Construct Piecewise Hermite Interpolation Surface with Blending Methods

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Abstract. Three methods are proposed to construct a piecewise Hermite interpolation surface (PHIS), which is a piecewise algebraic surface interpolating a set of given points with associated normal directions. The surface is obtained by blending together some low-degree surface patches. Both the first and the second methods are completely local and give a surface with $G^0$-continuity. In the third construction, we reduce the number of surfaces patches by joining as many cubic patches as possible. This method gives a global solution with $G^1$-continuity. These three different methods can be used to meet different requirements of the designers.

1. Introduction

To construct a piecewise smooth surface interpolating a set of scattered data with a prescribed triangulation is of great importance in solid modeling or free-form modeling [Baj3, Ber, Hop]. It is paid great attention that the normal direction of these data is given. We call such interpolation surface a piecewise Hermite interpolation surface (PHIS). The use of implicit surfaces to construct PHIS has been described in many papers [Baj1, Baj2, Dah1, Dah2, Shi]. Implicit surfaces have some important advantages over parametric surfaces. Specifically, the set of algebraic surfaces is closed under certain geometric operations such as intersection and offset. Furthermore, the implicit algebraic equation presentation captures all elements of that set. Sederberg introduced the Bernstein-Beziér (BB) form of a tri-variate polynomial within a tetrahedron such that it’s easy to control the shape of the surface [Sed]. With thus form, Dahmen proposed an approach with quadrics patches for constructing tangent plane continuous (i.e. $G^1$-continuous) surface to interpolate a set of scattered data with given normal directions [Dah1]. In order to decrease the number of the patches and overcome some shape control problems such as multi-sheets and self-intersection, several improvements have been proposed in later publications [Baj2, Dah2, Guo, Xu]. Cubic surfaces instead of quadrics were used in [Dah2]. Guo considered the patches within a bitetrahedron to give a single sheeted, smooth and singly connected surface [Guo]. Bajaj etc. defined a smooth and single-sheeted algebraic surface patches named A-patches [Baj2], then Xu etc. extended it to rational functions [Xu]. On the other hand, Li gave a $C^2$-continuity interpolation surface based the idea of subdivision [Li H.].

In this paper, we will convert the construction of a PHIS into the design of a set of blending surfaces. There are various blending methods for implicit algebraic surfaces [Baj1, Har, Hof1, Roc, War, Wu T., Wu]. The methods in [Hof1] and [Roc] depent essentially on the representation of the

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surfaces to be blended and could be considered as extensions of LiMing’s skill [Li]. Hartmann generalized those tools as functional splines and used them to blend vertices with $G^n$-continuity [Har]. Warren gave an algebraic description for $G^n$-continuity and derived a general method for blending several surfaces simultaneously [War]. Several quadrics were specially considered in [Wu T.], in which the blending problem of implicit surfaces was transformed into solving a corresponding linear equation system. An algorithm called Hermite interpolation, which is based on Bezout’s Theorem, was presented in [Baj1]. Based on the idea of generic point, Wu gave a method to obtain blending surfaces with lowest degree [Wu]. Shi then extended this method to give a PHIS of degree six [Shi].

In this paper we’ll give three methods to construct a PHIS. The surface is obtained by blending together some low-degree surfaces. In the first method, each triangular patch, interpolating three points, is obtained by blending together three transition surfaces interpolating any two of these points. The second method uses more patches while reduces the degree of the surfaces. Here, the data are interpolated with cubic surfaces, then any two adjacent patches are blended together using a higher-degree surface. Both the first and the second methods are completely local and give a piecewise algebraic surface with $G^n$-continuity. Finally, in our third construction, we reduce the number of patches by joining as many cubic surfaces as possible, directly with $G^1$-continuity. As a difference from the previous two methods, this approach requires global computations. The limitation of quadrics in constructing a PHIS is also discussed.

The rest of the paper is organized as follows. In Section 2., we define some notations and introduce several results about the BB form and surface blending. Our methods are illustrated in Sections 3. and 4.. Examples are also given simultaneously. The conclusion is contained in Section 5..

2. Notations and Basic Results

The following notations will be used in the paper.

$P := \{ p_i \in \mathbb{R}^3 : i \in S \}$, a set of points.

$N := \{ n_i \in \mathbb{R}^3 : i \in S \}$, a set of normal directions.

$T := \text{a collection of triples } I \in \{1, \ldots, k\}^3$.

$[I] := \text{a planar triangle spanned by } p_i, \ i \in I$.

