}$, such that

$$C'_1(s_0) = a * C'_2(s'_0), C'(t_0) \neq b * C'_1(s_0), \quad (2.2)$$

Then S_1 and S_2 meet with G^1 continuity along C .

Fig. 1. proof of the theory

Making use of the definition of geometric continuity, the theorem and the corollary tell us a constructive method to construct blending surfaces.

A quadric surface is a surface defined by a polynomial with degree two. Here we don't discuss the surfaces or other graphs defined by quadric polynomials such as $x^2 = \pm a^2, x^2 + b^2 y^2 = 0$. And we mainly discussed the **closed** surfaces, whose planar sections are either ellipses or circles if the planes intersect with the surfaces appropriately. It is easy to show that they are six such surfaces: elliptic cylinder, elliptic cone, elliptic paraboloid, elliptic sphere, hyperboloid of one sheet and hyperboloid of two sheet. There are three quadric surfaces which are not closed. They are hyperbolic paraboloid, hyperbolic cylinder, parabolic cylinder.

In what below, we show how to construct the blending surface.

Then we can get the base curve defined by the following equation.

In fact, the base curve needs not always to be Bézier curve with 4 control points for our problem. For example, we can use arc of an ellipse to contact the two axes when $d_0 = 0$. To avoid the blending surface intersecting itself, the radius of curvature of the base curve at every point should be larger than the maximal radius (in the normal plane of the base curve) at the point, which means that the following inequality should hold:

$$1/\sqrt{(\frac{\partial^2 x(t)}{\partial t^2})^2 + (\frac{\partial^2 y(t)}{\partial t^2})^2 + (\frac{\partial^2 z(t)}{\partial t^2})^2} \geq \max_{\theta \in [0, 2\pi)} r(\theta, t), \forall t \in [0, 1] \quad (2.4)$$

Here $r(\theta, t)$ is defined by (2.5). We can adjust the value of l_1 and l_2 to satisfy this inequality for all t in $[0, 1]$. Changing the value of l_1 and l_2 also can be used to rectify the shape of the base curve.

2.2 Designing the radius function Now we have a base curve $(x(t), y(t), z(t)), t \in [0, 1]$. The second step is to construct the radius function. We will define a radius function: $r(\theta, t)$, $\theta \in [0, 2\pi)$, $t \in [0, 1]$. In the normal plane of the base curve at every point $P_t(x(t), y(t), z(t))$, there exists a one-to-one correspondence between the real numbers in $[0, 2\pi)$ and the radials from P_t . Let h_0 be the normal plane of the base curve at P_t , R_t be the radial from P_t in h_0 which is parallel to the XOY-plane, R_θ be the radial from P_t in h_0 which forms an angle θ with R_t , Q_t be the intersection of R_θ and the blending plane to be constructed. Then $r(\theta, t)$ is the distance from P_t to Q_t . Obversely, it should be positive. To the same θ , we can define a regular continuous space curve by Q_t when t changes from 0 to 1. Let $S_\theta(t)$ (It is curve $\widetilde{Q_0 Q_t Q_1}$ as shown in figure 2) denote the curve. In order to connect the given surfaces smoothly, the tangential line of the curve at the extreme points should be in the tangential plane of the given surfaces. As is shown in the theorem. Let θ change in $[0, 2\pi)$. We can get a collection of curves. All these curves form the blending surface. Each point on the intersection curve ($S_1 \cap h_1$ or $S_2 \cap h_2$) and the axis of the given surface (S_1 or S_2) defines a plane, which intersects the given surface at a planar curve $C_{s1}(s)$ (or $C_{s2}(s)$) (Here $C'_{s1}(s)(C'_{s2}(s))$ is equal to $M_1(M_2)$ at point $Q_0(Q_1)$). The tangent of angle between the tangential line of the curve at the point and the axis is $\tan \alpha_1$ (or $\tan \alpha_2$), where α_i is a function of θ . We can use the planar vector $\vec{M}'_i = (1, M'_i(\theta)) = (1, \tan \alpha_i)$ ($i=1, 2$) to denote it (if the tangential line is vertical to the axis, then $\vec{M}'_i = (0, \pm 1)$). From what we state above, we know $C_{s1}(s)$ (or $C_{s2}(s)$) connecting $S_\theta(t)$ with G^1 continuity. When θ changes from 0 to 2π , $S_\theta(t)$ forms the blending surface $S(\theta, t)$. The corollary ensures that $S(\theta, t)$ connects S_1 (or S_2) with G^1 continuity. We can get the following representation about the radius function: $r(\theta, t) = \overline{P_t Q_t}, t \in [0, 1], \theta \in [0, 2\pi)$. Let $r_1(\theta)$ (or $r_2(\theta)$) be the distance from A (or B) to the point on the intersection curve which corresponds to $\theta \in [0, 2\pi)$. The radius function to be constructed should satisfy $r(\theta, 0) = r_1(\theta)$, $r(\theta, 1) = r_2(\theta)$, $\frac{\partial r(\theta, 0)}{\partial t} = M'_1(\theta)$, $\frac{\partial r(\theta, 1)}{\partial t} = M'_2(\theta)$. We can use Hermite interpolation method to get the radius function:

$$r(\theta, t) = (M'_1 - M'_2 - 2r_2 + 2r_1)t^3 + (3r_2 - 3r_1 - 2M'_1 + M'_2)t^2 + M'_1 t + r_1 \quad (2.5)$$

Here $M'_i = M'_i(\theta)$, $r_i = r_i(\theta)$, and $t \in [0, 1]$ increases along the base curve from A to B. We will show how to calculate the radius function of an elliptic cone and a cylinder for example. The standard functions of them are (It is not denoted the corresponding position in the coordinate system): $\frac{y^2}{a^2} + \frac{z^2}{b^2} - cx^2 = 0$, $y^2 + z^2 - 0.25 = 0$. In order to simplify the calculation, we consider the parametric form of the first surface: $(x(\theta, t), y(\theta, t), z(\theta, t)) = (t, a \cos \theta f(t), b \sin \theta f(t))$. Here $f(t) = \sqrt{c}t, t > 0$. Then $r_1(\theta) = \sqrt{c}d_1 \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$, $M'_1(\theta) = \sqrt{c} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$. In the same way, we can get the following result: $r_2(\theta) = 0.5$, $M'_2(\theta) = 0$.

2.3 Getting the parametric blending surface From the discussion above, we can get the expression of the blending surface. One can prove that it connects the given surfaces with tangent plane continuity. $(x(t), y(t), z(t))$ in the formula is the base curve which can be defined by (2.3).

$$\left\{ \begin{array}{l} x(\theta, t) = r(\theta, t) \left(\frac{-\frac{\partial x(t)}{\partial t} \frac{\partial z(t)}{\partial t}}{\sqrt{((\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2 + (\frac{\partial z(t)}{\partial t})^2)((\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2)}} \sin \theta \right. \\ \quad \left. + \frac{-\frac{\partial y(t)}{\partial t}}{\sqrt{(\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2}} \cos \theta \right) + x(t) \\ y(\theta, t) = r(\theta, t) \left(\frac{-\frac{\partial y(t)}{\partial t} \frac{\partial z(t)}{\partial t}}{\sqrt{((\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2 + (\frac{\partial z(t)}{\partial t})^2)((\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2)}} \sin \theta \right. \\ \quad \left. + \frac{\frac{\partial x(t)}{\partial t}}{\sqrt{(\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2}} \cos \theta \right) + y(t) \\ z(\theta, t) = r(\theta, t) \frac{(\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2}{\sqrt{((\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2 + (\frac{\partial z(t)}{\partial t})^2)((\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2)}} \sin \theta + z(t) \end{array} \right. \quad (2.6)$$

3. Examples

Example 1 In this example, we consider the connecting of an elliptic cylinder ($\frac{y^2}{a_2^2} + \frac{z^2}{b_2^2} - c_2x = 0$) and an elliptic paraboloid ($\frac{y^2}{a_1^2} + \frac{z^2}{b_1^2} - 1 = 0$). Then we can get the blending surface defined by (2.6). Here

$$r_2(\theta) = \sqrt{c_2 d_2} \sqrt{a_2^2 \sin^2 \theta + b_2^2 \cos^2 \theta}, r_1(\theta) = \sqrt{a_1^2 \sin^2 \theta + b_1^2 \cos^2 \theta} \quad (3.1)$$

$$M_2'(\theta) = \sqrt{a_2^2 \sin^2 \theta + b_2^2 \cos^2 \theta} \frac{c}{2\sqrt{cd_2}}, M_1'(\theta) = 0 \quad (3.2)$$

And the parameters of the blending surface shown in Figure 3 are given below: $l_1 = l_2 = 0.5$, $d_0 = 0.3$, $d_1 = 0.5$, $d_2 = 0.6$, $\alpha = 5\pi/6$, $a_1 = 0.25$, $b_1 = 0.3$, $a_2 = 0.3$, $b_2 = 0.35$, $c_2 = 0.3$.