$T$ is said to be a surface triangulation of $P$ if the intersection $[I] \cap [J]$, for any $I, J \in T$, is empty or a vertex or a common edge of $[I]$ and $[J]$.

Two points are said adjacent if they are contained in an edge of a triangle and two elements $I_1, I_2 \in T$ are adjacent if $[I_1]$ and $[I_2]$ share a common edge.

The problem can be stated as follows:

**PHIS:** With $P, N, T, S$ stated as above, construct a smooth piecewise algebraic surface $f = 0$ such that

1. $f(p_i) = 0, \ i \in S$.
2. $\nabla f(p_i) = c_i n_i, \ i \in S$, for some nonzero constants $c_i$, where $\nabla f = (\partial f / \partial x, \partial f / \partial y, \partial f / \partial z)$.

In the following, a surface $f = 0 \text{ interpolates a point } p_i \in P$ means that it not only contains point $p_i$ but also has the desired normal direction $n_i$ at $p_i$. We also say that a surface $f = 0 \text{ interpolates } I \in T$ if it interpolates the three vertices of $[I]$. 


Bernstein-Bézier form: Let \( p_1, p_2, p_3, p_4 \) be affine independent points in \( \mathbb{R}^3 \) and \( V = [p_1 p_2 p_3 p_4] \) a tetrahedron with vertices \( p_1, p_2, p_3, p_4 \). \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) is said to be the barycentric coordinates of \( p \) if

\[
p = \sum_{i=1}^{4} \lambda_i p_i, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{4} \lambda_i = 1.
\]

The barycentric coordinates relate to the Cartesian coordinates via

\[
\begin{pmatrix}
x \\
y \\
z \\
1
\end{pmatrix} = \begin{pmatrix}
x_1 & x_2 & x_3 & x_4 \\
y_1 & y_2 & y_3 & y_4 \\
z_1 & z_2 & z_3 & z_4 \\
1 & 1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{pmatrix}
\]

Any polynomial \( f(p) \) with degree \( n \) has the Bernstein-Bézier form over \( V \) as

\[
f(p) = \sum_{|\alpha|=n} c_{\alpha} B_{\alpha}^n(\lambda), \quad |\alpha| = \sum_{i=1}^{4} \alpha_i, \quad \alpha_i \geq 0,
\]

where

\[
B_{\alpha}^n(\lambda) = \frac{n!}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3} \lambda_4^{\alpha_4}
\]

is Bernstein polynomial, \( c_{\alpha} \) are called control points. It can be easily seen that the surface \( f(p) = 0 \) contains point \( p_i \) if and only if \( c_{\alpha e_i} = 0 \), where \( e_i \) is the \( i-th \) unite vector.

The following lemma will be used to determine the control points of an interpolation surface around a vertex from the normal at the interpolation point.

**Theorem 2.1** [Far1] If \( f(p) = \sum_{|\alpha|=n} c_{\alpha} B_{\alpha}^n(\lambda) \) then for \( j = 1, 2, 3, 4, \ j \neq i \),

\[
b_{(n-1)e_i+e_j} = b_{ne_i} - \frac{1}{n} (p_j - p_i) \cdot \nabla f(p_i),
\]

Based on the ideal theory, Warren gave an algebraic description of the blending surfaces.

**Definition 2.2** [War] Surfaces \( f = 0 \) and \( f_0 = 0 \) intersect transversally if there exists a point \( p \) on each irreducible component of \( f = 0, f_0 = 0 \) such that the tangent planes to \( f = 0 \) and \( f_0 = 0 \) at \( p \) exist and are distinct.

**Theorem 2.3** [War] Let \( f \) and \( f_0 \) be distinct irreducible polynomials. If surface \( f = 0 \) and \( f_0 = 0 \) intersect transversally in a single irreducible curve, then any polynomial \( F \) such that \( F = 0 \) meets \( f = 0 \) with \( G^n \)-continuity along \( f_0 = 0 \) must be in the ideal \( (f, f_0^{n+1}) \), i.e. the ideal generated by \( f \) and \( f_0^{n+1} \).

Hartmann used the following elliptic functional splines to blend surfaces [Har].

**Theorem 2.4** [Har] Let \( f = 0, f_0 = 0 \) be two intersecting \( C^n \)-continuous regular implicit surfaces. Then the implicit surface

\[
F := (1 - \mu)f - \mu f_0^{n+1} = 0, \quad 0 < \mu < 1.
\]

meets \( f = 0 \) with \( G^n \)-continuity along \( f_0 = 0 \).
Remark 2.5 To blend two or more surfaces \( f_i = 0 \), one may choose \( f = f_1 f_2 \ldots f_k \) and a suitable surface \( f_0 = 0 \).

Theorem 2.6 [Har] Let \( f_1 = 0, f_{10} = 0 \) and \( f_2 = 0, f_{20} = 0 \) be two pairs of intersecting \( C^n \)-continuous regular implicit surfaces. Intersection curve \( f_i = 0, f_{i0} = 0 \) must not be contained in surfaces \( f_k = 0 \) and \( f_{k0} = 0 \) for \( i \neq k \).

Then the implicit blending surface

\[
F := (1 - \mu) f_1 f_{20}^{n+1} - \mu f_2 f_{10}^{n+1} = 0, \quad 0 < \mu < 1.
\]

meets \( f_i = 0 \) with \( G^n \)-continuity along \( f_{i0} = 0 \), \( i = 1, 2 \).

3. Solution via Transition Surfaces

3.1. Method Description

Here we’ll provide a completely local procedure to construct a PHIS. First, for any two adjacent points \( p_i, p_j \in P \), an interpolating surface \( f_{ij} = 0 \), called transition surface, a plane \( p_{ij} = 0 \) containing \( p_i, p_j \), named cutting plane, are constructed. Then, on each triangle \([(i, j, k)]\), a blending surface is constructed to blend together the three surfaces: \( f_{ij} = 0, f_{ik} = 0, f_{jk} = 0 \) along planes \( p_{ij} = 0, p_{ik} = 0, p_{jk} = 0 \) respectively.

Transition Surfaces: The transition surfaces have great impact on the properties of the blending surfaces and play important roles in the final model of a PHIS. We may get it just as the construction of the interpolation surface in section 4.1. But it seems complex for the construction of the transition surfaces. In this part, however, we intend not to go deep into this construction. We’ll illustrate two methods to give transition surfaces with quadrics in section 3.2..

Cutting Planes: The equation of line \( \overline{p_ip_j} \) is

\[
\frac{x - x_i}{x_j - x_i} = \frac{y - y_i}{y_j - y_i} = \frac{z - z_i}{z_j - z_i}.
\]

Let

\[
g_1 = (x - x_i)(y_j - y_i) - (y - y_i)(x_j - x_i),
g_2 = (z - z_i)(y_j - y_i) - (y - y_i)(z_j - z_i).
\]

where \( p_i = (x_i, y_i, z_i) \). The cutting plane \( p_{ij} \) containing line \( \overline{p_ip_j} \) lies in a plane group of the form:

\[
p_{ij} := \alpha g_1 + (1 - \alpha) g_2 = 0, \quad 0 \leq \alpha \leq 1.
\]

Remark 3.1 As a default choice, we can choose the cutting plane \( p_{ij} = 0 \) such that it has an equal biangle with the two planes which share the common line \( \overline{p_ip_j} \). Fig 1 shows a transition surface cut by a cutting plane. Fig 2 illustrate three lines obtained by three transition surfaces cut by three cutting planes respectively.
Fig. 1.

**Blending Surfaces:** For $I = (i, j, k) \in T$, we can get three transition surfaces $f_{ij} = 0, f_{ik} = 0, f_{jk} = 0$ and cutting planes $p_{ij} = 0, p_{ik} = 0, p_{jk} = 0$ from the above procedure. Let the blending surface

$$f_{ijk} := (1 - u)f_{ij}f_{ik}f_{jk} - u(p_{ij}p_{ik}p_{jk})^{n+1} = 0, \ 0 < u < 1.$$  

From Theorem 2.4, $f_{ijk} = 0$ is a desired PHIS and we get the theorem below:

**Theorem 3.2** We can construct a PHIS with $G^n$-continuity of degree $\max\{d^3, 3(n + 1)\}$ from the above procedure, where $d$ is the degree of the transition surface $f_{ij}$. The number of the surface patches is just as many as that of the elements in the triangulation $T$.

Proof: For any two adjacent elements $I_1 = (i, j, k), I_2 = (i, j, l) \in T$, suppose that $f_{ij}, f_{ik}, f_{jk}, f_{ijk}$ and $f_{ijl}$ are constructed from the above procedure. The interpolation property of $f_{ijk} = 0$ at points $p_i, p_j, p_k$ is evident from its containment of $f_{ij}, f_{ik}, f_{jk}$. On the other hand, since both $f_{ijk} = 0$ and $f_{ijl} = 0$ meet $f_{ij} = 0$ with $G^n$-continuity along $p_{ij} = 0$, $f_{ijk} = 0$ and $f_{ijl} = 0$ meet with $G^n$-continuity. Then the composite surface $f_{ijk}, (i, j, k) \in T$, is a PHIS with $G^n$-continuity.