Example 2 Let us assume that the surfaces to be blended are two cylinders with intersecting axes. The two axes form an angle α . The radii of the cylinders are r_1 and r_2 respectively. Using the method introduced in section 2, we construct the blending surface as the following form by (2.6):

$$\left\{ \begin{array}{l} x(\theta, t) = r(\theta, t) \left(\frac{-\frac{\partial y(t)}{\partial t}}{\sqrt{(\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2}} \cos \theta \right) + x(t) \\ y(\theta, t) = r(\theta, t) \left(\frac{\frac{\partial x(t)}{\partial t}}{\sqrt{(\frac{\partial x(t)}{\partial t})^2 + (\frac{\partial y(t)}{\partial t})^2}} \cos \theta \right) + y(t) \\ z(\theta, t) = r(\theta, t) \sin \theta + z(t) \end{array} \right. \quad (3.3)$$

Here $r(\theta, t) = 2(r_1 - r_2)t^3 - 3(r_1 - r_2)t^2 + r_1$. And $(x(t), y(t), z(t)) = (d_1 \cos \alpha, d_1 \sin \alpha, 0) \cdot B_{2,0}(t) + (0, 0, 0) \cdot B_{2,1}(t) + (d_2, 0, 0) \cdot B_{2,2}(t)$. In order to get a “good” blending surface, we should think about inequality (2.4). The problem is easily transformed to the following form:

$1/\sqrt{d_1^2 + d_2^2 + 2d_1d_2 \cos \alpha} \geq 2\max\{r_1, r_2\}$. The parameters of the graph shown in Figure 4 are : $r_1 = 0.2$, $r_2 = 0.3$, $d_1 = 0.4$, $d_2 = 0.3$, $\alpha = 5\pi/6$.

Fig. 3. elliptic cylinder jointing with elliptic paraboloid

Fig. 4. connecting of two cylinders

Example 3 In this example, we will show that our method can be modified to construct blending surfaces along non-planar blending boundaries. The axes of two cylinders are vertical and intersectant. Cylinder($y^2 + z^2 - r_1^2 = 0$) cuts in cylinder($x^2 + z^2 - r_2^2 = 0$). Cylinder($y^2 + z^2 - r^2 = 0, r < r_2$) intersects the second cylinder at a space curve. We can get the blend surface with the following representation:

$$S(\theta, t) = (d_0 - d(\theta, t), R(\theta, t) \cos \theta, R(\theta, t) \sin \theta), t \in [0, b(\theta)] \quad (3.4)$$

Here $b(\theta) = d_0 - \sqrt{r_2^2 - r^2 \sin^2 \theta}$,

$$R(\theta, t) = (r_1 - r)(2t^3 - 3t^2) + 3l_2(t^3 - t^2) \sqrt{\frac{r_2^2 - r^2 \sin^2 \theta}{r_2^2 - r^2 \sin^2 \theta \cos^2 \theta}} + r_1 \quad (3.5)$$

$$d(\theta, t) = 3l_1 t^2(t - 2) + 3l_2 t^2(t - 1) \frac{r \sin^2 \theta}{\sqrt{r_2^2 - r^2 \sin^2 \theta \cos^2 \theta}} - t^2(2t - 3)b(\theta) + 3l_1 t \quad (3.6)$$

Here the parameters l_1, l_2 are used to adjust the shape of the blending surface. The parameters of the figure shown in Figure 5 are the following: $d_0 = 0.6$, $r_1 = 0.2$, $r_2 = 0.3$, $r = 0.2$, $l_1 = 0.1$, $l_2 = 0.1$.

Example 4 Five cylinders whose axes jointing at the same point connecting a sphere with G^1 continuity are shown in this example. As is shown in Figure 6.

4. Conclusion

A method for connecting two surfaces G^1 -continuously is introduced. It is based on a G^1 -continuous parametric regular curve. Obviously, this method can be extended to connecting general regular surfaces.

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Fig. 5. two cylinders inserting

Fig. 6. five cylinders jointing

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