### 3.2. Solution with Surfaces of Degree Six and Its Limitation

Choosing transition surface $f_{ij} = 0$ as a quadric, we may obtain a $G^1$-continuous PHIS with degree six. In this section we’ll illustrate two methods to construct the transition surfaces and cutting plane [Li M.]. One of them considers the functional splines [Har] and the other benefits from the property of the triangular Bézier patches.

![Fig. 3. A quadric via functional splines](image)

**Quadrics via functional splines:** For two adjacent points $p_i, p_j \in P$, suppose $t_i = 0, t_j = 0$ be the tangent planes at point $p_i, p_j$ respectively and $s_{ij} = 0$ a cutting plane, just as shown in Figure 3.
Let
\[ f_{ij} := (1 - \lambda)t_it_j - \lambda s_{ij}^2 = 0, \quad 0 < \lambda < 1 \]

The quadric surface \( f_{ij} = 0 \) meets planes \( s_i = 0, s_j = 0 \) along \( s_{ij} = 0 \) with \( G^1 \)-continuity. Its interpolation property at point \( p_i, p_j \) is a direct consequence of our construction.

Triangular Bézier patches: In order to give an interactive control of the transition surfaces and the cutting planes, we will take the triangular Bézier patches into consideration [Far1]. For \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \), a triangular Bézier patch with \( b_{\alpha} \) as its control points is defined as
\[
b(\lambda) = \sum_{|\alpha|=n} b_{\alpha} B^n_{\alpha}(\lambda), \quad |\alpha| = \sum_{i=1}^{3} \alpha_i, \quad \alpha_i \geq 0,
\]

Where
\[
B^n_{\alpha}(\lambda) = \frac{n!}{\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3}}, \quad \sum_{i=1}^{3} \lambda_i = 1.
\]

For \( I = (i, j, k) \in T \), let \( t_i = 0, t_j = 0, t_k = 0 \) be the tangent planes at points \( p_i, p_j, p_j \) and \( s_{ij}, s_{jk}, s_{ik} \) the common points of the corresponding tangent planes. See Figure 4. The Bézier curve in \( R^3 \) with \( p_i, s_{ij}, p_j \) as its control points is of the form below,
\[
b_{ij}(t) := p_i t^2 + 2s_{ij} t(1-t) + p_j (1-t)^2 = 0.
\]

It’s evident that \( b_{ij}(t) = 0 \) is a plane conics and can be implicitized as a quadric \( f_{ij} = 0 \) cut by a plane \( p_{ij} = 0 \) in \( R^3 \), which corresponds to our transition surface and cutting plane respectively.

A PHIS with transition surface: With above procedures, \( f_{ij}, f_{ik}, f_{jk} \) and \( p_{ij}, p_{ik}, p_{jk} \) are obtained. Let
\[
f_{ijk} := (1 - u)f_{ij}f_{ik}f_{jk} - u(p_{ij}p_{ik}p_{jk})^2 = 0.
\]

\( f_{ijk} = 0 \) is a surface of degree six and the composite surface \( f_{ijk} = 0, (i, j, k) \in T \) is a PHIS with \( G^1 \)-continuity.

Fig 5 shows a blending surface with a transition surface, which keep \( G^1 \) continuity. We give an example in Fig 6 with a piecewise surface with degree 6, the data from which is gotten from a sphere. Since their lack of enough degrees of freedom, the abilities of quadrics to express free-form surfaces are limited. For instance, it is impossible to avoid intersections and cusps at the same time if using only one quadric patch to interpolate two adjacent points. We will construct a 2D piecewise Hermite Interpolation Curve (PHIC) to illustrate this. This is reasonable because 2D interpolation
problem is a special case of its 3D counterpart. To see this, we may construct a 3D interpolation problem such that the points in a subset of $P$ and their normal directions are all in a plane. Then a solution to this problem will lead to a PHIC.

**PHIC:** Let $P := \{p_i \in \mathbb{R}^2, i \in S\}$ with give order and $N := \{n_i \in \mathbb{R}^2, i \in S\}$ a set of normal directions associated with $P$. Construct a piecewise smooth algebraic curve such that for $i \in S$

1. $f(p_i) = 0$,
2. $\nabla f(p_i) = c_i n_i$ for some nonzero constants $c_i$, where $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

Farin and Pottmann considered similar plane curve interpolation problems while the curvatures demands is added [Far2, Pot]. Using conics in rational Bézier form will lead to a PHIC [Fud, Li M.]. However, as shown in Figures 7 and 8, in certain cases, it is impossible to avoid intersections and cusps at the same time [Li M.].

To cope with such unexpected case, we may add the number of the conics as [Pot] or elevate the degree of the curve [Li M.]. Figure 9 shows a PHIC constructed with cubic curve segments. Acted in a similar way in $\mathbb{R}^3$, we need to interpolate $p_i, p_j \in P$ with more than one quadric patches or using cubic surfaces. With the latter consideration, the degree of the blending surface will be nine, which
is too high. In the next section, we will present two PHIS’s with lower degrees.

\[ \textbf{4. Solutions Based on Cubic Surfaces} \]

In this section we’ll provide two methods based on cubic surfaces to construct a PHIS with blending methods. The first is completely local and the second is global while with less cubic surfaces as compared with previous publications [Baj2, Dah2].

\[ \textbf{4.1. Three Basic Constructions} \]

\textit{Interpolation Surfaces}: For \( I_0 = (i, j, k) \in T \), add a new point \( v \) such that the three tangent planes at point \( p_i, p_j, p_k \) are locally contained in tetrahedron \( V = [p_i p_j p_k v] \). One can refer [Baj2] for more details. Rewrite the vertices as \( p_1 = p_i, p_2 = p_j, p_3 = p_k, p_4 = v \) and assume the interpolation surface \( f_0(\lambda) = 0 \) in BB form:

\[ f_0(\lambda) = \sum_{|\alpha|=3} c_\alpha B_\alpha(\lambda). \]

Let \( c_{3000} = 0, c_{0300} = 0, c_{0030} = 0 \), then

\[ f_0(p_i) = 0, \quad i = 1, 2, 3. \]

Moreover for \( i = 1, 2, 3 \), choose

\[ c_{2e_i + e_j} = \frac{c_i}{n} (p_j - p_i) \cdot n_i, \quad j = 1, 2, 3, 4, \quad j \neq i, \]

where \( c_i \) is a nonzero constant. From Theorem 2.1,

\[ \nabla f_0(p_i) = c_i n_i. \]

Now the following theorem is evident and Figure 11 shows an example of an interpolation surface.

\textbf{Theorem 4.1} The cubic interpolation surface \( f_0(\lambda) = 0 \) determined above interpolates \( p_i, p_j, p_k \) with given normal directions \( n_i, n_j, n_k \).
Propagation Surfaces: Let $I_1 = (i, j, k), I_2 = (i, j, l) \in T$ and $f_1 = 0$ a cubic surface interpolating $I_1$. We will construct a cubic surface $f_2 = 0$, which interpolates $I_2$ and meets surface $f_1 = 0$ with $G^1$-continuity.

Take a cutting plane $p_{ij} = 0$ and set

$$f_2 = f_1 - (ax + by + cz + d)p_{ij}^2, \quad 0 < u < 1.$$ 

It can be seen that $f_2 \in (f_1, p_{ij}^2)$, the ideal generated by $f_1$ and $p_{ij}^2$. From Theorem 2.3, surface $f_2 = 0$ meets $f_1 = 0$ with $G^1$-continuity along the plane $p_{ij} = 0$. To ensure $f_2 = 0$ interpolate point $p_l$ with normal direction $n_l = (n_{1l}, n_{2l}, n_{3l})$, let

$$f_2(p_l) = 0,$$

$$f_{2x}(p_l)n_{1l}^2 - f_{2y}(p_l)n_{1l}^1 = 0,$$

$$f_{2x}(p_l)n_{1l}^2 - f_{2z}(p_l)n_{1l}^1 = 0,$$

$$f_{2y}(p_l)n_{1l}^3 - f_{2z}(p_l)n_{1l}^2 = 0$$

where,

$$f_{2x} = \frac{\partial f_2}{\partial x}, \quad f_{2y} = \frac{\partial f_2}{\partial y}, \quad f_{2z} = \frac{\partial f_2}{\partial z}.$$ 

Thus we get a system of linear equations in $a, b, c, d$. Considering the linear dependence of the last three equations, there will remain a free coefficient which can be used as a shape parameter. Figure 10 shows an example of a propagation surface.

**Theorem 4.2** Suppose $I_1 = (i, j, k), I_2 = (i, j, l)$ are two adjacent elements in $T$ and $f_1 = 0$ an interpolation surface to $I_1$. Then the cubic surface $f_2 = 0$ determined from above procedure will have $G^1$-continuity with $f_1 = 0$ and interpolate $I_2$.

Blending Surfaces: Suppose that $f_1 = 0, f_2 = 0$ are two cubic surfaces interpolating $I_1 = (i, j, k), I_2 = (i, j, l) \in T$ without continuity. Let $p_1, p_2$ be two cutting planes both containing line $p_ip_j$ and let

$$G := (1 - u)f_1p_2^{n+1} - uf_2p_1^{n+1} = 0, \quad 0 < u < 1.$$ 

From Theorem 2.6, the blending surface $G = 0$ joins the two cubic surfaces $f_1 = 0, f_2 = 0$ with $G^n$-continuity along $p_1 = 0, p_2 = 0$ respectively.

**Theorem 4.3** For two interpolation cubic surfaces $f_1 = 0, f_2 = 0$ intersecting transversally, the surface $G = 0$ of degree $n + 4$ constructed above will join them with $G^n$-continuity.
4.2. Local Solution with $G^n$-continuity

For every element $I \in \mathcal{T}$, construct a cubic surface $f_I = 0$ interpolating $I$ with Theorem 4.1. Then, based on Theorem 4.3, for every two surfaces $f_I = 0$ and $f_J = 0$, where $I, J$ are adjacent in $\mathcal{T}$, a blending surface $f_{IJ} = 0$ will be got, which joins $f_I$ and $f_J$ with $G^n$-continuity, $n \geq 1$. The derived composite surface is a PHIS with $G^n$-continuity. Particularly, with $G^1$-continuity, the highest degree of the surfaces involved is five, with $G^2$-continuity, the degree is six.

4.3. A Solution Mainly with Cubic Surfaces

In this section we’ll describe our propagation method to generate a PHIS, mainly with cubic surface. To give an appropriate order of this propagation, we represent the triangulation $\mathcal{T}$ with a graph $\mathcal{G}$ such that the vertices of $\mathcal{G}$ represent the triangles in $\mathcal{T}$ and an edge $(i, j)$ in the graph means that $I_i, I_j$ in $\mathcal{T}$ are adjacent. For example, Figure 13 is the corresponding graph of the triangulation in Figure 12.

Propagation Algorithm:

1. Choose any element $I_0 \in \mathcal{T}$ and construct an interpolation cubic surface $f_0$ for $I_0$ with Theorem 4.1. The surface is set as an initial surface.

2. Generate a graph representation $\mathcal{G}$ for $\mathcal{T}$ and a spanning tree $\mathcal{T}$ for $\mathcal{G}$ with $I_0$ as the root.

3. Start from $I_0$, for any edge $(I_i, I_j)$ in the tree $\mathcal{T}$, construct a propagation surface with Theorem 4.2.

4. For those edges in $\mathcal{G} - \mathcal{T}$, construct the blending surfaces of degree five with Theorem 4.3.

Remark 4.4  Suppose that the number of the vertices and edges of the graph $\mathcal{G}$ is $v$ and $e$ respectively. We will use $v$ cubic surfaces and $e - v + 1$ quintic surfaces in all.

4.4. Shape Control and Examples

It is a difficult problem to control the shape of an implicit surface. It corresponding to control the remaining free coefficients, which serve as shape parameters, in our construction procedures. For an interactive shape control, one may specify some additional data and then approximate these points in the least square sense. Taken as a default shape control, a method may be taken to keep the surface patch close to a quadric patch [Dah2]. Figure 17 shows a PHIS corresponding the triangulation in Figure 14 with the propagation method. The data is selected without constraints. In this example,
the initial surface is shown in Figure 15, then we propagate it in Figure 16 with three cubic surfaces. In the end, one quintic surface is used to give a blending in Figure 17.

5. Conclusion

A better PHIS should use less surface patches with lower degrees. We shows that using quadrics to construct a PHIS generally leads to cusps or intersections. To obtain a smooth piecewise interpolation surface with quadrics, it is expected that a large number of surface patches are needed. Cubic surfaces are general enough to construct a PHIS. But using cubic surfaces alone, we also need to construct many patches. In this paper, we try to construct a PHIS with as less number patches as possible, and most of the surface patches are cubic. A challenging problem is that: can we use $n$ cubic patches to construct...
References


